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HADAMARD TYPE INEQUALITIES FOR (s,r)-PREINVEX FUNCTIONS IN THE FIRST SENSE

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Abstract. In this paper we study a new concept of (s,r)- preinvex functions in the first sens. Some new Hadamard-type integral inequalities are introduced. Which are compared with some existing inequalities in the literature.

1. Introduction

It is well-known that if the function f: $[a,b] \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) is reversed (see [26]).

The inequality (1) is called Hermite-Hadamard integral inequality in the literature. The above inequality has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature we can mention the works [1, 4, 5, 8, 9, 12, 13, 16, 18, 21-25, 28-33, 36] and the references cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson [10], introduced a new class of generalized convex functions, called invex functions. In [6], the authors gave the concept of preinvex function which is special case of invexity. Pini [27], Noor [19, 20], Yang and Li [35] and Weir [34], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

In [18] Ngoc et al. proved the following theorem for r-convex functions

Theorem 1.[18, Theorem 2.1] Let $f:[a,b] \to (0,\infty)$ be r-convex function on $[a,b]$ with $a<b$, then the following inequality holds for $0<r\leq 1$:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\tfrac{r}{r+1}\right)\left\{f^r(a)+f^r(b)\right\}^{\frac{1}{r}}.$$

In [23] Park gave the following theorems

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Theorem 2.[23, Theorem 2.2] Let $f : [a, b] \to (0, \infty)$ be an (s, r)-convex function in the first sense on $[a, b]$ with $a < b$, then for $r, s \in (0, 1]$ the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left\{ \left(\tfrac{r}{s+r}\right)^{\frac{1}{r}} f^r(a) + \frac{\Gamma\left(1+\frac{1}{r}\right)\Gamma\left(1+\frac{1}{s}\right)}{\Gamma\left(1+\frac{1}{r}+\frac{1}{s}\right)} f^r(b) \right\}^{\frac{1}{r}}.$$

Theorem 3.[23, Theorem 2.3] Let $f, g : [a, b] \to (0, \infty)$ be, respectively (s_1, r_1)-convex and (s_2, r_2)-convex functions in the first sense on $[a, b]$ with $a < b$, then for $0 < r_1, r_2 \leq 2$ the following inequality holds:

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{2}\Bigg[&\left\{ \left(\tfrac{r_1}{r_1+2s_1}\right)^{\frac{r_1}{2}} f^{r_1}(a) + \left(\frac{\Gamma\left(1+\frac{2}{r_1}\right)\Gamma\left(1+\frac{1}{s_1}\right)}{\Gamma\left(1+\frac{2}{r_1}+\frac{1}{s_1}\right)}\right)^{\frac{r_1}{2}} f^{r_1}(b) \right\}^{\frac{2}{r_1}} \\ &+ \left\{ \left(\tfrac{r_2}{r_2+2s_2}\right)^{\frac{r_2}{2}} g^{r_2}(a) + \left(\frac{\Gamma\left(1+\frac{2}{r_2}\right)\Gamma\left(1+\frac{1}{s_2}\right)}{\Gamma\left(1+\frac{2}{r_2}+\frac{1}{s_2}\right)}\right)^{\frac{r_2}{2}} g^{r_2}(b) \right\}^{\frac{2}{r_2}} \Bigg].\end{aligned}$$

In [36] Zabandan et al. proved the following theorems

Theorem 4.[36, Theorem 2.1] Let $f : [a, b] \to (0, \infty)$ be r-convex and $r \geq 1$. Then the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left\{ \tfrac{f^r(a)+f^r(b)}{2} \right\}^{\frac{1}{r}}.$$

Theorem 5.[36, Theorem 2.8] Let $f, g : [a, b] \to (0, \infty)$ be r-convex and s-convex functions respectively on $[a, b]$ and $r, s > 0$. Then for the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{2}\left(\tfrac{r}{r+2}\right)\tfrac{f^{r+2}(b)-f^{r+2}(a)}{f^r(b)-f^r(a)} + \frac{1}{2}\left(\tfrac{s}{s+2}\right)\tfrac{g^{s+2}(b)-g^{s+2}(a)}{g^s(b)-g^s(a)},$$

with $f(b) \neq f(a)$ and $g(b) \neq g(a)$.

In [30] W. Ul-Haq and J. Iqbal proved the following Hadamard's inequalities for r-preinvex function

Theorem 6.[30, Theorem 4] Let $f : K = [a, a + \eta(b, a)] \to (0, \infty)$ be an r-preinvex function on the interval of real numbers K° (interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$, then the following inequality holds:

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \left[\tfrac{f^r(a)+f^r(b)}{2}\right]^{\frac{1}{r}}, \quad r \geq 1.$$

Theorem 7.[30, Theorem 6] Let $f : K = [a, a + \eta(b, a)] \to (0, \infty)$ be an r-preinvex function with $(r \geq 0)$ on the interval of real numbers K° (interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$, then the following inequality holds:

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \begin{cases} \tfrac{r}{r+1}\left[\tfrac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)}\right], & r \neq 0 \\ \tfrac{f(a)-f(b)}{\ln f(a)-\ln f(b)}, & r = 0, \end{cases}$$

with $f(b) \neq f(a)$.

Theorem 8.[30, Theorem 11] Let $f, g : K = [a, a + \eta(b, a)] \to (0, \infty)$ be an r-preinvex and s-preinvex functions respectively with $r, s > 0$ on the interval of real numbers K° (interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$, then the following inequality holds:

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{1}{2}\frac{r}{r+2}\left[\frac{f^{r+2}(a)-f^{r+2}(b)}{f^r(a)-f^r(b)}\right] + \frac{1}{2}\frac{s}{s+2}\left[\frac{g^{s+2}(a)-g^{s+2}(b)}{g^s(a)-g^s(b)}\right],$$

with $f(b) \neq f(a)$ and $g(b) \neq g(a)$.

In [21, 22] Noor proved the following Hadamard's inequality for *log*-preinvex function and product of two log-preinvex functions

Theorem 9.[22, Theorem 2.8] Let f be a *log*-preinvex function on the interval $[a, a + \eta(b, a)]$, then

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)-f(b)}{\ln f(a) - \ln f(b)},$$

with $f(b) \neq f(a)$.

Theorem 10.[21, Theorem 3.1] Let $f, g : K = [a, a + \eta(b, a)] \to (0, \infty)$ be preinvex functions on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$, then the following inequality holds.

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{1}{4}\left(\frac{\left[f^2(b)-f^2(a)\right]}{\ln f(b) - \ln f(a)} + \frac{\left[g^2(b)-g^2(a)\right]}{\ln g(b) - \ln g(a)}\right),$$

with $f(b) \neq f(a)$ and $g(b) \neq g(a)$.

Motivated by the above results, in this paper we introduce a new class of preinvex functions which is called (s, r)-preinvex functions in the first sense, then we establish some new Hadamard type inequalities where the function f be in this novel class of functions.

2. Preliminaries

In this section we recall some concepts of convexity which are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 1.[26] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.[26] A positive function $f : I \to \mathbb{R}$ is said to logarithmically convex, if

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{(1-t)}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 3.[24] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the first sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 4.[1] A positive function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-logarithmically convex in the first sense on I, for some $s \in (0,1]$, if

$$f(tx+(1-t)y) \le [f(x)]^{t^s} [f(y)]^{(1-t^s)}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 5.[25] A positive function $f : I \to \mathbb{R}$ is said to be r-convex on I, where $r \ge 0$, if

$$f(tx+(1-t)y) \le \begin{cases} [tf^r(x)+(1-t)f^r(y)]^{\frac{1}{r}}, & r \ne 0 \\ [f(x)]^{1-t}[f(y)]^t, & r = 0 \end{cases}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 6.[34] A set K is said to be invex at x with respect to η, if

$$x + t\eta(y,x) \in K$$

holds for all $x, y \in K$ and $t \in [0,1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 7.[34] A function f on the invex set K is said to be preinvex with respect to η, if

$$f(x+t\eta(y,x)) \le (1-t)f(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 8.[19] A positive function f on the invex set K is said to be logarithmically preinvex with respect to η, if

$$f(x+t\eta(y,x)) \le [f(x)]^{(1-t)}[f(y)]^t$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 9.[32] A nonnegative function f on the invex set K is said to be s-preinvex in the first sense with respect to η, if

$$f(x+t\eta(y,x)) \le (1-t^s)f(x) + t^s f(y)$$

for some fixed $s \in (0,1]$ and all $x, y \in K$ and $t \in [0,1]$.

Definition 10.[33] The function f on the invex set K is said to be s-log-preinvex in the first sense with respect to η, if

$$f(x+t\eta(y,x)) \le [f(x)]^{(1-t^s)}[f(y)]^{t^s}$$

for some fixed $s \in (0,1]$ and all $x, y \in K$ and $t \in [0,1]$.

Definition 11.[2] A positive function f on the invex set K is said to be r-preinvex with respect to η, where $r \ge 0$, if

$$f(x+t\eta(y,x)) \le \begin{cases} [(1-t)f^r(x)+tf^r(y)]^{\frac{1}{r}}, & r \ne 0 \\ [f(x)]^{1-t}[f(y)]^t, & r = 0 \end{cases}$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Lemma 1.[15] For $a \ge 0$ and $b \ge 0$, the following algebraic inequalities are true

$$(a+b)^\lambda \le 2^{\lambda-1}\left(a^\lambda + b^\lambda\right), \qquad \text{for } \lambda \ge 1$$

and

$$(a+b)^\lambda \le a^\lambda + b^\lambda, \qquad \text{for } 0 \le \lambda \le 1.$$

Lemma 2.[11] Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then for any $\varepsilon > 0$ we have

$$a^{\frac{q}{p}} \leq \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} a + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}.$$

We also recall that the Euler Beta function is defined as follows

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

3. Main results

In the following definition, we introduce a new concept of (s,r)-preinvex function in the first sense.

Definition 12. A positive function f on the invex set K, is said to be (s,r)-preinvex function in the first sense, if

$$f(x+t\eta(y,x)) \leq \begin{cases} \left[(1-t^s) f^r(x) + t^s f^r(y)\right]^{\frac{1}{r}}, & r \neq 0 \\ [f(x)]^{(1-t^s)}[f(y)]^{t^s}, & r = 0 \end{cases}$$

holds for some fixed $s \in (0,1]$, $r \in \mathbb{R}$ and all $x, y \in K$, and $t \in [0,1]$.
Now we set off to establish some Hadamard type inequalities for (s,r)-preinvex functions in the first sense.

Theorem 11. Let $f : [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s,r)-preinvex function in the first sense with respect to η with $\eta(b,a) > 0$, If $f \in L_1([a, a+\eta(b,a)])$, then the following inequality

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \left[\left(1 - \frac{1}{s+1}\right) f^r(a) + \frac{1}{s+1} f^r(b)\right]^{\frac{1}{r}} \tag{2}$$

holds for some fixed $s \in (0,1]$, and $r \geq 1$.
Proof. For $x = a + t\eta(b,a)$, we have

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx = \int_0^1 f(a+t\eta(b,a))dt. \tag{3}$$

Let $\varphi(x) = x^r$, obviously φ is convex function since $r \geq 1$, then

$$\varphi\Big(\int_0^1 f(a+t\eta(b,a))dt\Big) \leq \int_0^1 \varphi\left(f(a+t\eta(b,a))\right)dt, \tag{4}$$

we can restate (4) as

$$\left[\int_0^1 f(a+t\eta(b,a))dt\right]^r \leq \int_0^1 \left(f(a+t\eta(b,a))\right)^r dt. \tag{5}$$

Now using the (s,r)-preinvexity in the first sense of f, we deduce

$$\begin{aligned}\int_0^1 (f(a+t\eta(b,a)))^r\,dt &\leq \int_0^1 [(1-t^s)\,f^r(a)+t^s f^r(b)]\,dt \\ &= f^r(a)\int_0^1 (1-t^s)\,dt + f^r(b)\int_0^1 t^s dt \\ &= \left(1-\frac{1}{s+1}\right) f^r(a) + \frac{1}{s+1} f^r(b). \quad (6)\end{aligned}$$

The substitution of (6) into (5), gives the desired result. The proof is completed.

Remark 1. For $s=1$, Theorem 11 becomes Theorem 4 from [30]. Moreover if we choose $\eta(b,a)=b-a$, we obtain Theorem 2.1 from [36].

Theorem 12. Let $f:[a,a+\eta(b,a)]\to\mathbb{R}^+$ be (s,r)-preinvex function in the first sense with respect to η, with $\eta(b,a)>0$. If $f\in L_1([a,a+\eta(b,a)])$, then the following inequality

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \left[\frac{1}{s^r} f^r(a)\left(\beta\left(\frac{1}{s},\frac{1}{r}+1\right)\right)^r + \left(\frac{r}{s+r}\right)^r f^r(b)\right]^{\frac{1}{r}} \quad (7)$$

holds for all $a,b\in K$ and $s,r\in(0,1]$.

Proof. From the (s,r)-preinvexity in the first sense of f, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a+t\eta(b,a))dt \\ &\leq \int_0^1 [(1-t^s)\,f^r(a)+t^s f^r(b)]^{\frac{1}{r}}\,dt. \quad (8)\end{aligned}$$

Since $0<r\leq 1$, using Minkowski's inequality, we get

$$\int_0^1 [(1-t)^s f^r(a)+t^s f^r(b)]^{\frac{1}{r}}\,dt \leq \left[\left(\int_0^1 (1-t^s)^{\frac{1}{r}} f(a)\,dt\right)^r + \left(\int_0^1 t^{\frac{s}{r}} f(b)dt\right)^r\right]^{\frac{1}{r}}$$

$$\begin{aligned}&= \left[f^r(a)\left(\int_0^1 (1-t^s)^{\frac{1}{r}}\,dt\right)^r + f^r(b)\left(\int_0^1 t^{\frac{s}{r}}dt\right)^r\right]^{\frac{1}{r}} \\ &= \left[f^r(a)\left(\frac{1}{s}\int_0^1 (1-u)^{\frac{1}{r}} u^{\frac{1}{s}-1}du\right)^r + \left(\frac{r}{s+r}\right)^r f^r(b)\right]^{\frac{1}{r}} \\ &= \left[\frac{1}{s^r} f^r(a)\left(\beta\left(\frac{1}{s},\frac{1}{r}+1\right)\right)^r + \left(\frac{r}{s+r}\right)^r f^r(b)\right]^{\frac{1}{r}}, \quad (9)\end{aligned}$$

which is the desired result. The proof is achieved.

Remark. If we choose $\eta(b,a) = b-a$ in Theorem 12, we obtain Theorem 2.2 from [23]. Moreover if we take $s=1$ then we obtain Theorem 2.1 from [18].

Theorem 13. Let $f : [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s,r)-preinvex function in the first sense with respect to η with $\eta(b,a) > 0$. If $f \in L_1([a, a+\eta(b,a)])$, then the following inequality

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \begin{cases} \frac{2^{\frac{1-r}{r}}}{s} f(a)\,\beta\left(\frac{1}{s}, \frac{1}{r}+1\right) + 2^{\frac{1-r}{r}} \frac{r}{s+r} f(b) & \text{if } 0 < r \leq 1 \\ \frac{1}{s} f(a)\,\beta\left(\frac{1}{s}, \frac{1}{r}+1\right) + \frac{r}{s+r} f(b) & \text{if } r \geq 1 \end{cases} \tag{10}$$

holds for some fixed $s \in (0,1]$, and $r > 0$.

Proof. Since f is (s,r)-preinvex function in the first sense, we have

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a+t\eta(b,a))dt \\ &\leq \int_0^1 \left[(1-t^s) f^r(a) + t^s f^r(b)\right]^{\frac{1}{r}} dt. \end{aligned} \tag{11}$$

From Lemma 1, we have

$$\left[(1-t^s) f^r(a) + t^s f^r(b)\right]^{\frac{1}{r}} \leq \begin{cases} 2^{\frac{1-r}{r}} \left((1-t^s)^{\frac{1}{r}} f(a) + t^{\frac{s}{r}} f(b)\right) & \text{if } 0 < r \leq 1 \\ (1-t^s)^{\frac{1}{r}} f(a) + t^{\frac{s}{r}} f(b) & \text{if } r \geq 1 \end{cases}, \tag{12}$$

integrating (12) with respect to t on $[0,1]$, we get

$$\begin{aligned} \int_0^1 \left[(1-t^s) f^r(a) + t^s f^r(b)\right]^{\frac{1}{r}} dt &\leq \begin{cases} 2^{\frac{1-r}{r}} f(a) \int_0^1 (1-t^s)^{\frac{1}{r}} dt + 2^{\frac{1-r}{r}} f(b) \int_0^1 t^{\frac{s}{r}} dt & \text{if } 0 < r \leq 1 \\ f(a) \int_0^1 (1-t^s)^{\frac{1}{r}} dt + f(b) \int_0^1 t^{\frac{s}{r}} dt & \text{if } r \geq 1 \end{cases} \\ &= \begin{cases} \frac{2^{\frac{1-r}{r}}}{s} f(a)\,\beta\left(\frac{1}{s}, \frac{1}{r}+1\right) + 2^{\frac{1-r}{r}} \frac{r}{s+r} f(b) & \text{if } 0 < r \leq 1 \\ \frac{1}{s} f(a)\,\beta\left(\frac{1}{s}, \frac{1}{r}+1\right) + \frac{r}{s+r} f(b) & \text{if } r \geq 1, \end{cases} \end{aligned} \tag{13}$$

which is the desired result. The proof is completed.

Theorem 14. Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be (s,r)-preinvex function in the first sense with respect to η with $\eta(b,a) > 0$. If $f \in L_1([a, a+\eta(b,a)])$, then the following inequality

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \begin{cases} \frac{r}{(r+1)\theta} \left[(\alpha+\theta)^{\frac{1+r}{r}} - \alpha^{\frac{1+r}{r}}\right] & \text{if } r > 0 \\ f(a) & \text{if } r = 0 \text{ and } f(a) = f(b) \\ f(a) \left[\frac{f(b)}{f(a)}\right]^{(1-s)\varepsilon^s} \left[\frac{\left[\frac{f(b)}{f(a)}\right]^{s\varepsilon^{s-1}} - 1}{s\varepsilon^{s-1} \ln\left[\frac{f(b)}{f(a)}\right]}\right] & \text{if } r = 0 \text{ and } f(a) \neq f(b) \end{cases} \tag{14}$$

holds for some fixed $s \in (0,1]$ and $r \geq 0$, where

$$\begin{aligned} \alpha &= f^r(a) + (1-s)\varepsilon^s [f^r(b) - f^r(a)] \\ \theta &= s\varepsilon^{s-1}[f^r(b) - f^r(a)], \end{aligned} \tag{15}$$

and $\varepsilon > 0$.

Proof. Case 1 : $r > 0$.

Since f is (s,r)-preinvex function in the first sense, we get

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a + t\eta(b,a))dt \\ &\leq \int_0^1 [(1-t^s) f^r(a) + t^s f^r(b)]^{\frac{1}{r}} dt \\ &= \int_0^1 [\, f^r(a) + t^s [f^r(b) - f^r(a)]]^{\frac{1}{r}} dt. \end{aligned} \tag{16}$$

From Lemma 2, we have

$$t^s \leq s\varepsilon^{s-1} t + (1-s)\varepsilon^s, \ \varepsilon > 0. \tag{17}$$

Substituting (17) into (16), we obtain

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \int_0^1 (\alpha + \theta t)^{\frac{1}{r}} dt, \tag{18}$$

where α and θ are are given by (15).

Let $z = \alpha + \theta t$, then (18) becomes

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx &\leq \frac{1}{\theta} \int_\alpha^{\alpha+\theta} z^{\frac{1}{r}} dz \\ &= \frac{r}{(r+1)\theta} \left[(\alpha+\theta)^{\frac{1+r}{r}} - \alpha^{\frac{1+r}{r}} \right]. \end{aligned} \tag{19}$$

Case 2 :

If $r = 0$, then f is s-log-preinvex in the first sense, we have

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a + t\eta(b,a))dt \\ &\leq \int_0^1 [f(a)]^{(1-t^s)} [f(b)]^{t^s} dt \\ &= f(a) \int_0^1 \left[\frac{f(b)}{f(a)} \right]^{t^s} dt. \end{aligned} \tag{20}$$

If $f(a)=f(b)$, (20) gives

$$\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)dx \leq f(a), \tag{21}$$

and if $f(a)\neq f(b)$, using (17), (20) becomes

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)dx &\leq f(a)\left[\frac{f(b)}{f(a)}\right]^{(1-s)\varepsilon^{s}}\int_{0}^{1}\left[\frac{f(b)}{f(a)}\right]^{s\varepsilon^{s-1}t}dt \\ &= f(a)\left[\frac{f(b)}{f(a)}\right]^{(1-s)\varepsilon^{s}}\left[\frac{\left[\frac{f(b)}{f(a)}\right]^{s\varepsilon^{s-1}}-1}{s\varepsilon^{s-1}\ln\left[\frac{f(b)}{f(a)}\right]}\right]. \end{aligned}\tag{22}$$

From (19), (21) and (22), we get the desired result. The proof is completed.

Remark. If we take $s=1$, in Theorem 14, we obtain Theorem 6 from [30]. Moreover if we choose $r=0$ we obtain Theorem 2.8 from [21].

Theorem 15. Let $f,g:[a,a+\eta(b,a)]\to\mathbb{R}_{+}$ be (s_1,r_1) and (s_2,r_2)-preinvex functions in the first sense respectively with respect to η with $\eta(b,a)>0$, and let $(s_1,r_1),(s_2,r_2)\in(0,1]\times(0,2]$. If $fg\in L_1([a,a+\eta(b,a)])$, then the following inequality is valid

$$\begin{aligned}&\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)g(x)dx \\ &\leq \frac{1}{2}\left[\frac{f^{r_1}(a)}{s_1}\left(\beta\left(\frac{1}{s_1},\frac{2}{r_1}+1\right)\right)^{\frac{r_1}{2}}+\left(\frac{r_1}{2s_1+r_1}\right)^{\frac{r_1}{2}}f^{r_1}(b)\right]^{\frac{2}{r_1}} \\ &\quad+\frac{1}{2}\left[\frac{g^{r_2}(a)}{s_2}\left(\beta\left(\frac{1}{s_2},\frac{2}{r_2}+1\right)\right)^{\frac{r_2}{2}}+\left(\frac{r_2}{2s_2+r_2}\right)^{\frac{r_2}{2}}g^{r_2}(b)\right]^{\frac{2}{r_2}}. \end{aligned}\tag{23}$$

Proof. Since f and g are (s_1,r_1) and (s_2,r_2)-preinvex functions in the first sense respectively, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)g(x)dx &= \int_{0}^{1} f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \int_{0}^{1}\Big[\left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{1}{r_1}} \\ &\quad\times\left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{1}{r_2}}\Big]dt. \end{aligned}\tag{24}$$

Applying the AG inequality, we get

$$
\begin{aligned}
&\int_0^1 \left[\left(1-t^{s_1}\right) f^{r_1}(a)+t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}}\left[\left(1-t^{s_2}\right) g^{r_2}(a)+t^{s_2} g^{r_2}(b)\right]^{\frac{1}{r_2}} dt \\
\leq\ & \frac{1}{2}\int_0^1 \left[\left(1-t^{s_1}\right) f^{r_1}(a)+t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt \\
&+\frac{1}{2}\int_0^1 \left[\left(1-t^{s_2}\right) g^{r_2}(a)+t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt. \qquad (25)
\end{aligned}
$$

Now, using Minkowski's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \left[\left(1-t^{s_1}\right) f^{r_1}(a)+t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt+\frac{1}{2}\int_0^1 \left[\left(1-t^{s_2}\right) g^{r_2}(a)+t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt \\
\leq\ & \frac{1}{2}\left[\left(\int_0^1 \left(1-t^{s_1}\right)^{\frac{2}{r_1}} f^2(a)\, dt\right)^{\frac{r_1}{2}}+\left(\int_0^1 t^{\frac{2s_1}{r_1}} f^2(b)\, dt\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
&+\frac{1}{2}\left[\left(\int_0^1 \left(1-t^{s_2}\right)^{\frac{2}{r_2}} g^2(a)\, dt\right)^{\frac{r2}{2}}+\left(\int_0^1 t^{\frac{2s_2}{r_2}} g^2(b)\, dt\right)^{\frac{r_2}{2}}\right]^{\frac{2}{r_2}} \\
=\ & \frac{1}{2}\left[f^{r_1}(a)\left(\int_0^1 \left(1-t^{s_1}\right)^{\frac{2}{r_1}} dt\right)^{\frac{r_1}{2}}+f^{r_1}(b)\left(\int_0^1 t^{\frac{2s_1}{r_1}} dt\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
&+\frac{1}{2}\left[g^{r_2}(a)\left(\int_0^1 \left(1-t^{s_2}\right)^{\frac{2}{r_2}} dt\right)^{\frac{r2}{2}}+g^{r_2}(b)\left(\int_0^1 t^{\frac{2s_2}{r_2}} dt\right)^{\frac{r_2}{2}}\right]^{\frac{2}{r_2}} \\
=\ & \frac{1}{2}\left[\frac{f^{r_1}(a)}{s_1}\left(\int_0^1 (1-u)^{\frac{2}{r_1}} u^{\frac{1-s_1}{s_1}} dt\right)^{\frac{r_1}{2}}+f^{r_1}(b)\left(\int_0^1 t^{\frac{2s_1}{r_1}} dt\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
&+\frac{1}{2}\left[\frac{g^{r_2}(a)}{s_2}\left(\int_0^1 (1-u)^{\frac{2}{r_2}} u^{\frac{1-s_2}{s_2}} dt\right)^{\frac{r2}{2}}+g^{r_2}(b)\left(\int_0^1 t^{\frac{2s_2}{r_2}} dt\right)^{\frac{r_2}{2}}\right]^{\frac{2}{r_2}} \\
=\ & \frac{1}{2}\left[\frac{f^{r_1}(a)}{s_1}\left(\beta\left(\frac{1}{s_1},\frac{2}{r_1}+1\right)\right)^{\frac{r_1}{2}}+\left(\frac{r_1}{2s_1+r_1}\right)^{\frac{r_1}{2}} f^{r_1}(b)\right]^{\frac{2}{r_1}} \\
&+\frac{1}{2}\left[\frac{g^{r_2}(a)}{s_2}\left(\beta\left(\frac{1}{s_2},\frac{2}{r_2}+1\right)\right)^{\frac{r2}{2}}+\left(\frac{r_2}{2s_2+r_2}\right)^{\frac{r_2}{2}} g^{r_2}(b)\right]^{\frac{2}{r_2}}.
\end{aligned}
$$

The proof is completed.

Remark. In Theorem 15, if we choose $\eta(b,a) = b - a$, and $s_1 = s_2 = 1$, we obtain Theorem 2.3 from [23].

Theorem 16. Let $f, g : [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s_1, r_1) and (s_2, r_2)-preinvex functions in the first sense respectively with respect to η with $\eta(b,a) > 0$, and let $(s_1, r_1), (s_2, r_2) \in (0,1] \times \mathbb{R}^+$. If $fg \in L_1([a, a+\eta(b,a)])$, then the following inequality

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq \left[\frac{1}{1+s_1}\left[f(b)\right]^{r_1} + \left(\frac{s_1}{1+s_1}\right)\left[f(a)\right]^{r_1}\right]^{\frac{1}{r_1}} \\ &\quad \times \left[\frac{1}{1+s_2}\left[g(b)\right]^{r_2} + \frac{s_2}{1+s_2}\left[g(a)\right]^{r_2}\right]^{\frac{1}{r_2}} \qquad (26) \end{aligned}$$

holds for $r_1 > 1$, and $\frac{1}{r_1} + \frac{1}{r_2} = 1$.

Proof. Since f and g are (s_1, r_1) and (s_2, r_2)-preinvex functions in the first sense respectively, we have

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \int_0^1 \left[\left[(1-t^{s_1})\left[f(a)\right]^{r_1} + t^{s_1}\left[f(b)\right]^{r_1}\right]^{\frac{1}{r_1}}\right. \\ &\quad \times \left.\left[(1-t^{s_2})\left[g(a)\right]^{r_2} + t^{s_2}\left[g(b)\right]^{r_2}\right]^{\frac{1}{r_2}}\right] dt, \end{aligned} \qquad (27)$$

using Hölder's inequality, we obtain

$$\begin{aligned} &\int_0^1 \left[(1-t^{s_1})\left[f(a)\right]^{r_1} + t^{s_1}\left[f(b)\right]^{r_1}\right]^{\frac{1}{r_1}} \left[(1-t^{s_2})\left[g(a)\right]^{r_2} + t^{s_2}\left[g(b)\right]^{r_2}\right]^{\frac{1}{r_2}} dt \\ \leq &\left[\int_0^1 \left[(1-t^{s_1})\left[f(a)\right]^{r_1} + t^{s_1}\left[f(b)\right]^{r_1}\right] dt\right]^{\frac{1}{r_1}} \\ &\times \left[\int_0^1 \left[(1-t^{s_2})\left[g(a)\right]^{r_2} + t^{s_2}\left[g(b)\right]^{r_2}\right] dt\right]^{\frac{1}{r_2}} \\ = &\left[\int_0^1 \left[\left[f(a)\right]^{r_1} + \left(\left[f(b)\right]^{r_1} - \left[f(a)\right]^{r_1}\right) t^{s_1}\right] dt\right]^{\frac{1}{r_1}} \end{aligned}$$

$$\times\left[\int_0^1\left[[g(a)]^{r_2}+\left([g(b)]^{r_2}-[g(a)]^{r_2}\right)t^{s_2}\right]dt\right]^{\frac{1}{r_2}}$$
$$=\left[\frac{1}{1+s_1}[f(b)]^{r_1}+\left(\frac{s_1}{1+s_1}\right)[f(a)]^{r_1}\right]^{\frac{1}{r_1}}\left[\frac{1}{1+s_2}[g(b)]^{r_2}+\frac{s_2}{1+s_2}[g(a)]^{r_2}\right]^{\frac{1}{r_2}}.$$

The proof is achieved.

Remark. In Theorem 16, if we choose $\eta(b,a)=b-a$, and $s_1=s_2=1$, we obtain Theorem 2.6 from [18].

Theorem 17. Let $f,g:[a,a+\eta(b,a)]\to\mathbb{R}_+$ be (s_1,r_1) and (s_2,r_2)-preinvex functions in the first sense respectively with respect to η with $\eta(b,a)>0$, and let $(s_1,r_1)\in(0,1]\times(0,2]$, and $(s_2,r_2)\in(0,1]\times[2,\infty)$. If $fg\in L_1([a,a+\eta(b,a)])$, then the following inequality is valid

$$\begin{aligned}&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)g(x)dx\\ \leq\;& 2^{\frac{2}{r_1}-1}\left[\frac{f^2(a)}{s_1}\beta\left(\frac{1}{s_1},\frac{r_1}{2}+1\right)+\frac{r_1}{2s_1+r_1}f^2(b)\right]\\ &+\frac{1}{2}\left[\frac{g^2(a)}{s_2}\beta\left(\frac{1}{s_2},\frac{r_2}{2}+1\right)+\frac{r_2}{2s_2+r_2}g^2(b)\right].\end{aligned}\tag{28}$$

Proof. Since f and g are (s_1,r_1) and (s_2,r_2)-preinvex functions in the first sense respectively, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt\\ &\leq \int_0^1\left[\left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{1}{r_1}}\right.\\ &\quad\left.\times\left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{1}{r_2}}\right]dt.\end{aligned}\tag{29}$$

Applying the AG inequality, we get

$$\begin{aligned}&\int_0^1\left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{1}{r_1}}\left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{1}{r_2}}dt\\ \leq\;&\frac{1}{2}\int_0^1\left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{2}{r_1}}dt\\ &+\frac{1}{2}\int_0^1\left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{2}{r_1}}dt.\end{aligned}\tag{30}$$

Now, using Lemma 1, we get

$$\begin{aligned}
& \frac{1}{2}\int_0^1 \left[\left(1-t^{s_1}\right) f^{r_1}(a)+t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt+\frac{1}{2}\int_0^1 \left[\left(1-t^{s_2}\right) g^{r_2}(a)+t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_1}} dt \\
\leq\ & 2^{\frac{2}{r_1}-1}\left[f^2(a)\int_0^1\left(1-t^{s_1}\right)^{\frac{2}{r_1}} dt+f^2(b)\int_0^1 t^{\frac{2 s_1}{r_1}} dt\right] \\
& +\frac{1}{2}\int_0^1\left[g^2(a)\int_0^1\left(1-t^{s_2}\right)^{\frac{2}{r_1}} dt+g^2(b)\int_0^1 t^{\frac{2 s_2}{r_2}} dt\right] \\
=\ & 2^{\frac{2}{r_1}-1}\left[\frac{f^2(a)}{s_1}\beta\left(\frac{1}{s_1},\frac{r_1}{2}+1\right)+\frac{r_1}{2 s_1+r_1} f^2(b)\right] \\
& +\frac{1}{2}\left[\frac{g^2(a)}{s_2}\beta\left(\frac{1}{s_2},\frac{r_2}{2}+1\right)+\frac{r_2}{2 s_2+r_2} g^2(b)\right]. \qquad (31)
\end{aligned}$$

The proof is achieved.

Theorem 18. Let $f,g:[a,a+\eta(b,a)]\rightarrow\mathbb{R}^+$ be (s_1,r_1)-preinvex function in the first sense and $(s_2,0)$- preinvex function respectively with respect to η with $\eta(b,a)>0$, and let $(s_1,r_1)\in(0,1]\times[2,\infty)$ and $s_2\in(0,1]$ and $g(a)\neq 0$, and $g(b)\neq 0$. If $fg\in L_1([a,a+\eta(b,a)])$, then the following inequality is valid

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \begin{cases} \frac{[f(a)]^2}{2s_1}\beta\left(\frac{1}{s_1},\frac{2}{r_1}+1\right)+\frac{r_1[f(b)]^2}{4s_1+2r_1} \\ +\frac{[g(a)]^2}{2}\left(\frac{g(b)}{g(a)}\right)^{2(1-s_2)\varepsilon^{s_2}} \frac{\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{s_2-1}}-1}{\ln\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{s_2-1}}} \\ \text{if } g(a)\neq g(b), \\ \frac{[f(a)]^2}{2s_1}\beta\left(\frac{1}{s_1},\frac{2}{r_1}+1\right)+\frac{r_1[f(b)]^2}{4s_1+2r_1} \\ +\frac{[g(a)]^2}{2} \text{ if } g(a)=g(b). \end{cases} \qquad (32)$$

Proof. Since f and g are (s_1,r_1), $(s_2,0)$-preinvex functions in the first sense respectively, we have

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\
&\leq \int_0^1\left[\left(1-t^{s_1}\right)f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{1}{r_1}}[g(a)]^{\left(1-t^{s_2}\right)}[g(b)]^{t^{s_2}}dt,
\end{aligned} \qquad (33)$$

applying the AG inequality, we get

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq \frac{1}{2}\int_0^1\left[\left(1-t^{s_1}\right)f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{2}{r_1}}dt \\
&\quad +\frac{[g(a)]^2}{2}\int_0^1\left[\left(\frac{g(b)}{g(a)}\right)^2\right]^{t^{s_2}}dt. \qquad (34)
\end{aligned}$$

In the case where $g(b) \neq g(a)$, using Lemma 2 and Lemma 1, (34) gives

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[(1-t^{s_1}) f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt + \frac{[g(a)]^2}{2}\int_0^1 \left[\left(\frac{g(b)}{g(a)}\right)^2\right]^{t^{s_2}} dt \\
\leq\ & \frac{[f(a)]^2}{2}\int_0^1 (1-t^{s_1})^{\frac{2}{r_1}} dt + \frac{[f(b)]^2}{2}\int_0^1 t^{\frac{2s_1}{r_1}} dt + \frac{[g(a)]^2}{2}\int_0^1 \left[\left(\frac{g(b)}{g(a)}\right)^2\right]^{t^{s_2}} dt \\
=\ & \frac{[f(a)]^2}{2s_1}\beta(\tfrac{1}{s_1}, \tfrac{2}{r_1}+1) + \frac{[f(b)]^2}{2}\frac{r_1}{2s_1+r_1} + \frac{[g(a)]^2}{2}\int_0^1 \left[\left(\frac{g(b)}{g(a)}\right)^2\right]^{s_2\varepsilon^{s_2-1}t+(1-s_2)\varepsilon^{s_2}} dt \\
\leq\ & \frac{[f(a)]^2}{2s_1}\beta(\tfrac{1}{s_1}, \tfrac{2}{r_1}+1) + \frac{[f(b)]^2}{2}\frac{r_1}{2s_1+r_1} \\
& + \frac{[g(a)]^2}{2}\left(\frac{g(b)}{g(a)}\right)^{2(1-s_2)\varepsilon^{s_2}} \int_0^1 \left[\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{s_2-1}}\right]^t dt \\
=\ & \frac{[f(a)]^2}{2s_1}\beta(\tfrac{1}{s_1}, \tfrac{2}{r_1}+1) + \frac{r_1[f(b)]^2}{4s_1+2r_1} + \frac{[g(a)]^2}{2}\left(\frac{g(b)}{g(a)}\right)^{2(1-s_2)\varepsilon^{s_2}} \frac{\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{s_2-1}}-1}{\ln\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{s_2-1}}}.
\end{aligned} \tag{35}$$

In the case where $g(b) = g(a)$, (34) becomes

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \quad \leq \quad & \frac{1}{2}\int_0^1 \left[(1-t^{s_1}) f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt \\
& + \frac{[g(a)]^2}{2}\int_0^1 dt,
\end{aligned} \tag{36}$$

using Lemma 1 for (36), we get

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \quad \leq \quad & \frac{[f(a)]^2}{2s_1}\beta(\frac{1}{s_1}, \frac{2}{r_1}+1) + \frac{r_1[f(b)]^2}{4s_1+2r_1} \\
& + \frac{[g(a)]^2}{2}.
\end{aligned} \tag{37}$$

The proof is achieved.

Remark. If we take $s_1 = s_2 = 1$, in Theorem 18, we obtain Theorem 11 from [30].

Theorem 19. Let $f, g : [a, a+\eta(b,a)] \to (0, +\infty)$ be $(s_1, 0)$ and $(s_2, 0)$-preinvex

functions in the first sense respectively with respect to η with $\eta(b,a) > 0$, and let

$s_1, s_2 \in (0,1]$. If $fg \in L_1([a, a+\eta(b,a)])$, then the following inequality is valid

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \begin{cases} \left(\frac{f(b)}{f(a)}\right)^{(s_1-1)\varepsilon^{s_1}} \left(\frac{g(b)}{g(a)}\right)^{(s_2-1)\varepsilon^{s_2}} f(a)\,g(a) \\ \quad \times \frac{\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}} \left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}} - 1}{\ln\left[\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}} \left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}}\right]} \\ \quad \text{if } f(b) \neq f(a) \text{ and } g(b) \neq g(a), \\ \left(\frac{g(b)}{g(a)}\right)^{(s_2-1)\varepsilon^{s_2}} f(a)\,g(a) \frac{\left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}} - 1}{\ln\left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}}} \\ \quad \text{if } f(b) = f(a) \text{ and } g(b) \neq g(a), \\ \left(\frac{f(b)}{f(a)}\right)^{(s_1-1)\varepsilon^{s_1}} f(a)\,g(a) \frac{\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}} - 1}{\ln\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}}} \\ \quad \text{if } f(b) \neq f(a) \text{ and } g(b) = g(a), \\ f(a)\,g(a) \text{ if } f(b) = f(a) \text{ and } g(b) = g(a), \end{cases} \tag{38}$$

where $\varepsilon > 0$.

Proof. Since f and g are $(s_1, 0)$ and $(s_2, 0)$-preinvex functions in the first sense respectively, we have

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \int_0^1 [f(a)]^{(1-t^{s_1})} [f(b)]^{t^{s_1}} [g(a)]^{(1-t^{s_2})} [g(b)]^{t^{s_2}} dt \\ &= f(a)\,g(a) \int_0^1 \left[\frac{f(b)}{f(a)}\right]^{t^{s_1}} \left[\frac{g(b)}{g(a)}\right]^{t^{s_2}} dt. \end{aligned} \tag{39}$$

If $f(b) \neq f(a)$ and $g(b) \neq g(a)$, from Lemma 2, (39) gives

$$\begin{aligned} & f(a)\,g(a) \int_0^1 \left[\frac{f(b)}{f(a)}\right]^{t^{s_1}} \left[\frac{g(b)}{g(a)}\right]^{t^{s_2}} dt \\ \leq\ & \left(\frac{f(b)}{f(a)}\right)^{(s_1-1)\varepsilon^{s_1}} \left(\frac{g(b)}{g(a)}\right)^{(s_2-1)\varepsilon^{s_2}} f(a)\,g(a) \int_0^1 \left[\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}} \left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}}\right]^t dt \\ =\ & \left(\frac{f(b)}{f(a)}\right)^{(s_1-1)\varepsilon^{s_1}} \left(\frac{g(b)}{g(a)}\right)^{(s_2-1)\varepsilon^{s_2}} f(a)\,g(a) \frac{\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}} \left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}} - 1}{\ln\left[\left(\frac{f(b)}{f(a)}\right)^{s_2\varepsilon^{s_2-1}} \left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}}\right]}. \end{aligned} \tag{40}$$

In the case where $f(b)=f(a)$, and $g(b)\neq g(a)$, we obtain

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq f(a)\,g(a)\int_0^1\left[\frac{g(b)}{g(a)}\right]^{t^{s_2}}dt\\ &= \left(\frac{g(b)}{g(a)}\right)^{(s_2-1)\varepsilon^{s_2}} f(a)\,g(a)\,\frac{\left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}}-1}{\ln\left(\frac{g(b)}{g(a)}\right)^{s_2\varepsilon^{s_2-1}}}.\end{aligned} \tag{41}$$

In the case where $f(b)\neq f(a)$ and $g(b)=g(a)$, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq f(a)\,g(a)\int_0^1\left[\frac{f(b)}{f(a)}\right]^{t^{s_1}}dt\\ &= \left(\frac{f(b)}{f(a)}\right)^{(s_1-1)\varepsilon^{s_1}} f(a)\,g(a)\,\frac{\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}}-1}{\ln\left(\frac{f(b)}{f(a)}\right)^{s_1\varepsilon^{s_1-1}}}.\end{aligned} \tag{42}$$

In the case where $f(b)=f(a)$, and $g(b)=g(a)$, we deduce

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx = f(a)\,g(a)\,. \tag{43}$$

From (40)-(43), we get the desired result. The proof is completed.

Theorem 20. Let $f,g:[a,a+\eta(b,a)]\to\mathbb{R}^+$ be (s_1,r_1) and (s_2,r_2)- preinvex functions in the first sense respectively with respect to η with $\eta(b,a)>0$, and let $(s_1,r_1),(s_2,r_2)\in(0,1]\times(0,\infty)$ and $f(b)\neq f(a)$, and $g(b)\neq g(a)$. If $fg\in L_1([a,a+\eta(b,a)])$, then the following inequality is valid

$$\begin{aligned}&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx\\ \leq\;& \frac{r_1}{2s_1\varepsilon^{s_1-1}[f^{r_1}(b)-f^{r_1}(a)](2+r_1)}\Big[\big[s_1\varepsilon^{s_1-1}\left[f^{r_1}(b)-f^{r_1}(a)\right]+f^{r_1}(a)\\ &+(s_1-1)\varepsilon^{s_1}\left[f^{r_1}(b)-f^{r_1}(a)\right]\big]^{\frac{2+r_1}{r_1}}-\left[f^{r_1}(a)+(s_1-1)\varepsilon^{s_1}\left[f^{r_1}(b)-f^{r_1}(a)\right]\right]^{\frac{2+r_1}{r_1}}\Big]\\ &+\frac{r_2}{2s_2\varepsilon^{s_2-1}[g^{r_2}(b)-g^{r_2}(a)](2+r_2)}\Big[\big[s_2\varepsilon^{s_2-1}\left[g^{r_2}(b)-g^{r_2}(a)\right]+g^{r_2}(a)\\ &+(s_2-1)\varepsilon^{s_2}\left[g^{r_2}(b)-g^{r_2}(a)\right]\big]^{\frac{2+r_2}{r_2}}-\left[g^{r_2}(a)+(s_2-1)\varepsilon^{s_2}\left[g^{r_2}(b)-g^{r_2}(a)\right]\right]^{\frac{2+r_2}{r_2}}\Big],\end{aligned} \tag{44}$$

where $\varepsilon>0$.

Proof. Since f and g are (s_1,r_1) and (s_2,r_2)-preinvex functions in the first sense respectively, we have

$$\begin{aligned} \frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)g(x)dx &= \int\limits_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \int\limits_0^1 \Big[\left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{1}{r_1}} \\ &\quad\times\left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{1}{r_2}}\Big]dt. \end{aligned} \tag{45}$$

Applying the AG inequality, we get

$$\begin{aligned} &\int\limits_0^1 \left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{1}{r_1}}\left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{1}{r_2}}dt \\ \leq\ & \frac{1}{2}\int\limits_0^1 \left[(1-t^{s_1})f^{r_1}(a)+t^{s_1}f^{r_1}(b)\right]^{\frac{2}{r_1}}dt \\ &+\frac{1}{2}\int\limits_0^1 \left[(1-t^{s_2})g^{r_2}(a)+t^{s_2}g^{r_2}(b)\right]^{\frac{2}{r_1}}dt \\ =\ & \frac{1}{2}\int\limits_0^1 \left[\left[f^{r_1}(b)-f^{r_1}(a)\right]t^{s_1}+f^{r_1}(a)\right]^{\frac{2}{r_1}}dt \\ &+\frac{1}{2}\int\limits_0^1 \left[\left[g^{r_2}(b)-g^{r_2}(a)\right]t^{s_2}+g^{r_2}(a)\right]^{\frac{2}{r_1}}dt. \end{aligned} \tag{46}$$

From Lemma 2, we can restate (46) as follows

$$\begin{aligned} &\frac{1}{2}\int\limits_0^1 \left[\left[f^{r_1}(b)-f^{r_1}(a)\right]t^{s_1}+f^{r_1}(a)\right]^{\frac{2}{r_1}}dt+\frac{1}{2}\int\limits_0^1 \left[\left[g^{r_2}(b)-g^{r_2}(a)\right]t^{s_2}+g^{r_2}(a)\right]^{\frac{2}{r_1}}dt \\ \leq\ & \frac{1}{2}\int\limits_0^1 \left[s_1\varepsilon^{s_1-1}\left[f^{r_1}(b)-f^{r_1}(a)\right]t+f^{r_1}(a)+(s_1-1)\varepsilon^{s_1}\left[f^{r_1}(b)-f^{r_1}(a)\right]\right]^{\frac{2}{r_1}}dt \\ &+\frac{1}{2}\int\limits_0^1 \left[s_2\varepsilon^{s_2-1}\left[g^{r_2}(b)-g^{r_2}(a)\right]t+g^{r_2}(a)+(s_2-1)\varepsilon^{s_2}\left[g^{r_2}(b)-g^{r_2}(a)\right]\right]^{\frac{2}{r_2}}dt \\ =\ & \tfrac{r_1}{2s_1\varepsilon^{s_1-1}[f^{r_1}(b)-f^{r_1}(a)](2+r_1)}\Big[\Big[s_1\varepsilon^{s_1-1}\left[f^{r_1}(b)-f^{r_1}(a)\right]+f^{r_1}(a) \\ &+(s_1-1)\varepsilon^{s_1}\left[f^{r_1}(b)-f^{r_1}(a)\right]\Big]^{\frac{2+r_1}{r_1}}-\left[f^{r_1}(a)+(s_1-1)\varepsilon^{s_1}\left[f^{r_1}(b)-f^{r_1}(a)\right]\right]^{\frac{2+r_1}{r_1}}\Big] \\ &+\tfrac{r_2}{2s_2\varepsilon^{s_2-1}[g^{r_2}(b)-g^{r_2}(a)](2+r_2)}\Big[\Big[s_2\varepsilon^{s_2-1}\left[g^{r_2}(b)-g^{r_2}(a)\right]+g^{r_2}(a) \\ &+(s_2-1)\varepsilon^{s_2}\left[g^{r_2}(b)-g^{r_2}(a)\right]\Big]^{\frac{2+r_2}{r_2}}-\left[g^{r_2}(a)+(s_2-1)\varepsilon^{s_2}\left[g^{r_2}(b)-g^{r_2}(a)\right]\right]^{\frac{2+r_2}{r_2}}\Big], \end{aligned} \tag{47}$$

which is the desired result.

Remark. If we take $s_1 = s_2 = 1$, in Theorem 20 we obtain Theorem 11 from [30]. Moreover if $\eta(b,a) = b - a$ then we obtain Theorem 2.8 from [36].

Theorem 21. Let $f, g : [a, a + \eta(b,a)] \to \mathbb{R}^+$ be $(s_1, 0)$ and $(s_2, 0)$-preinvex functions in the first sense respectively with respect to η with $\eta(b,a) > 0$, and let $s_1, s_2 \in (0,1]$. If $fg \in L_1([a, a + \eta(b,a)])$, then the following inequality is valid

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{[f(a)]^2}{2} \left(\frac{f(b)}{f(a)}\right)^{2(1-s_1)\varepsilon^{s_1}} \frac{\left(\frac{f(b)}{f(a)}\right)^{2s_1\varepsilon^{1-s_1}} - 1}{\ln\left(\frac{f(b)}{f(a)}\right)^{2s_1\varepsilon^{1-s_1}}}$$
$$+ \frac{[g(a)]^2}{2} \left(\frac{g(b)}{g(a)}\right)^{2(1-s_2)\varepsilon^{s_2}} \frac{\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{1-s_2}} - 1}{\ln\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{1-s_2}}}, \tag{48}$$

where $\varepsilon > 0$.

Proof. Since f and g are $(s_1, 0)$ and $(s_2, 0)$-preinvex functions in the first sense respectively, we have

$$\begin{aligned}
\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a + t\eta(b,a))g(a + t\eta(b,a))dt \\
&\leq \int_0^1 [f(a)]^{(1-t^{s_1})} [f(b)]^{t^{s_1}} [g(a)]^{(1-t^{s_2})} [g(b)]^{t^{s_2}} dt \\
&= f(a)\, g(a) \int_0^1 \left[\frac{f(b)}{f(a)}\right]^{t^{s_1}} \left[\frac{g(b)}{g(a)}\right]^{t^{s_2}} dt. \qquad (49)
\end{aligned}$$

Applying the AG inequality, we obtain

$$\begin{aligned}
\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq \frac{1}{2}\int_0^1 \left[[f(a)]^{(1-t^{s_1})} [f(b)]^{t^{s_1}}\right]^2 dt \\
&\quad + \frac{1}{2}\int_0^1 \left[[g(a)]^{(1-t^{s_2})} [g(b)]^{t^{s_2}}\right]^2 dt. \qquad (50)
\end{aligned}$$

Using Lemma 2 for (50) yields

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \\
\leq\ & \frac{[f(a)]^2}{2}\int_0^1\left[\left(\frac{f(b)}{f(a)}\right)^2\right]^{s_1\varepsilon^{1-s_1}\ t+(1-s_1)\varepsilon^{s_1}}dt \\
&+\frac{[g(a)]^2}{2}\int_0^1\left[\left(\frac{g(b)}{g(a)}\right)^2\right]^{s_2\varepsilon^{1-s_2}\ t+(1-s_2)\varepsilon^{s_2}}dt \\
=\ & \frac{[f(a)]^2}{2}\left(\frac{f(b)}{f(a)}\right)^{2(1-s_1)\varepsilon^{s_1}}\int_0^1\left[\left(\frac{f(b)}{f(a)}\right)^{2s_1\varepsilon^{1-s_1}}\right]^t dt \\
&+\frac{[g(a)]^2}{2}\left(\frac{g(b)}{g(a)}\right)^{2(1-s_2)\varepsilon^{s_2}}\int_0^1\left[\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{1-s_2}}\right]^t dt \\
=\ & \frac{[f(a)]^2}{2}\left(\frac{f(b)}{f(a)}\right)^{2(1-s_1)\varepsilon^{s_1}}\frac{\left(\frac{f(b)}{f(a)}\right)^{2s_1\varepsilon^{1-s_1}}-1}{\ln\left(\frac{f(b)}{f(a)}\right)^{2s_1\varepsilon^{1-s_1}}} \\
&+\frac{[g(a)]^2}{2}\left(\frac{g(b)}{g(a)}\right)^{2(1-s_2)\varepsilon^{s_2}}\frac{\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{1-s_2}}-1}{\ln\left(\frac{g(b)}{g(a)}\right)^{2s_2\varepsilon^{1-s_2}}}. \qquad (51)
\end{aligned}$$

The proof is achieved.

Remark. If we take $s_1 = s_2 = 1$, in Theorem 21, we obtain Theorem 3.1 from [22].

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