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HARDY-SOBOLEV-MAZ' YA INEQUALITY ON TIME SCALE AND APPLICATION TO THE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we will prove some new dynamic inequalities of Hardy-Sobolev-May'ze type on time scales. An application in the boundary value problems for dynamic equation.

1. Introduction

The classical Hardy inequality states that for $f \geq 0$ and integrable over any finite interval $(0, x)$ and f^p is integrable and onvergent over $(0, \infty)$ and $p > 1$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(t)\, dt \qquad (1)$$

holds and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. Inequality (1) which is usually referred to in the literature as the classical Hardy inequality, was proved in 1925 by Hardy [17]. More general Hardy integral inequalities have been studied in continuous. The inequalities of Hardy and Sobolev have a pivotal role in analysis and continue to be topics of intensive study. In its familiar basic form in $L^p(\Omega)$; the Hardy inequality takes the form

$$\int_\Omega |\nabla f(x)|^p\, dx \geq C(n, p) \int_\Omega \frac{|f(x)|^p}{|x|^p} dx, \qquad \text{for all } f \in W_0^{1,p}(\Omega), \qquad (2)$$

where Ω is a bounded domain in $\mathbb{R}^n$, containing the origin, $p > 1$ and $C(n, p)$ is constant > 0.

Indeed, Rupert l. Frank and Michael loss [20] have obtained the following improved Hardy inequalities valid for any $f \in W_0^{1,p}((a, b))$

$$\int_a^b \left| f'(x) \right|^2 dx \geq \frac{1}{4} \int_a^b \left| \frac{f(x)}{x} \right|^2 dx + K_p \|f\|^2_{L^p((a,b))}. \qquad (3)$$

where $a, b \in \mathbb{R}$, $a \leq 0 < b$, $p > 1$ and K_p is constant > 0.

Hardy type inequalities on time scales not only give a unification of continuous inequalities of Hardy type but also can be extended to different types of time scales.

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In 2005, Řehàk [8] stated that if $a > 0$, $P > 1$, and f be a nonnegative function such that the delta integral $\int_a^\infty f^p(s)\,\Delta s$ exists as a finite number, then

$$\int_a^\infty \left(\frac{1}{\sigma(t) - a}\int_a^{\sigma(t)} f(s)\,\Delta s\right)^p \le \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(t)\,\Delta t \tag{4}$$

unless $f \equiv 0$. If, in addition, $\dfrac{\mu(t)}{t} \to 0$ as $t \to \infty$, then the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

The aim of this paper is to extend a Hardy-Sobolev inequality (2) and Hardy-Sobolev-Maz'ye inequality (3) on time scales and we give an application of our extension of the Hardy inequality in the boundary value problems.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. (supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. If $\sigma(t) > t$ we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k := \mathbb{T}$.

The graininess function for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a real valued function on a time scale $\mathbb{T}$. Then, for $t \in \mathbb{T}^k$, we define $f^\Delta(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \le \varepsilon\,|\sigma(t) - s|.$$

We say that f is delta differentiable on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient $\dfrac{f}{g}$ (where $gg^\sigma \neq 0$) of two differentiable function f and g

$$(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta, \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{5}$$

A function $f : \mathbb{T} \to \mathbb{R}$ will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

The set of functions that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

We will work with the $L^p_\Delta([a,b]_\mathbb{T})$ spaces, where $[a,b]_\mathbb{T} = [a,b] \cap \mathbb{T}$, $a, b \in \mathbb{T}$, $a < b$, is an arbitrary closed subinterval of $\mathbb{T}$ and $[a,b)_\mathbb{T} = [a,b) \cap \mathbb{T}$; we state some of their properties whose proofs can be found in [6, 3, 10] .

Lemma 2.1. *The set of all right-scattered points of $\mathbb{T}$ is at most countable, that is, there are $I \subset \mathbb{N}$ and $\{t_i\}_{i\in I}$ such that*

$$\mathcal{R} := \{t \in \mathbb{T} : \sigma(t) > t\} = \{t_i\}_{i\in I}.$$

Proposition 2.2. *Let $A \subset \mathbb{T}$. Then A is a $\Delta-$measurable if and only if, A is Lebesgue measurable. If $b \notin A$, then*

$$\mu_\Delta (A) = \mu_L (A) + \sum_{i \in I_A} \mu (t_i),$$

where $I_E := \{i \in I : t_i \in E\}$.

Definition 2.3. *Let $E \subset T$ be a Δ-measurable set and let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$ and let $f : E \to \mathbb{R}$ be a Δ-measurable function. Say that f belongs to $L^p_\Delta (E)$ provided that either*

$$\int_E |f(s)|^\Delta \Delta s < \infty \qquad \text{if } p \in \mathbb{R},$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leq C \qquad \Delta - a.e. on\ E \text{ if } p = +\infty.$$

Theorem 2.4. *Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$. Then, the set $L^p_\Delta ([a, b]_\mathbb{T})$ is a Banach space together with the norm defined for every $f \in L^p_\Delta ([a, b]_\mathbb{T})$ as*

$$\|f\|_{L^p_\Delta} := \begin{cases} \left(\int_{[a,b)_\mathbb{T}} |f(t)|^\Delta \Delta t\right)^{\frac{1}{p}}, & \text{if } p \in \mathbb{R}, \\ \inf\{C \in \mathbb{R} : |f| \leq C\ \Delta - a.e. on\ [a, b]_\mathbb{T}\} & \text{if } p = +\infty. \end{cases}$$

Moreover, $L^2_\Delta ([a, b]_\mathbb{T})$ is a Hilbert space together with the inner product given for every $f, g \in L^p_\Delta ([a, b]_\mathbb{T})$ by

$$(f, g)_{L^2_\Delta} := \int_{[a,b)_\mathbb{T}} f(s) .g(s) \Delta s.$$

Definition 2.5. *Assume $n \in \mathbb{N}$, $n \geq 1$, $p \in \overline{\mathbb{R}}$ and $p \geq 1$. Let $f : [a, b]_\mathbb{T} \to \overline{\mathbb{R}}$. Say that f belongs to $W^{n,p}_\Delta ([a, b]_\mathbb{T})$ if and only if $f \in L^p_\Delta ([a, b]_\mathbb{T})$ and $f^{\Delta^j} \in L^p_\Delta \left([a, \rho^j (b)]_\mathbb{T}\right)$, for all $j \in [1, n-1]_\mathbb{Z}$.*
Where

$$\rho^j (b) = \rho\left(\rho^{j-1} (b)\right) \text{ and } f^{\Delta^j} = \left(f^{\Delta^j - 1}\right)^\Delta, \text{ for all } j \in [1, n-1]_\mathbb{Z}.$$

Theorem 2.6. *Assume $n \in \mathbb{N}$, $n \geq 1$, $p \in \overline{\mathbb{R}}$ and $p \geq 1$. The set $W^{1,p}_\Delta ([a, b]_\mathbb{T})$ is a Banach space together with the norm defined for every $f \in W^{n,p}_\Delta ([a, b]_\mathbb{T})$ as*

$$\|f\|_{W^{1,p}_\Delta} := \sum_{j=0}^n \left\|f^{\Delta^j}\right\|_{L^p_\Delta},$$

where $f^{\Delta^0} = f$. Furthermore, the set $H^n_\Delta ([a, b]_\mathbb{T}) = W^{n,2}_\Delta ([a, b]_\mathbb{T})$ is a Hilbert space together with the inner product given for every $f, g \in H^n_\Delta ([a, b]_\mathbb{T})$ by

$$(f, g)_{H^n_\Delta} := \sum_{j=0}^n \left(f^{\Delta^j}, g^{\Delta^j}\right)_{L^2_\Delta}.$$

Definition 2.7. *Assume $n \in \mathbb{N}$, $n \geq 1$, $p \in \overline{\mathbb{R}}$ and $p \geq 1$, define the set $W^{n,p}_{0,\Delta} ([a, b]_\mathbb{T})$ as the closure of the $C^n_{0,rd} ([a, b]_\mathbb{T})$ in $W^{1,p}_\Delta ([a, b]_\mathbb{T})$.*
Denote as $H^n_{0,\Delta} ([a, b]_\mathbb{T}) = W^{n,2}_{0,\Delta} ([a, b]_\mathbb{T})$.
Where

$$C^n_{0,rd} ([a, b]_\mathbb{T}) = \left\{f \in C^n_{rd} ([a, b]_\mathbb{T}) : f(a) = f\left(\rho^j (b)\right) = 0, \text{ for all } j \in [1, n-1]_\mathbb{Z}\right\}.$$

Proposition 2.8. *Assume $n \in \mathbb{N}$, $n \geq 1$, $p \in \overline{\mathbb{R}}$ and $p \geq 1$. Let $f \in W_{\Delta}^{n,p}([a,b]_{\mathbb{T}})$. Then, $f \in W_{0,\Delta}^{n,p}([a,b]_{\mathbb{T}})$ if and only if $f(a) = f\left(\rho^{j}(b)\right) = 0$, for all $j \in [1, n-1]_{\mathbb{Z}}$.*

Proposition 2.9. *Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$. Then, there exists a constant $L > 0$, only dependent on $(b-a)$, such that*

$$\|f\|_{W_{\Delta}^{1,p}} \leq L. \left\|f^{\Delta}\right\|_{L_{\Delta}^{p}}, \qquad \text{for all } f \in W_{0,\Delta}^{1,p}([a,b]_{\mathbb{T}}).$$

that is, in $W_{0,\Delta}^{1,p}([a,b]_{\mathbb{T}})$, the norm defined for every $f \in W_{0,\Delta}^{1,p}([a,b]_{\mathbb{T}})$ as $\left\|f^{\Delta}\right\|_{L_{\Delta}^{p}}$ is equivalent to the norm $\|f\|_{W_{\Delta}^{1,p}}$.

3. Main Results

In this paper, we suppose that $\mathbb{T}$ is a particular time scale, $a < b < \infty$ are points in $\mathbb{T}$.

Now, we are ready to state and prove the main results in this paper. We generalize the Hardy-Sobolev-Maz'ya inequality (3) on time scales.

Theorem 3.1. *Let $q \geq 2$. Then there exist constant C_q only on q such that the inequality*

$$\int_a^b \left|f^{\Delta}(t)\right|^2 \Delta t \geq \frac{1}{4}\int_a^b \frac{|f(t)|^2}{(b-t)^2}\Delta t + C_q\left(\int_a^b |f(t)|^q \Delta t\right)^{\frac{2}{q}}, \qquad \text{(HSM)}$$

holds for all $f \in W_{0,\Delta}^{1,q}([a,b]_{\mathbb{T}})$.
If, in addition, $t \to \dfrac{\mu(t)}{b-t}$ is a function nonincreasing.

Proof. Let g is function define by:

$$f(t) = \eta(t)\, g(t), \qquad t \in [a,b]_{\mathbb{T}}.$$

Where $\eta(t) = \sqrt{b-t}$, for all $t \in [a,b]_{\mathbb{T}}$. Then $\eta \in C_{rd}^{1}([a,b]_{\mathbb{T}})$ and

$$\eta^{\Delta}(t) = \frac{-1}{\eta(t) + \eta^{\sigma}(t)}. \tag{6}$$

Using propertie (6), we obtain that

$$\eta^{\sigma}(t)\, g^{\Delta}(t) = f^{\Delta}(t) + \frac{f(t)}{\eta^2(t) + \eta(t)\,\eta^{\sigma}(t)}. \tag{7}$$

By (6), we have $\eta^{\sigma}(t) \leq \eta(t)$, and

$$\begin{aligned}
\left|\eta^{\sigma}(t)\, g^{\Delta}(t)\right|^2 &= \left|f^{\Delta}(t)\right|^2 + \frac{f^2(t)}{\left(\eta^2(t) + \eta(t)\,\eta^{\sigma}(t)\right)^2} + \frac{2f^{\Delta}(t)\, f(t)}{\eta^2(t) + \eta(t)\,\eta^{\sigma}(t)} \\
&\leq \left|f^{\Delta}(t)\right|^2 + \frac{2f(t)}{\left(\eta^2(t) + \eta(t)\,\eta^{\sigma}(t)\right)}\left\{\frac{f(t)}{\left(\eta^2(t) + \eta(t)\,\eta^{\sigma}(t)\right)} + f^{\Delta}(t)\right\} - \frac{|f(t)|^2}{4(b-t)^2} \\
&\leq \left|f^{\Delta}(t)\right|^2 + \xi(t)\, g^{\Delta}(t)\, g(t) - \frac{|f(t)|^2}{4(b-t)^2}.
\end{aligned} \tag{8}$$

Where $\xi(t) := -2\eta^{\Delta}(t)\,\eta^{\sigma}(t)$, for all $t \in [a, \rho(b)]_{\mathbb{T}}$. Then ξ is Δ-differentiable for all the points right-scattered. Let $t \in [a,b]_{\mathbb{T}}$ such that t is point right-dense, then t is point accumulation, we have two cases.

(a) First case, there exists $c, d \in [a,b]_{\mathbb{T}}$ such that $t \in [c,d] \subset [a,b]_{\mathbb{T}}$, then ξ is Δ-differentiable in t and $\xi^{\Delta}(t) = 0$.

(b) Second case, there exists a sequence $(t_k)_{k\in\mathbb{N}} \in \mathcal{R} \cap [a,b]_{\mathbb{T}}$, such that, for all $k \in \mathbb{N}$ one has t_k is point isolat and $t_k \longrightarrow t$ as $k \longrightarrow \infty$. In this case, $\xi^{\Delta}(t)$ do not exist.

By the proposition 2.2, we get

$$\mu_{\Delta}\left(\left\{t \in [a,b]_{\mathbb{T}} : \sigma(t) = t \text{ and } t = \lim_{k\longrightarrow\infty} t_k,\ (t_k)_{k\in\mathbb{N}} \subset \mathcal{R}\right\}\right) = 0.$$

Consequently, we obtain that ξ^{Δ} is Δ-differentiable a.e on $[a,b]_{\mathbb{T}}$.
Let $t, s \in [a, \rho(b)]_{\mathbb{T}}$ shch that $t > s$, we have

$$\begin{aligned}
\xi(t) - \xi(s) &= \frac{1}{2}\xi(t)\xi(s)\left\{\frac{\eta(s)}{\eta^{\sigma}(s)} - \frac{\eta(t)}{\eta^{\sigma}(t)}\right\} \\
&= \frac{1}{2}\xi(t)\xi(s)\left\{\sqrt{1 + \frac{\mu(s)}{b - \sigma(s)}} - \sqrt{1 + \frac{\mu(t)}{b - \sigma(t)}}\right\}.
\end{aligned}$$

Then ξ is function increasing.
Therefore,

$$\begin{aligned}
\int_a^b \xi(t)\, g^{\Delta}(t)\, g(t)\, \Delta t &= -\int_a^b [\xi.g]^{\Delta}(t)\, g^{\sigma}(t)\, \Delta t \\
&= -\int_a^b \xi^{\Delta}(t)\, |g^{\sigma}(t)|^2\, \Delta t - \int_a^b \xi(t)\, g^{\Delta}(t)\, g^{\sigma}(t)\, \Delta t \\
&\leq -\int_a^b \xi(t)\, g^{\Delta}(t)\, g(t)\, \Delta t - \int_a^b \xi(t)\, \mu(t) \left|g^{\Delta}(t)\right|^2 \Delta t \\
&\leq -\int_a^b \xi(t)\, g^{\Delta}(t)\, g(t)\, \Delta t.
\end{aligned}$$

Using the above inequality we have

$$\int_a^b \left|\eta^{\sigma}(t)\, g^{\Delta}(t)\right|^2 \Delta t \leq \int_a^b \left(\left|f^{\Delta}(t)\right|^2 - \frac{|f(t)|^2}{4(b-t)^2}\right) \Delta t \tag{9}$$

Bötzsche rule [1], we see that

$$\begin{aligned}
|g(t)|^{\frac{q+2}{2}} &\leq \frac{q+2}{2} \left|g^{\Delta}(t)\right| \int_0^1 |hg(t) + (1-h)\, g^{\sigma}(t)|^{\frac{q}{2}}\, dh \\
&\leq \frac{q+2}{2} \left|g^{\Delta}(t)\right| |g_1(t)|^{\frac{q}{2}}.
\end{aligned}$$

Using the fact that η is decreasing and we find that

$$\begin{aligned}
|f(t)|^{\frac{q+2}{2}} &= |\eta(t)|^{\frac{q+2}{2}} \int_a^t \left(|g(s)|^{\frac{q+2}{2}}\right)^{\Delta} \Delta s \\
&\leq \frac{q+2}{2} \int_a^t |\eta(t)|^{\frac{q+2}{2}} \left|g^{\Delta}(s)\right| |G(s)|^{\frac{q}{2}}\, \Delta s \\
&\leq \frac{q+2}{2} \int_a^b \left|g^{\Delta}(s)\right| |G(s)|^{\frac{q}{2}}\, |\eta(s)|^{\frac{q+2}{2}}\, \Delta s.
\end{aligned}$$

Where $G := \max\{|g|, |g^{\sigma}|\}$.
Using the Hölder inequality we find

$$\begin{aligned} |f(t)|^{q+2} &\leq m_q \left(\int_a^b \left| g^{\Delta}(t) \right|^2 \eta^2(t) \Delta t \right) \left(\int_a^b |G(t)|^q \eta^q(t) \Delta t \right) \\ &\leq m_q \int_a^b \left(\left| f^{\Delta}(t) \right|^2 - \frac{|f(t)|^2}{4(b-t)^2} \right) \Delta t \left(\int_a^b |F(t)|^q \Delta t \right). \end{aligned}$$

Where $m_q = \frac{1}{4}(q+2)^2$ and $F := \max\{|f|, |f^{\sigma}|\}$.
Then

$$\int_a^b |f_1(t)|^q \Delta t \leq (m_q)^{\frac{q}{q+2}} \left(\int_a^b \left(\left| f^{\Delta}(t) \right|^2 - \frac{|f(t)|^2}{4(b-t)^2} \right) \Delta t \right)^{\frac{q}{q+2}} \left(\int_a^b |F(t)|^q \Delta t \right)^{\frac{q}{q+2}}.$$

Thus

$$\int_a^b \left| f^{\Delta}(t) \right|^2 \geq \frac{1}{4} \int_a^b \frac{|f(t)|^2}{(b-t)^2} \Delta t + \frac{1}{m_q} \left(\int_a^b |F(t)|^q \Delta t \right)^{\frac{2}{q}}.$$

The intended inequality (HSM) is proved. □

4. Application

We are concerned with the existence of positive solutions of the p-Laplacian dynamic equation on a time scale

$$\begin{cases} \left[r\phi_p \left(u^{\Delta} \right) \right]^{\Delta} + \dfrac{\xi}{(\sigma(t) - a)^p} \phi_p(u^{\sigma}) = -f \text{ in } \left[a, \rho^2(b) \right]_{\mathbb{T}}, \\ u(a) = u(b) = 0, \end{cases} \quad (10)$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-1} s$, $p > 1$, $f \in L^q_{\Delta}([a,b]_{\mathbb{T}})$, $\frac{1}{p} + \frac{1}{q} = 1$, $r \in C_{rd}([a,b]_{\mathbb{T}})$ and $\alpha\xi \geq C_p$ (Define in the Theorem 3.1).
Consider again the functional

$$E_p(u) := \frac{1}{p} \int_a^b r \left| u^{\Delta} \right|^p \Delta t - \frac{1}{p} \int_a^b h |u^{\sigma}|^p \Delta t - \int_a^b f u^{\sigma} \Delta t,$$

is then well de ned on the Sobolev space $W^{1,p}_{0,\Delta}([a,b]_{\mathbb{T}})$. The (weak) solutions of the problem (10) are then the critical points of the functional (E_p).
The classical results in the Calculus of Variations characterize the weak. Then, the problem (10) has weak solution in $W^{1,p}_{0,\Delta}([a,b]_{\mathbb{T}}) \cap W^{2,p}_{0,\Delta}([a,b]_{\mathbb{T}})$.

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