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A STUDY ON THE POSITIVE SOLUTIONS OF AN EXPONENTIAL TYPE DIFFERENCE EQUATION

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Abstract. The aim of this paper is to study the dynamical behavior of positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-k}},\ n = 0, 1, 2, ...$$

with the parameters α, β, γ are positive real numbers and the initial conditions x_{-k}, x_{-k+1},. . .,x_0 are non-negative real numbers where k is an even number.

1. Introduction

Recently, the behaviors of positive solutions for exponential type difference equations have attracted great attention of many authors. That is because exponential type difference equations have many applications in population dynamics. For example, In [5], El-Metwally et al. investigated the global stability of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},\ n = 0, 1, 2, ... \tag{1}$$

where the parameters α, β are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary non-negative numbers, which is a population model. Wang and Feng [13] investigated the global asymptotic stability of the recursive sequence

$$x_{n+1} = \alpha + \beta x_n e^{-x_{n-1}},\ n = 0, 1, 2, ... \tag{2}$$

where the parameters α, β are positive numbers and and the initial conditions x_{-1}, x_0 are arbitrary non-negative numbers, which is a further population model. Ozturk et al. [11] investigated the convergence, the boundedness and the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-1}},\ n = 0, 1, 2, ... \tag{3}$$

where the parameters α, β, γ are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary non-negative numbers. For more works, see [1], [3], [5], [6], [7], [10], [11], [12], [13] and the references cited therein.

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Motivated by all above mentioned work, in this paper we investigate the convergence, the boundedness and the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-k}}, \ n = 0, 1, 2, ... \tag{4}$$

with the parameters α, β, γ are positive real numbers and the initial conditions x_{-k}, x_{-k+1},. . .,x_0 are non-negative real numbers where k is an even number.

2. Preliminaries

Now we present some definitions and results which will be useful in our investigation, for more details we refer to [2], [4], [8] and [9].

Definition 1. *Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function where k is a non-negative integer. Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \ n = 0, 1, 2, ... \tag{5}$$

with the initial conditions $x_{-k}, ..., x_0 \in I$. A point $\overline{x}$ is called an equilibrium point of Eq.(5) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

Definition 2. *Let $\overline{x}$ be an equilibrium point of Eq.(5).*

(i) *The equilibrium $\overline{x}$ is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0, ..., x_{-k} \in I$ with $|x_0 - \overline{x}| + ... + |x_{-k} - \overline{x}| < \delta$, then*
$$|x_n - \overline{x}| < \varepsilon, \ \forall n \geq -k.$$
(ii) *The equilibrium $\overline{x}$ is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $x_0, ..., x_{-k} \in I$ with $|x_0 - \overline{x}| + ... + |x_{-k} - \overline{x}| < \gamma$, then*
$$\lim_{n\to\infty} x_n = \overline{x}.$$
(iii) *The equilibrium $\overline{x}$ is called global attractor if for every $x_0, ..., x_{-k} \in I$,*
$$\lim_{n\to\infty} x_n = \overline{x}.$$
(iv) *The equilibrium $\overline{x}$ is called globally asymptotically stable if it is locally asymptotically stable and is a global attractor.*
(v) *The equilibrium $\overline{x}$ is called unstable if is not stable.*

Definition 3. *Let $a_i = \frac{\partial f}{\partial u_i}(\overline{x}, ..., \overline{x})$ for each $i = 0, 1, ..., k$ denote the partial derivatives of $f(u_0, u_1, ..., u_k)$ evaluated at an equilibrium $\overline{x}$ of Eq.(5). Then*

$$z_{n+1} = a_0 z_n + a_1 z_{n-1} + ... + a_k z_{n-k}, \ n = 0, 1, 2, ... \tag{6}$$

is called the linearized equation of Eq.(5) about the equilibrium point $\overline{x}$.

Theorem 4. *(Clark's theorem). Consider Eq.(6). Then*

$$\sum_{i=0}^{k} |a_i| < 1 \tag{7}$$

is a sufficient condition for the locally asymptotically stability of Eq.(5).

Definition 5. *Eq.(5) is said to be permanent, if there exist numbers P and Q with $0 < P \leq Q < \infty$ such that for any initial conditions $x_{-k}, ..., x_0$ there exists a positive integer N which depends on the initial conditions such that $P \leq x_n \leq Q$ for $n \geq N$.*

Definition 6. *A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(5) is called periodic with period p if $x_{n+p} = x_n$ for $n \geq -k$.*

Definition 7. *A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(5) is called nonoscillatory if there exists $N \geq -k$, such that either*

$$x_n > \overline{x},\ \forall n \geq N \text{ or } x_n < \overline{x},\ \forall n \geq N,$$

and it is called oscillatory if it is not nonoscillatory.

Lemma 8. *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-k}),\ n = 0, 1, 2, ..., \tag{8}$$

where $k \in \{1, 2, ...\}$. Let $I = [a, b]$ be some interval of real numbers, and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(i) *$f(u, v)$ is a non-increasing function both in u and v;*
(ii) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$M = f(m, m) \text{ and } m = f(M, M)$$

then $m = M$. Then Eq.(8) has a unique positive equilibrium point $\overline{x}$ and every solution of Eq.(8) converges to $\overline{x}$.

3. Main Results

Firstly, we determine the existence and uniqueness of equilibrium point of Eq.(4). The equilibrium points of Eq.(4) are solutions of the equation

$$\overline{x} = \frac{\alpha + \beta e^{-\overline{x}}}{\gamma + \overline{x}}. \tag{9}$$

Consider the function

$$f(x) = \frac{\alpha + \beta e^{-x}}{\gamma + x} - x. \tag{10}$$

Then, we obtain

$$f(0) = \frac{\alpha + \beta}{\gamma} > 0,\ \lim_{x \to \infty} f(x) = -\infty$$

and

$$f'(x) = \frac{e^{-x}\beta(-\gamma - x - 1) - \alpha}{(\gamma + x)^2} - 1,$$

it follows that Eq.(9) has exactly one solution $\overline{x}$.

In the following theorem, the locally asymptotically stability of the equilibrium point $\overline{x}$ of Eq.(4) is described.

Theorem 9. *The equilibrium point $\overline{x}$ of Eq.(4)is locally asymptotically stable if*

$$\beta < \gamma e^{\frac{-\gamma + \sqrt{\gamma^2 + 4(\alpha + \gamma)}}{2}}. \tag{11}$$

Proof. The linearized equation associated with Eq.(4) about $\overline{x}$ is

$$x_{n+1}+\frac{\beta e^{-\overline{x}}}{\gamma+\overline{x}}x_n+\frac{\overline{x}}{\gamma+\overline{x}}x_{n-k}=0. \tag{12}$$

By using a sufficient condition for the locally asymptotically stability in (7) of Theorem 4, we obtain the inequality

$$\left|\frac{\beta e^{-\overline{x}}}{\gamma+\overline{x}}\right|+\left|\frac{\overline{x}}{\gamma+\overline{x}}\right|<1 \tag{13}$$

and some computations show that

$$\beta e^{-\overline{x}}<\gamma. \tag{14}$$

Substituting $\beta e^{-\overline{x}}$ obtained (9) into (14) we have

$$\overline{x}^2+\gamma\overline{x}-(\alpha+\gamma)<0. \tag{15}$$

Eq.(15) is in a quadratic form and have two real equilibrium points. In the sequel we use the positive one which is

$$\overline{x}_1=\frac{-\gamma+\sqrt{\gamma^2+4(\alpha+\gamma)}}{2}. \tag{16}$$

Subtituting $\overline{x}_1$ into Eq.(14), it is easily obtained

$$\beta<\gamma e^{\frac{-\gamma+\sqrt{\gamma^2+4(\alpha+\gamma)}}{2}},$$

which completes the proof. □

Theorem 10. *Every positive solution of Eq.(4) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be an arbitrary positive solution of system (4). From (4), we can write the inequality

$$x_{n+1}=\frac{\alpha+\beta e^{-x_n}}{\gamma+x_{n-k}}\leq\frac{\alpha+\beta}{\gamma},\ n\geq 0. \tag{17}$$

In addition, from (4) and (17) we have

$$x_{n+1}=\frac{\alpha+\beta e^{-x_n}}{\gamma+x_{n-k}}\geq\frac{\alpha\gamma+\beta\gamma e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma^2+\alpha+\beta},\quad n\geq 1. \tag{18}$$

From (17) and (18), we get

$$\frac{\alpha\gamma+\beta\gamma e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma^2+\alpha+\beta}\leq x_n\leq\frac{\alpha+\beta}{\gamma},\ n\geq 2,$$

which completes the proof. □

Theorem 11. *Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of Eq.(4) such that:*

(i) $x_{-k},x_{-k+2},...,x_{-2},x_0\geq\overline{x}$ *and* $x_{-k+1},x_{-k+3},...,x_{-3},x_{-1}<\overline{x}$ *or*

(ii) $x_{-k},x_{-k+2},...,x_{-2},x_0<\overline{x}$ *and* $x_{-k+1},x_{-k+3},...,x_{-3},x_{-1}\geq\overline{x}$.

Then $\{x_n\}_{n=-k}^{\infty}$ oscillates about the equilibrium point $\overline{x}$ with semicycles of length one.

Proof. Assume that (i) holds. (The case where (ii) holds is similar and will be omitted.) From Eq.(4), we have

$$\begin{aligned} x_1 &= \frac{\alpha+\beta e^{-x_0}}{\gamma+x_{-k}} \leq \frac{\alpha+\beta e^{-\overline{x}}}{\gamma+\overline{x}} = \overline{x} \\ x_2 &= \frac{\alpha+\beta e^{-x_1}}{\gamma+x_{-k+1}} > \frac{\alpha+\beta e^{-\overline{x}}}{\gamma+\overline{x}} = \overline{x} \\ x_3 &= \frac{\alpha+\beta e^{-x_2}}{\gamma+x_{-k+2}} \leq \frac{\alpha+\beta e^{-\overline{x}}}{\gamma+\overline{x}} = \overline{x} \\ x_4 &= \frac{\alpha+\beta e^{-x_3}}{\gamma+x_{-k+3}} > \frac{\alpha+\beta e^{-\overline{x}}}{\gamma+\overline{x}} = \overline{x}. \end{aligned}$$

Then the result follows by induction. □

Lemma 12. *Let* $f(u,w) = \frac{\alpha+\beta e^{-u}}{\gamma+w}$ *and* $u,w \in [0,\infty)$*, then* $f(u,w)$ *is a non-increasing function both in* u *and* w*.*

Proof. The proof is simple and will be omitted. □

Theorem 13. *Assume that (11) holds and that* $\beta < \gamma$*. Then the equilibrium* $\overline{x}_1$ *of Eq.(4) is global asymptotically stable.*

Proof. From Lemma 12, $f(u,w)$ is non-increasing in each of its aguments. Then for any $u,w \in [0,\infty)$, we have

$$0 < f(u,w) = \frac{\alpha+\beta e^{-u}}{\gamma+w} < \frac{\alpha+\beta}{\gamma}.$$

Let $\lambda = \lim_{n\to\infty} \inf x_n$, $\Lambda = \lim_{n\to\infty} \sup x_n$ and $\varepsilon > 0$ such that $\varepsilon < \min\{\frac{\alpha+\beta}{\gamma} - \Lambda, \lambda\}$. Then there exist $n_0 \in \mathbb{N}$ such that $\lambda - \varepsilon \leq x_n \leq \Lambda + \varepsilon$. Thus

$$\frac{\alpha+\beta e^{-(\Lambda+\varepsilon)}}{\gamma+(\Lambda+\varepsilon)} \leq x_{n+1} \leq \frac{\alpha+\beta e^{-(\lambda-\varepsilon)}}{\gamma+(\lambda-\varepsilon)},\ n \geq n_0+1.$$

Therefore,

$$\frac{\alpha+\beta e^{-(\Lambda+\varepsilon)}}{\gamma+(\Lambda+\varepsilon)} \leq \lambda \leq \Lambda \leq \frac{\alpha+\beta e^{-(\lambda-\varepsilon)}}{\gamma+(\lambda-\varepsilon)}.$$

This inequality yields

$$\frac{\alpha+\beta e^{-\Lambda}}{\gamma+\Lambda} \leq \lambda \leq \Lambda \leq \frac{\alpha+\beta e^{-\lambda}}{\gamma+\lambda},$$

which implies that

$$\alpha+\beta e^{-\Lambda} - \gamma\lambda \leq \lambda\Lambda \leq \alpha+\beta e^{-\lambda} - \gamma\Lambda.$$

From the last inequality and the assumption $\beta < \gamma$, it follows that $\Lambda \leq \lambda$. Hence, $\lambda = \Lambda = \overline{x}_1$. From Lemma 8, Eq.(4) has a positive unique equilibrium point and every solution of Eq.(4) converges to $\overline{x}_1$. That shows $\lim_{n\to\infty} x_n = \overline{x}_1$, which completes the proof. □

Theorem 14. *Eq.(4) has no positive solutions of prime period* $k+1$*.*

Proof. Let

$$..., \phi_1, \phi_2, ..., \phi_{k+1}, \phi_1, \phi_2, ..., \phi_{k+1}, ...$$

be a period $k+1$ solutions of Eq.(4). Then

$$\phi_1 = \frac{\alpha + \beta e^{-\phi_{k+1}}}{\gamma + \phi_1}, \ \phi_2 = \frac{\alpha + \beta e^{-\phi_1}}{\gamma + \phi_2} \text{ and } \phi_{k+1} = \frac{\alpha + \beta e^{-\phi_k}}{\gamma + \phi_{k+1}}.$$

Note that

$$\alpha = \phi_1^2 + \gamma\phi_1 - \beta e^{-\phi_{k+1}} = \phi_2^2 + \gamma\phi_2 - \beta e^{-\phi_1} = ... = \phi_{k+1}^2 + \gamma\phi_{k+1} - \beta e^{-\phi_k}.$$

Set

$$F(z) = z^2 + \gamma z - \beta e^{-z} - \alpha.$$

It is easy to see that $F(\overline{x}) = 0$ and $F'(z) > 0$. This implies that the function $F(z)$ is increasing, which completes the proof. □

4. Numerical Examples

As a confirmation of the result obtained above, we present the following examples:

Example 15. *We illustrate the solution which corresponds to the initial conditions $x_{-2} = 1.2$, $x_{-1} = 0.4$, $x_{-6} = 0.9$ of Eq.(4) for $k = 2$ in Figure 1, Figure 2 and Figure 3.*

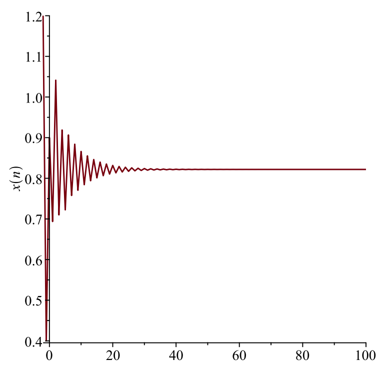

Figure 1. $\alpha = 1$, $\beta = 3$, $\gamma = 2$.

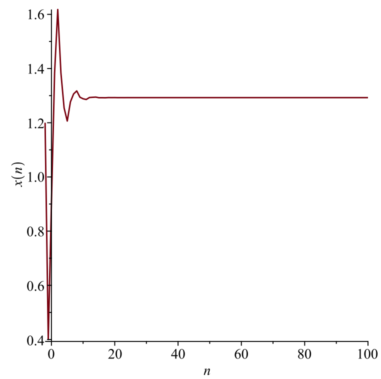

FIGURE 2. $\alpha = 5$, $\beta = 2$, $\gamma = 3$.

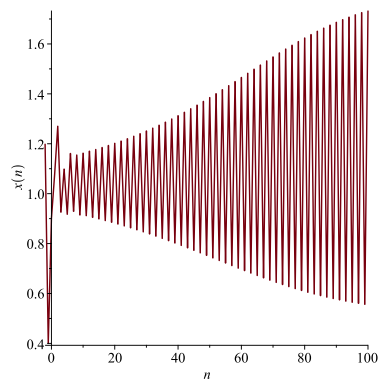

FIGURE 3. $\alpha = 1$, $\beta = 6$, $\gamma = 2$.

Example 16. *We illustrate the solution which corresponds to the initial conditions* $x_{-4} = 2.02$*,* $x_{-3} = 0.45$*,* $x_{-2} = 1.2$*,* $x_{-1} = 0.4$*,* $x_{-6} = 0.9$ *of Eq.(4) for* $k = 4$ *in Figure 4, Figure 5 and Figure 6.*

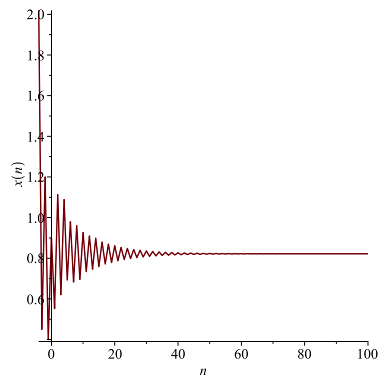

FIGURE 4. $\alpha = 1$, $\beta = 3$, $\gamma = 2$.

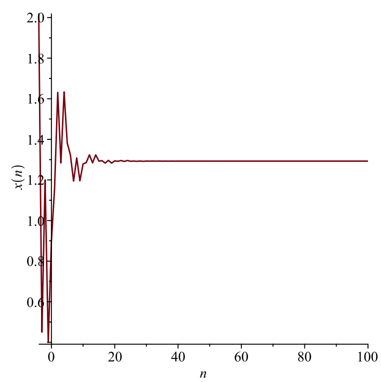

FIGURE 5. $\alpha = 5$, $\beta = 2$, $\gamma = 3$.

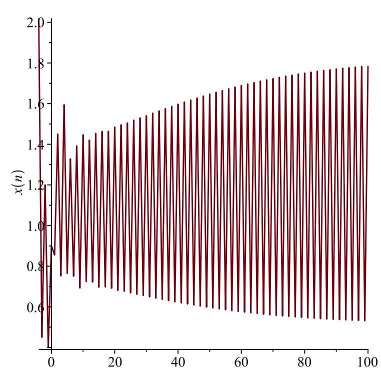

FIGURE 6. $\alpha = 1$, $\beta = 6$, $\gamma = 2$.

Example 17. *We illustrate the solution which corresponds to the initial conditions* $x_{-6} = 1.71$*,* $x_{-5} = 2.01$*,* $x_{-4} = 2.02$*,* $x_{-3} = 0.45$*,* $x_{-2} = 1.2$*,* $x_{-1} = 0.4$*,* $x_{-6} = 0.9$ *of Eq.(4) for* $k = 6$ *in Figure 7, Figure 8 and Figure 9.*

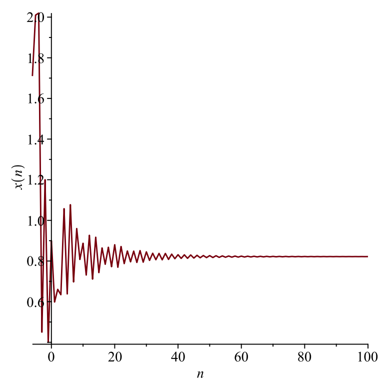

FIGURE 7. $\alpha = 1$, $\beta = 3$, $\gamma = 2$.

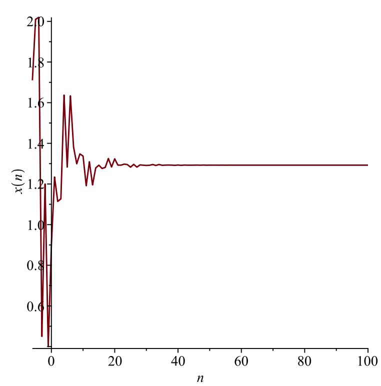

FIGURE 8. $\alpha = 5$, $\beta = 2$, $\gamma = 3$.

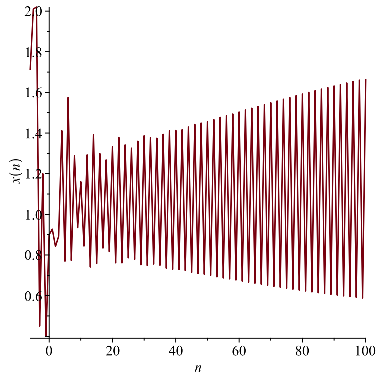

FIGURE 9. $\alpha = 1$, $\beta = 6$, $\gamma = 2$.

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