Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 106-113.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

FIXED POINTS AND CONTINUITY OF BOYD AND WONG TYPE CONTRACTIVE MAPS

FAUSTINE NZIKU AND SANTOSH KUMAR

Abstract. In this paper, we have established and proved a fixed point theorem for Boyd and Wong [2] type contraction in partial metric spaces. In particular, we have generalized the results due to Pant and Pant [6] for Boyd and Wong type contraction condition into partial metric spaces in which a contractive mapping which posses a fixed point but not continuous at the fixed point is used. In addition to that we have presented a common fixed point theorem for a pair of maps. We have concluded our results by providing an illustrative example to demonstrate our results.

1. Introduction and Preliminaries

Continuity is an ideal property which is sometimes difficult to be fulfilled especially in some daily life applications. For instance, most of neural network systems like bar code scanning, speech recognition and hand written digits recognition. These neural network systems are some excellent prototype for learning discontinuity phenomena. Actually, different kinds of day to day real world phenomena are transformed into threshold functions which satisfies some desirable continuity of weaker forms and some new type of contraction to provide solution to some daily life applications. Therefore it is desirable to relax continuity assumptions because in some applications the function may not be continuous.

One of the fundamental tool for non linear analysis is the Banach fixed point theorem [1]. As a result of its usefulness and applications, this theorem has been massively investigated and generalized by different researchers. One of the important generalization of the Banach fixed point theorem is the Boyd and Wong [2] fixed point theorem. A mapping T satisfying,

$$d(Tx, Ty) \leq \chi(d(x, y)), \forall x, y \in M, \tag{1}$$

whereby (M, d) is a complete metric space and a mapping $\chi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right on $[0, \infty)$ such that $\chi(t) < t$, $\forall\ t > 0$. Consequently, T has a unique fixed point $z \in M$ and $d(T^n x, z) \to 0$ as $n \to \infty$, $\forall$ $x \in M$.

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Metric spaces, Partial metric spaces, Fixed point, Boyd and Wong type contractive condition, k-continuous maps.
Submitted Aug. 25, 2018.

Matthews [5] introduced the study of partial metric spaces as an important subject in the approach of formalizing the meaning of programming languages by formulating mathematical objects called denotations. Partial metric was introduced to ensure that partial order semantics should have a metric based tools for program verification.

The following definitions are due to Bukatin et. al. [3].

Definition 1 [3] Let X be a non-empty set. A function $p : X \times X \to [0, \infty)$ is called a partial metric on X if it satisfies the followings axioms:

(PM0): $0 \leq p(x,x) \leq p(x,y)$ (non-negativity and small self-distance).
(PM1): $p(x,y) = p(x,x) = p(y,y) \Rightarrow x = y$ (indistancy implies equality).
(PM2): $p(x,y) = p(y,x)$ (symmetric).
(PM3): $p(x,y) + p(z,z) \leq p(x,z) + p(z,y)$ (triangularity), for all $x, y, z \in X$.

(X,p) is called a partial metric space.

Note that $p(x,y) = 0$ implies $x = y$ (by $PM0$ through $PM2$), the converse is always not true. Therefore, a metric space is a partial metric space with all self-distances zero.

Definition 2 [3] Let $\{x_n\}$ be a sequence in a partial metric space (X,p), then,

(i) A sequence $\{x_n\} \in X$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n\to\infty} p(x_n,x_n)$.
(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if there exists $\epsilon > 0$ such that for all $n, m > N$, we have $p(x_n, x_m) < \epsilon$ for some integers $N \geq 0$, that is $\lim_{n,m\to+\infty} p(x_n, x_m)$ exists and it is finite.
(iii) A partial metric space (X,p) is complete if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

The following definitions are due to Pant and Pant [6].

Definition 3 [6] A mapping $T : X \to X$ is called k-continuous for $k = 1, 2, 3, ...$ if $T^k x_n \to Tt$ whenever a sequence $\{x_n\}$ is in X such that $T^{k-1} x_n \to t$.

We denote $p(X)$ and $p(T(X))$ to represent the diameter of a set X and the diameter of the range of T respectively.

Pant and Pant [6] proved the following theorem for Boyd and Wong type fixed point theorem in complete metric spaces:

Theorem 1 [6] Let T be a mapping of a complete metric space (X,d) into itself satisfying,

$$d(Tx, Ty) \leq \phi(\max\{d(x,Tx), d(y,Ty)\}), \tag{2}$$

for all $x, y \in X$, where the function $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) < t$ for each $t > 0$. If ϕ is upper semi-continuous in the open interval $(0, d(T^k(X)))$, then T has a unique fixed point.

2. Main Results

Now, we state and prove our main results in partial metric spaces which is a generalization of Theorem 1 in partial metric spaces and then provide an illustrative example to demonstrate our results.

Theorem 2 Let X be a non-empty set and let p be a partial metric on X. Let $T : X \to X$ be a mapping of a complete partial metric space (X,p) satisfying,

$$p(Tx, Ty) \leq \phi(\max\{p(x,Tx), p(y,Ty)\}), \tag{3}$$

for all $x, y \in X$, where the function $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) < t$ for all $t > 0$. If ϕ is upper semi-continuous in the open interval $(0, p(T^k(X)))$ for k=0,1,2,3,..., then T has a unique fixed point.

Proof. For the case when $k = 0$, we see that the mapping ϕ is upper semi-continuous on the interval $(0, p(X))$ which is analogous to the generalization of Boyd and Wong fixed point theorem in complete partial metric space.

We now consider the case when $k = 1$. Let $x_0 \in X$. We define a sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n$ for all integers $n \geq 0$.

If we suppose that there exists an integer $n \geq 0$ such that $x_{n+1} = x_n$ then T has a fixed point x_n and the proof is complete. Otherwise, suppose that $x_{n+1} \neq x_n$ for all integer $n \geq 0$, then from (3) we have,

$$\begin{aligned} p(x_n, x_{n+1}) =& p(Tx_{n-1}, Tx_n) \\ \leq& \phi(\max\{p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\}). \end{aligned} \tag{4}$$

Suppose that $\max\{p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\} = p(x_n, Tx_n)$, then,

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \phi(p(x_n, Tx_n)) < p(x_n, x_{n+1}) \tag{5}$$

which is a contradiction. Hence,

$$\max\{p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\} = p(x_{n-1}, Tx_{n-1}). \tag{6}$$

Therefore, $p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \phi(p(x_{n-1}, Tx_{n-1})) < p(x_{n-1}, x_n)$. Thus, the sequence $\{p(x_n, x_{n+1})\}$ is a decreasing sequence. It is obvious that the sequence $p(x_n, x_{n+1})$ decreases to the real number $r \geq 0$. We claim that $r = 0$. In contrary suppose that $r > 0$, since ϕ is upper semi-continuous in $(0, p(T(X)))$ and

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \phi(p(x_{n-1}, x_n)). \tag{7}$$

Taking limit as $n \to \infty$ in (7) we obtain $r \leq \phi(r) < r$ which is a contradiction. Hence, $r = 0$ and

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{8}$$

Now we show that a sequence $\{x_n\} \in X$ is a Cauchy sequence. For any positive integer m, we have,

$$\begin{aligned} p(x_n, x_{n+m}) &= p(Tx_{n-1}, Tx_{n+m-1}) \\ &\leq \phi(\max\{p(x_{n-1}, Tx_{n-1}), p(x_{n+m-1}, T_{n+m-1})\}) \\ &= \phi(\max\{p(x_{n-1}, x_n), p(x_{n+m-1}, x_{n+m})\}) \\ &= \phi(p(x_{n-1}, x_n)) \\ &< p(x_{n-1}, x_n) \end{aligned} \tag{9}$$

Taking limit as $n \to \infty$ in (9) and considering (8) we obtain,

$$\lim_{n\to\infty} p(x_n, x_{n+m}) = 0. \tag{10}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, then there exists a point $z \in X$ such that $x_n \to z$. We shall show that a point $z \in X$ is a fixed point of a mapping T.

In contrary, suppose that z is not a fixed point of T. Then,

$$p(x_{n+1}, Tz) = p(Tx_n, Tz) \leq \phi(\max\{p(x_n, Tx_n), p(z, Tz)\}) \tag{11}$$

as $n \to \infty$ in (11) we obtain,

$$p(z,Tz) \le \phi(\max\{p(z,Tz), p(z,Tz)\}) = \phi(p(z,Tz)) < p(z,Tz) \tag{12}$$

which is a contradiction, hence z is a fixed point of T.

Now, we shall show that z is a unique fixed point of T. Suppose that there exists another point $y \neq z$ which is a fixed point of T. Then,

$$\begin{aligned} 0 < p(z,y) &= p(Tz,Ty) \\ &\le \phi(\max\{p(z,Tz), p(y,Ty)\}) \\ &= \phi(\max\{p(z,z), p(y,y)\}) \\ &= \phi(p(z,z)) < p(z,z) \end{aligned} \tag{13}$$

which is a contradiction. Hence $y = z$. □

Now, we will demonstrate our example to explain the above theorem:

Example Let $X = [0,2]$ with partial metric $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Let a mapping $T : X \to X$ defined by,

$$Tx = \begin{cases} 0, & x \in [0,1) \\ 1, & x \in [1,2]. \end{cases}$$

Also define $\phi : [0,\infty) \to [0,\infty)$ as,

$$\phi(t) = \begin{cases} \frac{1+t}{2}, & t > 1 \\ \frac{t}{2}, & t \le 1. \end{cases}$$

It is clear that the mapping T satisfies the criteria of Theorem 2 with a unique fixed point $T = 1$ but it is discontinuous at this fixed point. Also we observe that $p(T(X)) = 1$ and ϕ is continuous on $(0,1)$.

Here we present an extension of Theorem 2 to a pair of maps to obtain a unique common fixed point.

Theorem 3 Let X be a non-empty set and let p to be a partial metric on X. Let T and S be self mappings of a complete partial metric space (X,p) satisfying:

$$p(Tx, Sy) \le \phi\{P(x,y)\}, \tag{14}$$

for all $x, y \in X$, where the mapping $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) < t$ for all $t > 0$ and

$$P(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Sy), \frac{p(x,Sy) + p(y,Tx)}{2}\right\}. \tag{15}$$

If ϕ is upper semi-continuous on $(0, p(T^k(X)))$ and $(0, p(S^k(X)))$ for $k = 0,1,2,...$, then T and S have a unique common fixed point and any fixed point of T is also a fixed of S and conversely.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\} \in X$ as $x_{n+1} = Tx_n$ and $x_{n+2} = Sx_{n+1}$, for all integers $n \ge 0$.

If we assume that there exists a non-negative integer n_0 such that, $x_{n_0} = x_{n_0+1}$, then $x_n = x_{n+1} = Tx_n$, this implies that x_n is a fixed point of T. Similarly, if there exists an integer $N \ge 0$ such that $x_{N+1} = x_{N+2}$, then x_{n+1} is a fixed point of S. This concludes the proof.

Otherwise, we suppose that $x_n \neq x_{n+1}$, for all integers $n \ge 0$. Let $\delta_n = p(x_n, x_{n+1})$,

obvious $\delta_{n+1} = p(x_{n+1}, x_{n+2})$.
From (14) we have:

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq \phi(P(x_n, x_{n+1})), \tag{16}$$

where,
$P(x_n, x_{n+1})$

$$= \max\left\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Sx_{n+1}), \frac{p(x_n, Sx_{n+1}) + p(x_{n+1}, Tx_n)}{2}\right\},$$

$$= \max\left\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2}\right\},$$

Since,
$\frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2}$

$$\leq \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) + p(x_{n+1}, x_{n+1})}{2}$$
$$= \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})}{2},$$

then,

$$P(x_n, x_{n+1}) = \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}.$$

Thus,

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &= p(Tx_n, Sx_{n+1}) \\ &\leq \phi(\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}) \end{aligned} \tag{17}$$

If we take $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2})$, then,

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq \phi\{p(x_{n+1}, x_{n+2})\} < p(x_{n+1}, x_{n+2}), \tag{18}$$

which is a contradiction. Hence $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_n, x_{n+1})$. Therefore,

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq \phi(p(x_n, x_{n+1})) < p(x_n, x_{n+1}), \tag{19}$$

which implies that the sequence $\{\delta_n\}$ is decreasing to a non-negative real number say δ, for all integers $n \geq 0$. We claim that $\delta = 0$. In contrary suppose that $\delta > 0$. Taking limit as $n \to \infty$ in (19) we obtain,

$$0 < \delta \leq \phi(\delta) < \delta, \tag{20}$$

which is a contradiction, hence we conclude that $\delta = 0$ and

$$\lim_{n\to\infty} (\delta_n) = \lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{21}$$

Now, we need to show that a sequence $\{x_n\} \in X$ is a Cauchy sequence. We claim otherwise. Therefore, there exists $\epsilon > 0$ and a sequence of integers $m(r), n(r)$ such that,

$$p(x_{n(r)}, x_{m(r)}) \geq \epsilon, \tag{22}$$

for all $n(r) > m(r) \geq r$ for some $r \geq 0$.
Furthermore, suppose that $m(r)$ is the smallest integer which is chosen in such away that (22) holds so that we have,

$$p(x_{(r)}, x_{m(r)-1}) < \epsilon. \tag{23}$$

Now, for all $n(r) > m(r)$ we have,

$$\begin{aligned} p(x_{n(r)}, x_{m(r)}) &\leq p(x_{n(r)}, x_{m(r)-1}) + p(x_{m(r)-1}, x_{m(r)}) - p(x_{m(r)-1}, x_{m(r)-1}) \\ &\leq p(x_{n(r)}, x_{m(r)-1}) + p(x_{m(r)-1}, x_{m(r)}). \end{aligned} \tag{24}$$

As $r \to \infty$ in (24) and considering (21) and (23) we see that,

$$p(x_{n(r)}, x_{m(r)}) \to \epsilon. \tag{25}$$

By similar computations we see that,

$$p(x_{n(r)-1}, x_{m(r)-1}) \to \epsilon. \tag{26}$$

Thus,

$$\begin{aligned} p(x_{n(r)}, x_{m(r)}) &= p(Tx_{n(r)-1}, Sx_{m(r)-1}) \\ &\leq \phi(P(x_{n(r)-1}, x_{m(r)-1})), \end{aligned} \tag{27}$$

where,

$$\begin{aligned} &P(x_{n(r)-1}, x_{m(r)-1}) \\ &= \max\Bigg\{p(x_{n(r)-1}, x_{m(r)-1}), p(x_{n(r)-1}, Tx_{n(r)-1}), p(x_{m(r)-1}, Sx_{m(r)-1}), \\ &\frac{p(x_{n(r)-1}, Sx_{m(r)-1}) + p(x_{m(r)-1}, Tx_{n(r)-1})}{2}\Bigg\} \\ &= \max\Bigg\{p(x_{n(r)-1}, x_{m(r)-1}), p(x_{n(r)-1}, x_{n(r)}), p(x_{m(r)-1}, x_{m(r)}), \\ &\frac{p(x_{n(r)-1}, x_{m(r)}) + p(x_{m(r)-1}, x_{n(r)})}{2}\Bigg\}, \end{aligned} \tag{28}$$

as $r \to \infty$ in (28) and considering (25) and (26), then (27) becomes,

$$0 < \epsilon \leq \phi(\epsilon) < \epsilon, \tag{29}$$

which is a contradiction. Hence, $\{x_n\} \in X$ is a Cauchy sequence and,

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{30}$$

Because X is complete, then we can pick a point $x_0 \in X$ such that,

$$\lim_{n\to\infty} p(x_n, x_0) = 0. \tag{31}$$

Here, we will prove that x_0 is a fixed point of S. Contrary suppose that $x_0 \neq Sx_0$. Now,

$$p(x_{n+1}, Sx_0) = \phi(p(Tx_n, Sx_0)) \leq \phi(P(x_n, x_0)). \tag{32}$$

where,

$$\begin{aligned} P(x_n, x_0) &= \max\left\{p(x_n, x_0), p(x_n, Tx_n), p(x_0, Sx_0), \frac{p(x_n, Sx_0) + p(x_0, Tx_n)}{2}\right\} \\ &= \max\left\{p(x_n, x_0), p(x_n, x_{n+1}), p(x_0, Sx_0), \frac{p(x_n, Sx_0) + p(x_0, x_{n+1})}{2}\right\}, \end{aligned} \tag{33}$$

as $n \to \infty$ in (33) we see that,

$$P(x_n, x_0) \to p(x_0, Sx_0). \tag{34}$$

Applying limit as $n \to \infty$ in (32) we have,

$$p(x_0, Sx_0) \leq \phi(p(x_0, Sx_0)) < p(x_0, Sx_0), \tag{35}$$

which is a contradiction. Hence $Sx_0 = x_0$.

Now, we will show that a point x_0 is a unique common fixed of T and S. In contrary, suppose that $x_0 \in X$ and $y_0 \in X$ are two different common fixed points of T and S respectively. Thus, $p(x_0, y_0) > 0$.
Now,

$$p(x_0, y_0) = p(Tx_0, Sy_0) \leq \phi(P(x_0, y_0)) \tag{36}$$

where,

$$\begin{aligned} P(x_0, y_0) &= \max\left\{p(x_0, y_0), p(x_0, Tx_0), p(y_0, Sy_0), \frac{p(x_0, Sy_0) + p(y_0, Tx_0)}{2}\right\} \\ &= \max\left\{p(x_0, y_0), p(x_0, x_0), p(y_0, y_0), \frac{p(x_0, y_0) + p(y_0, x_0)}{2}\right\} \\ &= p(x_0, y_0). \end{aligned} \tag{37}$$

Hence,

$$p(x_0, y_0) = p(Tx_0, Sy_0) \leq \phi(p(x_0, y_0)) < p(x_0, y_0).$$

which is a contradiction. Therefore, T and S have a unique common fixed point, that is $x_0 = y_0$.

To prove that any fixed point of T is also a fixed point of S and conversely, we suppose to the contrary that $x_0 = Tx_0$ and $x_0 \neq Sx_0$.
Now,

$$p(Tx_0, Sx_0) = p(x_0, Sx_0) \leq \phi(P(x_0, Sx_0)) \tag{38}$$

where,

$$\begin{aligned} &P(x_0, Sx_0) \\ &= \max\left\{p(x_0, Sx_0), p(x_0, Tx_0), p(Sx_0, S^2x_0), \frac{p(x_0, S^2x_0) + p(Sx_0, Tx_0)}{2}\right\} \\ &= \max\left\{p(x_0, Sx_0), p(x_0, x_0), p(Sx_0, S^2x_0), \frac{p(x_0, S^2x_0) + p(Sx_0, x_0)}{2}\right\} \\ &= p(x_0, Sx_0). \end{aligned} \tag{39}$$

Thus,

$$p(Tx_0, Sx_0) = p(x_0, Sx_0) \leq \phi(p(x_0, Sx_0)) < p(x_0, Sx_0) \tag{40}$$

which is a contradiction. Therefore, $x_0 = Tx_0 = Sx_0$. In similar way it is easy to show that any fixed point of S is also a fixed point of T. □

Remark If we let $T = S$ in the above theorem, we directly obtain Theorem 2.

References

[1] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fund. Math., **3** (1) (1922), 133-181.

[2] D.W. Boyd and J.S Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (2) (1969), 458-464.

[3] M. Bukatin, R. Kopperman, S. Matthews and H. Pajoohesh, *Partial metric spaces*. The Amer. Math. Monthly, **116** (8), (2009), 708-718.

[4] Z. Kadelburg, H.K. Nashine and S. Radenovic, *Fixed point results under various contractive conditions in partial metric spaces*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, **107** (2) (2013), 241-256.

[5] S. Matthews, *Partial metric topology in Papers on General Topology and Applications*, Eighth Summer Conference at Queens College. Eds. S. Andima et al., Annals of the New York Academy of Sciences, Vol. 728, (1994), 183–197.

[6] A. Pant and R.P Pant, *Fixed Points and Continuity of Contractive Maps*. Filomat, **31** (11), (2017), 3501-3506.

[7] S. Romaguera, *Fixed point theorems for generalized contractions on partial metric spaces*, Topology and Its Applications, **159** (1) (2012), 194-199.

Faustine Nziku

Depertment of Mathematics, College of Natural and Applied Sciences,University of Dar es Salaam, Tanzania.

E-mail address: faustinenziku@gmail.com, faustinenziku@yahoo.com

Santosh Kumar

Depertment of Mathematics, College of Natural and Applied Sciences,University of Dar es Salaam, Tanzania.

E-mail address: drsengar2002@gmail.com