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SOME DIFFERENT TYPE INTEGRAL INEQUALITIES PERTAINING GENERALIZED RELATIVE SEMI-m-$(r; h_1, h_2)$-PREINVEX MAPPINGS AND THEIR APPLICATIONS

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Abstract. In this article, we first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings. Secondly, a new identity pertaining twice differentiable mappings defined on **m**-invex set is derived. By using the notion of generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article. At the end, some applications to special means for different positive real numbers are provided as well.

1. Introduction

The following double inequality is known as Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the subsequent double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions the readers are refer to [[3]-[6],[8],[10]-[13],[17],[18],[20]-[22],[25],[28]] and the references mentioned in these papers. Also the following result is known in the literature as the Ostrowski inequality [16], which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

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Theorem 1.2. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{1.2}$$

The following inequality is well known in the literature as Simpson's inequality:

Theorem 1.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be four time differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b), that is $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}| < \infty$. Then, we have*

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5. \tag{1.3}$$

Inequality (1.3) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Ostrowski type inequalities see [[9],[16]]. For other recent results concerning Simpson type inequalities see [[15],[24]].

Let us recall the Gauss-Jacobi type quadrature formula, [26] as follows

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star|f|, \tag{1.4}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star|f|$. In [14], Liu obtained integral inequalities for P-function related to the left-hand side of (1.4), and in [23], Özdemir et al. also presented several integral inequalities concerning the left-hand side of (1.4) via some kinds of convexity.
Now, let us evoke some basic definitions as follows.

Definition 1.4. [1] A set $K \subseteq \mathbb{R}^n$ is said to be invex respecting the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Definition 1.5. [5] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0, 1].$$

Definition 1.6. [19] Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y, x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.5}$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Definition 1.7. [27] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, a function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K, if

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.6}$$

holds for all $x, y \in K$ and $t \in (0, 1)$.

Definition 1.8. [16] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if it is non-negative and $\forall\ x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.7}$$

Definition 1.9. [22] A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0,1)$, among $m \in (0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.8}$$

Definition 1.10. [6] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 1.11. In Definition 1.10, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

Definition 1.12. [24] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y, mx)\big) \le mh_1(t) f(x) + h_2(t) f(y) \tag{1.9}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

Definition 1.13. [17] Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$.
Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Motivated by the above literatures, the main objective of this article is to establish in Section 2 integral inequalities using two lemmas as auxiliary results for the left hand side of Gauss-Jacobi type quadrature formula and some new estimates on Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals associated with generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article. In Section 3, some applications to special means for different positive real numbers will be given.

2. Main results

The following definitions will be used in this section.

Definition 2.1. Let $\mathbf{m} : [0,1] \longrightarrow (0,1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as $\mathbf{m}$-invex with respect to the mapping $\Lambda : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi\Lambda(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\Lambda(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0,1]$ could reduce to $\Lambda(y, mx) = \eta(y, mx)$. For example when $\mathbf{m}(t) = m$ for all $t \in [0,1]$, then the $\mathbf{m}$-invex set degenerates an m-invex set on K.

We next introduce the notion of generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open $\mathbf{m}$-invex set with respect to the mapping $\Lambda : K \times K \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$, $\theta : I \longrightarrow K$ are continuous functions and $\mathbf{m} : [0,1] \longrightarrow (0,1]$. A mapping $f : K \longrightarrow (0, +\infty)$ is said to be generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex, if

$$f\big(\mathbf{m}(t)\theta(x) + \xi\Lambda(\theta(y), \mathbf{m}(t)\theta(x))\big) \leq \Big[\mathbf{m}(\xi)h_1(\xi)f^r(x) + h_2(\xi)f^r(y)\Big]^{\frac{1}{r}} \tag{2.1}$$

holds for all $x, y \in I$ and $t, \xi \in [0,1]$, where $r \neq 0$.

Remark 2.4. In Definition 2.3, if we choose $\mathbf{m} = m = r = 1$, these definition reduces to the definition considered by Noor in [20] and Preda et. al. in [7].

Remark 2.5. In Definition 2.3, if we choose $\mathbf{m} = m = r = 1$ and $\theta(x) = x$, then we get Definition 1.12.

Remark 2.6. For $r = 1$, let us discuss some special cases in Definition 2.3 as follows.

(I) If taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Breckner-preinvex mappings.
(II) If taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Godunova-Levin-Dragomir-preinvex mappings.
(III) If taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-$(\mathbf{m}, h)$-preinvex mappings.
(IV) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-$(\mathbf{m}, tgs)$-preinvex mappings.
(V) If taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-$\mathbf{m}$-MT-preinvex mappings.

We claim the following integral identity.

Lemma 2.7. *Let $\theta : I \longrightarrow K$ be a continuous function and $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. Assume that $f : K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + \Lambda(\theta(b), \boldsymbol{m}(t)\theta(a))] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\Lambda : K \times K \longrightarrow \mathbb{R}$ for $\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ and $\forall\, t \in [0,1]$. Then for any fixed $p, q > 0$, we have*

$$\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))} (x - \boldsymbol{m}(t)\theta(a))^p(\boldsymbol{m}(t)\theta(a) + \Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) - x)^q f(x)dx$$

$$= \Lambda^{p+q+1}(\theta(b), \boldsymbol{m}(t)\theta(a)) \int_0^1 \xi^p(1-\xi)^q f(\boldsymbol{m}(t)\theta(a) + \xi\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)))d\xi.$$

Proof. We observe that

$$\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}(x-\mathbf{m}(t)\theta(a))^p(\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))-x)^q f(x)dx$$

$$= \Lambda(\theta(b),\mathbf{m}(t)\theta(a))\int_0^1(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a))-\mathbf{m}(t)\theta(a))^p$$
$$\times(\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))-\mathbf{m}(t)\theta(a)-\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a)))^q$$
$$\times f(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a)))d\xi$$
$$= \Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\int_0^1\xi^p(1-\xi)^q f(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a)))d\xi.$$

This completes the proof of the lemma. □

Remark 2.8. In Lemma 2.7, if we choose $\mathbf{m}(t)\equiv 1$ for any $t\in[0,1]$, $\Lambda(\theta(b),\mathbf{m}(t)\theta(a)) = \theta(b)-\mathbf{m}(t)\theta(a)$ and $\theta(x)=x$ for all $x\in I$, then we get the left hand side of (1.4).

With the help of Lemma 2.7, we have the following results.

Theorem 2.9. *Let $k>1$ and $0<r\leq 1$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$, $\theta: I\longrightarrow K$ are continuous functions and $\boldsymbol{m}:[0,1]\longrightarrow(0,1]$. Assume that $f: K=[\boldsymbol{m}(t)\theta(a),\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))]\longrightarrow(0,+\infty)$ is a continuous mapping on K° with respect to $\Lambda: K\times K\longrightarrow\mathbb{R}$ for $\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))>0$ and $\forall\, t\in[0,1]$. If $f^{\frac{k}{k-1}}$ is generalized relative semi-$\boldsymbol{m}$-$(r;h_1,h_2)$-preinvex mapping on an open $\boldsymbol{m}$-invex set K, then for any fixed $p,q>0$, we have*

$$\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}(x-\boldsymbol{m}(t)\theta(a))^p(\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))-x)^q f(x)dx$$
$$\leq\Lambda^{p+q+1}(\theta(b),\boldsymbol{m}(t)\theta(a))\sqrt[k]{\beta(kp+1,kq+1)} \tag{2.2}$$
$$\times\Big[f^{\frac{rk}{k-1}}(a)I^r(h_1(\xi);\boldsymbol{m}(\xi),r)+f^{\frac{rk}{k-1}}(b)I^r(h_2(\xi);r)\Big]^{\frac{k-1}{rk}},$$

where

$$I(h_1(\xi);\boldsymbol{m}(\xi),r):=\int_0^1\boldsymbol{m}^{\frac{1}{r}}(\xi)h_1^{\frac{1}{r}}(\xi)d\xi,\quad I(h_2(\xi);r):=\int_0^1 h_2^{\frac{1}{r}}(\xi)d\xi.$$

Proof. Let $k>1$ and $0<r\leq 1$. Since $f^{\frac{k}{k-1}}$ is generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvex mapping on K, combining with Lemma 2.7, Hölder inequality, Minkowski inequality and properties of the modulus, we get

$$\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}(x-\mathbf{m}(t)\theta(a))^p(\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))-x)^q f(x)dx$$

$$\leq|\Lambda(\theta(b),\mathbf{m}(t)\theta(a))|^{p+q+1}\left[\int_0^1\xi^{kp}(1-\xi)^{kq}d\xi\right]^{\frac{1}{k}}$$
$$\times\left[\int_0^1\left|f(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a)))\right|^{\frac{k}{k-1}}d\xi\right]^{\frac{k-1}{k}}$$
$$\leq\Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\sqrt[k]{\beta(kp+1,kq+1)}$$

$$\times\left[\int_0^1 \left[\mathbf{m}(\xi)h_1(\xi)f^{\frac{rk}{k-1}}(a)+h_2(\xi)f^{\frac{rk}{k-1}}(b)\right]^{\frac{1}{r}}d\xi\right]^{\frac{k-1}{k}}$$
$$\leq \Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\sqrt[k]{\beta(kp+1,kq+1)}$$
$$\times\left\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{1}{r}}(\xi)f^{\frac{k}{k-1}}(a)d\xi\right)^r+\left(\int_0^1 h_2^{\frac{1}{r}}(\xi)f^{\frac{k}{k-1}}(b)d\xi\right)^r\right\}^{\frac{k-1}{rk}}$$
$$= \Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\sqrt[k]{\beta(kp+1,kq+1)}$$
$$\times\left[f^{\frac{rk}{k-1}}(a)I^r(h_1(\xi);\mathbf{m}(\xi),r)+f^{\frac{rk}{k-1}}(b)I^r(h_2(\xi);r)\right]^{\frac{k-1}{rk}}.$$

So, the proof of this theorem is completed. □

We point out some special cases of Theorem 2.9.

Corollary 2.10. *In Theorem 2.9 for $k=2$, we get*

$$\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}(x-\boldsymbol{m}(t)\theta(a))^p(\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),\boldsymbol{m}(t)\theta(a))\sqrt{\beta(2p+1,2q+1)} \tag{2.3}$$
$$\times\sqrt[2r]{f^{2r}(a)I^r(h_1(\xi);\boldsymbol{m}(\xi),r)+f^{2r}(b)I^r(h_2(\xi);r)}.$$

Corollary 2.11. *In Theorem 2.9 for $h_1(t)=h_2(t)=1$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\sqrt[k]{\beta(kp+1,kq+1)}\left[mf^{\frac{rk}{k-1}}(a)+f^{\frac{rk}{k-1}}(b)\right]^{\frac{k-1}{rk}}. \tag{2.4}$$

Corollary 2.12. *In Theorem 2.9 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\sqrt[k]{\beta(kp+1,kq+1)}I^{\frac{k-1}{k}}(h(\xi);r) \tag{2.5}$$
$$\times\left[mf^{\frac{rk}{k-1}}(a)+f^{\frac{rk}{k-1}}(b)\right]^{\frac{k-1}{rk}}.$$

Corollary 2.13. *In Corollary 2.12 for $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\sqrt[k]{\beta(kp+1,kq+1)}\left(\frac{r}{r+s}\right)^{\frac{k-1}{k}} \tag{2.6}$$
$$\times\left[mf^{\frac{rk}{k-1}}(a)+f^{\frac{rk}{k-1}}(b)\right]^{\frac{k-1}{rk}}.$$

Corollary 2.14. *In Corollary 2.12 for* $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *and* $0 < s < r$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))} (x - m\theta(a))^p (m\theta(a) + \Lambda(\theta(b), m\theta(a)) - x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b), m\theta(a)) \sqrt[k]{\beta(kp+1, kq+1)} \left(\frac{r}{r-s}\right)^{\frac{k-1}{k}} \tag{2.7}$$
$$\times \left[m f^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b) \right]^{\frac{k-1}{rk}}.$$

Corollary 2.15. *In Theorem 2.9 for* $h_1(t) = h_2(t) = t(1-t)$ *and* $\boldsymbol{m}(t) = m \in (0,1]$ *for all* $t \in [0,1]$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))} (x - m\theta(a))^p (m\theta(a) + \Lambda(\theta(b), m\theta(a)) - x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b), m\theta(a)) \sqrt[k]{\beta(kp+1, kq+1)} \beta^{\frac{k-1}{k}} \left(1 + \frac{1}{r}, 1 + \frac{1}{r}\right) \tag{2.8}$$
$$\times \left[m f^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b) \right]^{\frac{k-1}{rk}}.$$

Corollary 2.16. *In Corollary 2.12 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r \in \left(\frac{1}{2}, 1\right]$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))} (x - m\theta(a))^p (m\theta(a) + \Lambda(\theta(b), m\theta(a)) - x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b), m\theta(a)) \sqrt[k]{\beta(kp+1, kq+1)} \beta^{\frac{k-1}{k}} \left(1 - \frac{1}{2r}, 1 + \frac{1}{2r}\right) \tag{2.9}$$
$$\times \left[m f^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b) \right]^{\frac{k-1}{rk}}.$$

Theorem 2.17. *Let* $l \geq 1$ *and* $0 < r \leq 1$. *Suppose* $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\theta : I \longrightarrow K$ *are continuous functions and* $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. *Assume that* $f : K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + \Lambda(\theta(b), \boldsymbol{m}(t)\theta(a))] \longrightarrow (0,+\infty)$ *is a continuous mapping on* K° *with respect to* $\Lambda : K \times K \longrightarrow \mathbb{R}$ *for* $\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ *and* $\forall\, t \in [0,1]$. *If* f^l *is generalized relative semi-*$\boldsymbol{m}$*-*$(r; h_1, h_2)$*-preinvex mapping on an open* $\boldsymbol{m}$*-invex set* K, *then for any fixed* $p, q > 0$, *we have*

$$\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))} (x - \boldsymbol{m}(t)\theta(a))^p (\boldsymbol{m}(t)\theta(a) + \Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) - x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b), \boldsymbol{m}(t)\theta(a)) \beta^{\frac{l-1}{l}}(p+1, q+1) \tag{2.10}$$
$$\times \sqrt[rl]{f^{rl}(a) I^r(h_1(\xi); \boldsymbol{m}(\xi), r, p, q) + f^{rl}(b) I^r(h_2(\xi); r, p, q)},$$

where

$$I(h_1(\xi); \boldsymbol{m}(\xi), r, p, q) := \int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi) \xi^p (1-\xi)^q h_1^{\frac{1}{r}}(\xi) d\xi;$$

$$I(h_2(\xi);r,p,q) := \int_0^1 \xi^p(1-\xi)^q h_2^{\frac{1}{r}}(\xi)d\xi.$$

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvex mapping on K, combining with Lemma 2.7, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we get

$$\int_{\mathbf{m}(t)\theta(a)}^{\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))} (x-\mathbf{m}(t)\theta(a))^p(\mathbf{m}(t)\theta(a)+\Lambda(\theta(b),\mathbf{m}(t)\theta(a))-x)^q f(x)dx$$

$$= \Lambda(\theta(b),\mathbf{m}(t)\theta(a))^{p+q+1}\int_0^1 \Big[\xi^p(1-\xi)^q\Big]^{\frac{l-1}{l}}\Big[\xi^p(1-\xi)^q\Big]^{\frac{1}{l}}$$
$$\times f(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a)))d\xi$$
$$\leq |\Lambda(\theta(b),\mathbf{m}(t)\theta(a))|^{p+q+1}\left[\int_0^1 \xi^p(1-\xi)^q d\xi\right]^{\frac{l-1}{l}}$$
$$\times\left[\int_0^1 \xi^p(1-\xi)^q\Big|f(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(b),\mathbf{m}(t)\theta(a)))\Big|^l d\xi\right]^{\frac{1}{l}}$$
$$\leq \Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\left[\int_0^1 \xi^p(1-\xi)^q\Big[\mathbf{m}(\xi)h_1(\xi)f^{rl}(a)+h_2(\xi)f^{rl}(b)\Big]^{\frac{1}{r}}d\xi\right]^{\frac{1}{l}}$$
$$\leq \Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\left\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)\xi^p(1-\xi)^q h_1^{\frac{1}{r}}(\xi)f^l(a)d\xi\right)^r+\left(\int_0^1 \xi^p(1-\xi)^q h_2^{\frac{1}{r}}(\xi)f^l(b)d\xi\right)^r\right\}^{\frac{1}{rl}}$$
$$= \Lambda^{p+q+1}(\theta(b),\mathbf{m}(t)\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\sqrt[rl]{f^{rl}(a)I^r(h_1(\xi);\mathbf{m}(\xi),r,p,q)+f^{rl}(b)I^r(h_2(\xi);r,p,q)}.$$

So, the proof of this theorem is completed. □

Let us discuss some special cases of Theorem 2.17.

Corollary 2.18. *In Theorem 2.17 for $l=1$, we get*

$$\int_{\boldsymbol{m}(t)\theta(a)}^{\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))} (x-\boldsymbol{m}(t)\theta(a))^p(\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),\boldsymbol{m}(t)\theta(a)) \tag{2.11}$$
$$\times\sqrt[r]{f^r(a)I^r(h_1(\xi);\boldsymbol{m}(\xi),r,p,q)+f^r(b)I^r(h_2(\xi);r,p,q)}.$$

Corollary 2.19. *In Theorem 2.17 for $h_1(t)=h_2(t)=1$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))} (x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\beta(p+1,q+1)\sqrt[rl]{mf^{rl}(a)+f^{rl}(b)}. \tag{2.12}$$

Corollary 2.20. *In Theorem 2.17 for* $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ *and* $\boldsymbol{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.13}$$
$$\times\sqrt[rl]{mf^{rl}(a)I^r(h(\xi);r,q,p)+f^{rl}(b)I^r(h(\xi);r,p,q)}.$$

Corollary 2.21. *In Corollary 2.20 for* $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.14}$$
$$\times\sqrt[rl]{mf^{rl}(a)\beta^r\left(q+\frac{s}{r}+1,p+1\right)+f^{rl}(b)\beta^r\left(p+\frac{s}{r}+1,q+1\right)}.$$

Corollary 2.22. *In Corollary 2.20 for* $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ *and* $0<s\leq r$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.15}$$
$$\times\sqrt[rl]{mf^{rl}(a)\beta^r\left(q-\frac{s}{r}+1,p+1\right)+f^{rl}(b)\beta^r\left(p-\frac{s}{r}+1,q+1\right)}.$$

Corollary 2.23. *In Theorem 2.17 for* $h_1(t)=h_2(t)=t(1-t)$ *and* $\boldsymbol{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1)\sqrt[l]{\beta\left(p+\frac{1}{r}+1,q+\frac{1}{r}+1\right)} \tag{2.16}$$
$$\times\sqrt[rl]{mf^{rl}(a)+f^{rl}(b)}.$$

Corollary 2.24. *In Corollary 2.20 for* $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r\in\left[\frac{1}{2},1\right]$, *we get*

$$\int_{m\theta(a)}^{m\theta(a)+\Lambda(\theta(b),m\theta(a))}(x-m\theta(a))^p(m\theta(a)+\Lambda(\theta(b),m\theta(a))-x)^q f(x)dx$$
$$\leq \Lambda^{p+q+1}(\theta(b),m\theta(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.17}$$

$$\times \sqrt[rl]{mf^{rl}(a)\beta^r\left(q+\frac{1}{2r}+1, p-\frac{1}{2r}+1\right)+f^{rl}(b)\beta^r\left(p+\frac{1}{2r}+1, q-\frac{1}{2r}+1\right)}.$$

For establishing our second main results regarding generalizations of Hermite-Hadamard, Ostrowski and Simpson type inequalities associated with generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity via fractional integrals, we need the following lemma.

Lemma 2.25. *Let $\theta : I \longrightarrow K$ be a continuous function and $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. Suppose $K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + \Lambda(\theta(b), \boldsymbol{m}(t)\theta(a))] \subseteq \mathbb{R}$ be an open $\boldsymbol{m}$-invex subset with respect to $\Lambda : K \times K \longrightarrow \mathbb{R}$ and let $\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ for all $t \in [0,1]$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable mapping on K° and $f'' \in L_1(K)$. Then for any $\lambda \in [0,1]$ and $\alpha > 0$, the following identity holds:*

$$\frac{\lambda-1}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\Big\{\Lambda^{\alpha+1}(\theta(x),\boldsymbol{m}(t)\theta(a))f'(\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a)))$$
$$+\Lambda^{\alpha+1}(\theta(x),\boldsymbol{m}(t)\theta(b))f'(\boldsymbol{m}(t)\theta(b)+\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b)))\Big\}$$
$$+\frac{1+\alpha-\lambda}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\Big\{\Lambda^{\alpha}(\theta(x),\boldsymbol{m}(t)\theta(a))f(\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a)))$$
$$+\Lambda^{\alpha}(\theta(x),\boldsymbol{m}(t)\theta(b))f(\boldsymbol{m}(t)\theta(b)+\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b)))\Big\}$$
$$+\frac{\lambda}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}$$
$$\times\Big\{\Lambda^{\alpha}(\theta(x),\boldsymbol{m}(t)\theta(a))f(\boldsymbol{m}(t)\theta(a))+\Lambda^{\alpha}(\theta(x),\boldsymbol{m}(t)\theta(b))f(\boldsymbol{m}(t)\theta(b))\Big\}$$
$$-\frac{\Gamma(\alpha+2)}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}$$
$$\times\Big[J^{\alpha}_{(\boldsymbol{m}(t)\theta(a)+\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a)))^-}f(\boldsymbol{m}(t)\theta(a))+J^{\alpha}_{(\boldsymbol{m}(t)\theta(b)+\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b)))^-}f(\boldsymbol{m}(t)\theta(b))\Big]$$
$$=\frac{\Lambda^{\alpha+2}(\theta(x),\boldsymbol{m}(t)\theta(a))}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\int_0^1\xi(\lambda-\xi^\alpha)f''(\boldsymbol{m}(t)\theta(a)+\xi\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a)))d\xi$$
$$+\frac{\Lambda^{\alpha+2}(\theta(x),\boldsymbol{m}(t)\theta(b))}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\int_0^1\xi(\lambda-\xi^\alpha)f''(\boldsymbol{m}(t)\theta(b)+\xi\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b)))d\xi. \quad (2.18)$$

We denote

$$I_{f,\Lambda,\theta,\boldsymbol{m}}(x;\lambda,\alpha,a,b):=\frac{\Lambda^{\alpha+2}(\theta(x),\boldsymbol{m}(t)\theta(a))}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}$$
$$\times\int_0^1\xi(\lambda-\xi^\alpha)f''(\boldsymbol{m}(t)\theta(a)+\xi\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a)))d\xi$$
$$+\frac{\Lambda^{\alpha+2}(\theta(x),\boldsymbol{m}(t)\theta(b))}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\int_0^1\xi(\lambda-\xi^\alpha)f''(\boldsymbol{m}(t)\theta(b)+\xi\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b)))d\xi. \quad (2.19)$$

Proof. A simple proof of the equality (2.18) can be done by performing two integration by parts in the integrals above and changing the variables. The details are left to the interested reader. This completes the proof of the lemma. □

Using Lemma 2.25, we now state the following theorems for the corresponding version for power of second derivative.

Theorem 2.26. *Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\theta : I \longrightarrow K$ are continuous functions and $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. Let $K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a) + \Lambda(\theta(b), \boldsymbol{m}(t)\theta(a))] \subseteq \mathbb{R}$ be an open $\boldsymbol{m}$-invex subset with respect to $\Lambda : K \times K \longrightarrow \mathbb{R}$ and let $\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ for all $t \in [0,1]$. Assume that $f : K \longrightarrow (0,+\infty)$ be a twice differentiable mapping on K°. If $0 < r \leq 1$ and $(f''(x))^q$ is generalized relative semi-$\boldsymbol{m}$-$(r; h_1, h_2)$-preinvex mapping on K, $q > 1$, $p^{-1} + q^{-1} = 1$, then for any $\lambda \in [0,1]$ and $\alpha > 0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}\left|I_{f,\Lambda,\theta,\boldsymbol{m}}(x;\lambda,\alpha,a,b)\right| \leq \frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))} \\ \times \Bigg\{ |\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a))|^{\alpha+2} \sqrt[rq]{(f''(a))^{rq} I^r(h_1(\xi);\boldsymbol{m}(\xi),r) + (f''(x))^{rq} I^r(h_2(\xi);r)} \\ + |\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b))|^{\alpha+2} \sqrt[rq]{(f''(b))^{rq} I^r(h_1(\xi);\boldsymbol{m}(\xi),r) + (f''(x))^{rq} I^r(h_2(\xi);r)} \Bigg\},\end{aligned} \tag{2.20}$$

where

$$A(\alpha,\lambda,p) := \int_0^1 |\xi(\lambda - \xi^\alpha)|^p d\xi$$

and $I(h_1(\xi);\boldsymbol{m}(\xi),r)$, $I(h_2(\xi);r)$ are defined as in Theorem 2.9.

Proof. Suppose that $q > 1$ and $0 < r \leq 1$. Using relation (2.19), generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvexity of $(f''(x))^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}&\left|I_{f,\Lambda,\theta,\mathbf{m}}(x;\lambda,\alpha,a,b)\right| \\ \leq &\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{|\Lambda(\theta(b),\mathbf{m}(t)\theta(a))|} \int_0^1 |\xi(\lambda-\xi^\alpha)||f''(\mathbf{m}(t)\theta(a) + \xi\Lambda(\theta(x),\mathbf{m}(t)\theta(a)))|d\xi \\ &+ \frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{|\Lambda(\theta(b),\mathbf{m}(t)\theta(a))|} \int_0^1 |\xi(\lambda-\xi^\alpha)||f''(\mathbf{m}(t)\theta(b) + \xi\Lambda(\theta(x),\mathbf{m}(t)\theta(b)))|d\xi \\ &\leq \frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))} \left(\int_0^1 |\xi(\lambda-\xi^\alpha)|^p d\xi\right)^{\frac{1}{p}} \\ &\times \left(\int_0^1 (f''(\mathbf{m}(t)\theta(a) + \xi\Lambda(\theta(x),\mathbf{m}(t)\theta(a))))^q d\xi\right)^{\frac{1}{q}} \\ &+ \frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))} \left(\int_0^1 |\xi(\lambda-\xi^\alpha)|^p d\xi\right)^{\frac{1}{p}} \\ &\times \left(\int_0^1 (f''(\mathbf{m}(t)\theta(b) + \xi\Lambda(\theta(x),\mathbf{m}(t)\theta(b))))^q d\xi\right)^{\frac{1}{q}} \\ &\leq \frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))} \sqrt[p]{A(\alpha,\lambda,p)} \\ &\times \left(\int_0^1 \left[\mathbf{m}(\xi)h_1(\xi)(f''(a))^{rq} + h_2(\xi)(f''(x))^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}} \\ &+ \frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))} \sqrt[p]{A(\alpha,\lambda,p)}\end{aligned}$$

$$\times\left(\int_0^1\left[\mathbf{m}(\xi)h_1(\xi)(f''(b))^{rq}+h_2(\xi)(f''(x))^{rq}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$\leq\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}\sqrt[p]{A(\alpha,\lambda,p)}$$

$$\times\left\{\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{1}{r}}(\xi)(f''(a))^q d\xi\right)^r+\left(\int_0^1 h_2^{\frac{1}{r}}(\xi)(f''(x))^q d\xi\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}\sqrt[p]{A(\alpha,\lambda,p)}$$

$$\times\left\{\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{1}{r}}(\xi)(f''(b))^q d\xi\right)^r+\left(\int_0^1 h_2^{\frac{1}{r}}(\xi)(f''(x))^q d\xi\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}$$

$$\times\left\{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}\sqrt[rq]{(f''(a))^{rq}I^r(h_1(\xi);\mathbf{m}(\xi),r)+(f''(x))^{rq}I^r(h_2(\xi);r)}\right.$$

$$\left.+|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}\sqrt[rq]{(f''(b))^{rq}I^r(h_1(\xi);\mathbf{m}(\xi),r)+(f''(x))^{rq}I^r(h_2(\xi);r)}\right\}.$$

So, the proof of this theorem is completed. □

Let us discuss some special cases of Theorem 2.26.

Corollary 2.27. *In Theorem 2.26 for $p=q=2$, we get*

$$|I_{f,\Lambda,\theta,\boldsymbol{m}}(x;\lambda,\alpha,a,b)|\leq\frac{1}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\sqrt{\frac{\lambda^2}{3}+\frac{1}{2\alpha+3}-\frac{2\lambda}{\alpha+3}}$$

$$\times\left\{|\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a))|^{\alpha+2}\sqrt[2r]{(f''(a))^{2r}I^r(h_1(\xi);\boldsymbol{m}(\xi),r)+(f''(x))^{2r}I^r(h_2(\xi);r)}\right.$$

$$\left.+|\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b))|^{\alpha+2}\sqrt[2r]{(f''(b))^{2r}I^r(h_1(\xi);\boldsymbol{m}(\xi),r)+(f''(x))^{2r}I^r(h_2(\xi);r)}\right\}. \quad (2.21)$$

Corollary 2.28. *In Theorem 2.26, if we choose $\Lambda(\theta(y),\boldsymbol{m}(t)\theta(x))=\theta(y)-\boldsymbol{m}(t)\theta(x)$, $x=\frac{a+b}{2}$ and $\boldsymbol{m}(t)\equiv 1,\forall\, t\in[0,1]$, we get the following generalized Simpson type inequality for fractional integrals:*

$$\left|I_{f,\theta}\left(\frac{a+b}{2};\lambda,\alpha,a,b\right)\right|=\left|\frac{\lambda-1}{(\theta(b)-\theta(a))}\right.$$

$$\times\left\{\left(\left(\theta\left(\frac{a+b}{2}\right)-\theta(a)\right)^{\alpha+1}+\left(\theta(b)-\theta\left(\frac{a+b}{2}\right)\right)^{\alpha+1}\right)f'\left(\theta\left(\frac{a+b}{2}\right)\right)\right\}$$

$$+\frac{1+\alpha-\lambda}{(\theta(b)-\theta(a))}$$

$$\times\left\{\left(\left(\theta\left(\frac{a+b}{2}\right)-\theta(a)\right)^{\alpha}+\left(\theta(b)-\theta\left(\frac{a+b}{2}\right)\right)^{\alpha}\right)f\left(\theta\left(\frac{a+b}{2}\right)\right)\right\}$$

$$+\frac{\lambda}{(\theta(b)-\theta(a))}$$

$$\times\left\{\left(\theta\left(\frac{a+b}{2}\right)-\theta(a)\right)^{\alpha}f(\theta(a))+\left(\theta(b)-\theta\left(\frac{a+b}{2}\right)\right)^{\alpha}f(\theta(b))\right\}$$

$$-\frac{\Gamma(\alpha+2)}{(\theta(b)-\theta(a))}\times\left[J^{\alpha}_{(\theta(\frac{a+b}{2}))^{-}}f(\theta(a))+J^{\alpha}_{(\theta(\frac{a+b}{2}))^{-}}f(\theta(b))\right]\Bigg|$$

$$\le\frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\theta(b)-\theta(a)}\times\left\{\left(\theta\left(\frac{a+b}{2}\right)-\theta(a)\right)^{\alpha+2}\right.$$

$$\times\sqrt[rq]{(f''(a))^{rq}I^{r}(h_1(\xi);r)+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}I^{r}(h_2(\xi);r)}$$

$$+\left(\theta(b)-\theta\left(\frac{a+b}{2}\right)\right)^{\alpha+2} \tag{2.22}$$

$$\left.\times\sqrt[rq]{(f''(b))^{rq}I^{r}(h_1(\xi);r)+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}I^{r}(h_2(\xi);r)}\right\}.$$

Corollary 2.29. *In Theorem 2.26, if we choose* $\Lambda(\theta(y),\boldsymbol{m}(t)\theta(x))=\theta(y)-\boldsymbol{m}(t)\theta(x)$ *and* $\lambda=\boldsymbol{m}(t)\equiv 1,\forall\,t\in[0,1]$, *we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|I_{f,\theta}(x;1,\alpha,a,b)\right|=\Bigg|\frac{\alpha}{(\theta(b)-\theta(a))}\left\{\left((\theta(x)-\theta(a))^{\alpha}+(\theta(b)-\theta(x))^{\alpha}\right)f(\theta(x))\right\}$$

$$+\frac{1}{(\theta(b)-\theta(a))}\left\{(\theta(x)-\theta(a))^{\alpha}f(\theta(a))+(\theta(b)-\theta(x))^{\alpha}f(\theta(b))\right\}$$

$$-\frac{\Gamma(\alpha+2)}{(\theta(b)-\theta(a))}\times\left[J^{\alpha}_{(\theta(x))^{-}}f(\theta(a))+J^{\alpha}_{(\theta(x))^{-}}f(\theta(b))\right]\Bigg|$$

$$\le\frac{1}{(\theta(b)-\theta(a))}\sqrt[p]{\frac{\alpha}{2(\alpha+2)}}$$

$$\times\left\{(\theta(x)-\theta(a))^{\alpha+2}\sqrt[rq]{(f''(a))^{rq}I^{r}(h_1(\xi);r)+(f''(x))^{rq}I^{r}(h_2(\xi);r)}\right. \tag{2.23}$$

$$\left.+(\theta(b)-\theta(x))^{\alpha+2}\sqrt[rq]{(f''(b))^{rq}I^{r}(h_1(\xi);r)+(f''(x))^{rq}I^{r}(h_2(\xi);r)}\right\}.$$

Corollary 2.30. *In Theorem 2.26, if we choose* $\Lambda(\theta(y),\boldsymbol{m}(t)\theta(x))=\theta(y)-\boldsymbol{m}(t)\theta(x)$, $\lambda=0$ *and* $\boldsymbol{m}(t)\equiv 1,\forall\,t\in[0,1]$, *we get the following generalized Ostrowski type inequality for fractional integrals:*

$$\left|I_{f,\theta}(x;0,\alpha,a,b)\right|$$

$$=\Bigg|\frac{1}{(\theta(a)-\theta(b))}\left\{\left((\theta(x)-\theta(a))^{\alpha+1}+(\theta(b)-\theta(x))^{\alpha+1}\right)f'(\theta(x))\right\}$$

$$+\frac{1+\alpha}{(\theta(b)-\theta(a))}\Big\{\left((\theta(x)-\theta(a))^{\alpha}+(\theta(b)-\theta(x))^{\alpha}\right)f(\theta(x))\Big\}$$

$$-\frac{\Gamma(\alpha+2)}{(\theta(b)-\theta(a))}\times\left[J^{\alpha}_{(\theta(x))-}f(\theta(a))+J^{\alpha}_{(\theta(x))-}f(\theta(b))\right]\Bigg|$$

$$\leq\frac{1}{(\theta(b)-\theta(a))}\sqrt[p]{\frac{1}{p(\alpha+1)+1}}$$

$$\times\Bigg\{(\theta(x)-\theta(a))^{\alpha+2}\sqrt[rq]{(f''(a))^{rq}I^{r}(h_1(\xi);r)+(f''(x))^{rq}I^{r}(h_2(\xi);r)} \tag{2.24}$$

$$+(\theta(b)-\theta(x))^{\alpha+2}\sqrt[rq]{(f''(b))^{rq}I^{r}(h_1(\xi);r)+(f''(x))^{rq}I^{r}(h_2(\xi);r)}\Bigg\}.$$

Corollary 2.31. *In Theorem 2.26 for $h_1(t)=h_2(t)=1$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized relative semi-(m,P)-preinvex mappings:*

$$|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)|\leq\frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),m\theta(a))}$$

$$\times\Bigg\{|\Lambda(\theta(x),m\theta(a))|^{\alpha+2}\sqrt[rq]{m(f''(a))^{rq}+(f''(x))^{rq}} \tag{2.25}$$

$$+|\Lambda(\theta(x),m\theta(b))|^{\alpha+2}\sqrt[rq]{m(f''(b))^{rq}+(f''(x))^{rq}}\Bigg\}.$$

Corollary 2.32. *In Theorem 2.26 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized relative semi-(m,h)-preinvex mappings:*

$$|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)|\leq\frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),m\theta(a))}\sqrt[q]{I(h(\xi);r)}$$

$$\times\Bigg\{|\Lambda(\theta(x),m\theta(a))|^{\alpha+2}\sqrt[rq]{m(f''(a))^{rq}+(f''(x))^{rq}} \tag{2.26}$$

$$+|\Lambda(\theta(x),m\theta(b))|^{\alpha+2}\sqrt[rq]{m(f''(b))^{rq}+(f''(x))^{rq}}\Bigg\}.$$

Corollary 2.33. *In Corollary 2.32 for $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, we get the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)|\leq\frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),m\theta(a))}\sqrt[q]{\frac{r}{r+s}}$$

$$\times\Bigg\{|\Lambda(\theta(x),m\theta(a))|^{\alpha+2}\sqrt[rq]{m(f''(a))^{rq}+(f''(x))^{rq}} \tag{2.27}$$

$$+|\Lambda(\theta(x),m\theta(b))|^{\alpha+2}\sqrt[rq]{m(f''(b))^{rq}+(f''(x))^{rq}}\Bigg\}.$$

Corollary 2.34. *In Corollary 2.32 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $0 < s < r$, we get the following inequality for generalized relative semi-(m, s)-Godunova-Levin-Dragomir-preinvex mappings:*

$$
\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),m\theta(a))}\sqrt[q]{\frac{r}{r-s}} \\
&\times \Bigg\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2}\sqrt[rq]{m(f''(a))^{rq}+(f''(x))^{rq}} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2}\sqrt[rq]{m(f''(b))^{rq}+(f''(x))^{rq}} \Bigg\}.
\end{aligned}
\tag{2.28}
$$

Corollary 2.35. *In Theorem 2.26 for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for generalized relative semi-(m, tgs)-preinvex mappings:*

$$
\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),m\theta(a))}\sqrt[q]{\beta\left(1+\frac{1}{r},1+\frac{1}{r}\right)} \\
&\times \Bigg\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2}\sqrt[rq]{m(f''(a))^{rq}+(f''(x))^{rq}} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2}\sqrt[rq]{m(f''(b))^{rq}+(f''(x))^{rq}} \Bigg\}.
\end{aligned}
\tag{2.29}
$$

Corollary 2.36. *In Corollary 2.32 for $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $r \in \left(\frac{1}{2}, 1\right]$, we get the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$
\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\Lambda(\theta(b),m\theta(a))}\sqrt[q]{\beta\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)} \\
&\times \Bigg\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2}\sqrt[rq]{m(f''(a))^{rq}+(f''(x))^{rq}} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2}\sqrt[rq]{m(f''(b))^{rq}+(f''(x))^{rq}} \Bigg\}.
\end{aligned}
\tag{2.30}
$$

Theorem 2.37. *Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, $\theta : I \longrightarrow K$ are continuous functions and $\boldsymbol{m} : [0, 1] \longrightarrow (0, 1]$. Let $K = [\boldsymbol{m}(t)\theta(a), \boldsymbol{m}(t)\theta(a)+\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a))] \subseteq \mathbb{R}$ be an open $\boldsymbol{m}$-invex subset with respect to $\Lambda : K\times K \longrightarrow \mathbb{R}$ and let $\Lambda(\theta(b), \boldsymbol{m}(t)\theta(a)) > 0$ for all $t \in [0, 1]$. Assume that $f : K \longrightarrow (0, +\infty)$ be a twice differentiable mapping on K°. If $0 < r \leq 1$ and $(f''(x))^q$ is generalized relative semi-$\boldsymbol{m}$-$(r; h_1, h_2)$-preinvex mapping on K, $q \geq 1$, then for any $\lambda \in [0, 1]$ and $\alpha > 0$, the following* *inequality for fractional integrals holds:*

$$\begin{aligned}\left|I_{f,\Lambda,\theta,\boldsymbol{m}}(x;\lambda,\alpha,a,b)\right| &\leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}\\ &\times\Bigg\{|\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a))|^{\alpha+2}\\ &\times\sqrt[rq]{(f''(a))^{rq}F^r(h_1(\xi);\boldsymbol{m}(\xi),\lambda,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);\lambda,\alpha,r)}\\ &+|\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b))|^{\alpha+2}\\ &\times\sqrt[rq]{(f''(b))^{rq}F^r(h_1(\xi);\boldsymbol{m}(\xi),\lambda,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);\lambda,\alpha,r)}\Bigg\},\end{aligned} \tag{2.31}$$

where

$$C(\alpha,\lambda):=\frac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\frac{\lambda}{2};$$

$$F(h_1(\xi);\boldsymbol{m}(\xi),\lambda,\alpha,r):=\int_0^1\boldsymbol{m}^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^\alpha)|h_1^{\frac{1}{r}}(\xi)d\xi;$$

$$G(h_2(\xi);\lambda,\alpha,r):=\int_0^1|\xi(\lambda-\xi^\alpha)|h_2^{\frac{1}{r}}(\xi)d\xi.$$

Proof. Suppose that $q\geq 1$ and $0<r\leq 1$. Using relation (2.19), generalized relative semi-**m**-$(r;h_1,h_2)$-preinvexity of $(f''(x))^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}&\left|I_{f,\Lambda,\theta,\mathbf{m}}(x;\lambda,\alpha,a,b)\right|\\ &\leq\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{|\Lambda(\theta(b),\mathbf{m}(t)\theta(a))|}\int_0^1|\xi(\lambda-\xi^\alpha)||f''(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(x),\mathbf{m}(t)\theta(a)))|d\xi\\ &+\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{|\Lambda(\theta(b),\mathbf{m}(t)\theta(a))|}\int_0^1|\xi(\lambda-\xi^\alpha)||f''(\mathbf{m}(t)\theta(b)+\xi\Lambda(\theta(x),\mathbf{m}(t)\theta(b)))|d\xi\\ &\leq\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}\left(\int_0^1|\xi(\lambda-\xi^\alpha)|d\xi\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1|\xi(\lambda-\xi^\alpha)|\left(f''(\mathbf{m}(t)\theta(a)+\xi\Lambda(\theta(x),\mathbf{m}(t)\theta(a)))\right)^q d\xi\right)^{\frac{1}{q}}\\ &+\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}\left(\int_0^1|\xi(\lambda-\xi^\alpha)|d\xi\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1|\xi(\lambda-\xi^\alpha)|\left(f''(\mathbf{m}(t)\theta(b)+\xi\Lambda(\theta(x),\mathbf{m}(t)\theta(b)))\right)^q d\xi\right)^{\frac{1}{q}}\\ &\leq\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)\\ &\times\left(\int_0^1|\xi(\lambda-\xi^\alpha)|\Big[\mathbf{m}(\xi)h_1(\xi)(f''(a))^{rq}+h_2(\xi)(f''(x))^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\\ &+\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)\end{aligned}$$

$$\times\left(\int_0^1 |\xi(\lambda-\xi^\alpha)|\left[\mathbf{m}(\xi)h_1(\xi)(f''(b))^{rq}+h_2(\xi)(f''(x))^{rq}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$\leq \frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)$$

$$\times\Bigg\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^\alpha)|(f''(a))^q d\xi\right)^r$$

$$+\left(\int_0^1 h_2^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^\alpha)|(f''(x))^q d\xi\right)^r\Bigg\}^{\frac{1}{rq}}$$

$$+\frac{|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)$$

$$\times\Bigg\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^\alpha)|(f''(b))^q d\xi\right)^r$$

$$+\left(\int_0^1 h_2^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^\alpha)|(f''(x))^q d\xi\right)^r\Bigg\}^{\frac{1}{rq}}$$

$$= \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),\mathbf{m}(t)\theta(a))}$$
$$\times\Bigg\{|\Lambda(\theta(x),\mathbf{m}(t)\theta(a))|^{\alpha+2}$$
$$\times\sqrt[rq]{(f''(a))^{rq}F^r(h_1(\xi);\mathbf{m}(\xi),\lambda,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);\lambda,\alpha,r)}$$
$$+|\Lambda(\theta(x),\mathbf{m}(t)\theta(b))|^{\alpha+2}$$
$$\times\sqrt[rq]{(f''(b))^{rq}F^r(h_1(\xi);\mathbf{m}(\xi),\lambda,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);\lambda,\alpha,r)}\Bigg\}.$$

So, the proof of this theorem is completed. □

Let us discuss some special cases of Theorem 2.37.

Corollary 2.38. *In Theorem 2.37 for $q=1$, we get*

$$|I_{f,\Lambda,\theta,\boldsymbol{m}}(x;\lambda,\alpha,a,b)| \leq \frac{1}{\Lambda(\theta(b),\boldsymbol{m}(t)\theta(a))}$$
$$\times\Bigg\{|\Lambda(\theta(x),\boldsymbol{m}(t)\theta(a))|^{\alpha+2}$$
$$\times\sqrt[r]{(f''(a))^{r}F^r(h_1(\xi);\boldsymbol{m}(\xi),\lambda,\alpha,r)+(f''(x))^{r}G^r(h_2(\xi);\lambda,\alpha,r)} \tag{2.32}$$
$$+|\Lambda(\theta(x),\boldsymbol{m}(t)\theta(b))|^{\alpha+2}$$
$$\times\sqrt[r]{(f''(b))^{r}F^r(h_1(\xi);\boldsymbol{m}(\xi),\lambda,\alpha,r)+(f''(x))^{r}G^r(h_2(\xi);\lambda,\alpha,r)}\Bigg\}.$$

Corollary 2.39. *In Theorem 2.37, if we choose* $\Lambda(\theta(y),\boldsymbol{m}(t)\theta(x))=\theta(y)-\boldsymbol{m}(t)\theta(x)$, $x=\frac{a+b}{2}$ *and* $\boldsymbol{m}(t)\equiv 1,\forall\, t\in[0,1]$, *we get the following generalized Simpson type inequality for fractional integrals:*

$$\begin{aligned}\left|I_{f,\theta}\left(\frac{a+b}{2};\lambda,\alpha,a,b\right)\right| &\le \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\theta(b)-\theta(a)}\\ &\times\Bigg\{\left(\theta\left(\frac{a+b}{2}\right)-\theta(a)\right)^{\alpha+2}\\ &\times\sqrt[rq]{(f''(a))^{rq}G^r(h_1(\xi);\lambda,\alpha,r)+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}G^r(h_2(\xi);\lambda,\alpha,r)}\\ &+\left(\theta(b)-\theta\left(\frac{a+b}{2}\right)\right)^{\alpha+2}\\ &\times\sqrt[rq]{(f''(b))^{rq}G^r(h_1(\xi);\lambda,\alpha,r)+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}G^r(h_2(\xi);\lambda,\alpha,r)}\Bigg\}.\end{aligned}\tag{2.33}$$

Corollary 2.40. *In Theorem 2.37, if we choose* $\Lambda(\theta(y),\boldsymbol{m}(t)\theta(x))=\theta(y)-\boldsymbol{m}(t)\theta(x)$ *and* $\lambda=\boldsymbol{m}(t)\equiv 1,\forall\, t\in[0,1]$, *we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\begin{aligned}\left|I_{f,\theta}(x;1,\alpha,a,b)\right| &\le \left(\frac{\alpha}{2(\alpha+2)}\right)^{1-\frac{1}{q}}\frac{1}{(\theta(b)-\theta(a))}\\ &\times\Bigg\{(\theta(x)-\theta(a))^{\alpha+2}\sqrt[rq]{(f''(a))^{rq}G^r(h_1(\xi);1,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);1,\alpha,r)}\\ &+(\theta(b)-\theta(x))^{\alpha+2}\sqrt[rq]{(f''(b))^{rq}G^r(h_1(\xi);1,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);1,\alpha,r)}\Bigg\}.\end{aligned}\tag{2.34}$$

Corollary 2.41. *In Theorem 2.37, if we choose* $\Lambda(\theta(y),\boldsymbol{m}(t)\theta(x))=\theta(y)-\boldsymbol{m}(t)\theta(x)$, $\lambda=0$ *and* $\boldsymbol{m}(t)\equiv 1,\forall\, t\in[0,1]$, *we get the following generalized Ostrowski type inequality for fractional integrals:*

$$\begin{aligned}\left|I_{f,\theta}(x;0,\alpha,a,b)\right| &\le \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\frac{1}{(\theta(b)-\theta(a))}\\ &\times\Bigg\{(\theta(x)-\theta(a))^{\alpha+2}\sqrt[rq]{(f''(a))^{rq}G^r(h_1(\xi);0,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);0,\alpha,r)}\\ &+(\theta(b)-\theta(x))^{\alpha+2}\sqrt[rq]{(f''(b))^{rq}G^r(h_1(\xi);0,\alpha,r)+(f''(x))^{rq}G^r(h_2(\xi);0,\alpha,r)}\Bigg\}.\end{aligned}\tag{2.35}$$

Corollary 2.42. *In Theorem 2.37 for $h_1(t) = h_2(t) = 1$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for generalized relative semi-(m, P)-preinvex mappings:*

$$
\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{C(\alpha,\lambda)}{\Lambda(\theta(b),m\theta(a))} \\
&\times \left\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2} \sqrt[rq]{m(f''(a))^{rq} + (f''(x))^{rq}} \right. \\
&\left. + |\Lambda(\theta(x),m\theta(b))|^{\alpha+2} \sqrt[rq]{m(f''(b))^{rq} + (f''(x))^{rq}} \right\}.
\end{aligned} \tag{2.36}
$$

Corollary 2.43. *In Theorem 2.37 for $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for generalized relative semi-(m, h)-preinvex mappings:*

$$
\begin{aligned}
&|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| \leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),m\theta(a))} \\
&\times \left\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2} \right. \\
&\times \sqrt[rq]{m(f''(a))^{rq} G^r(h(1-\xi);\lambda,\alpha,r) + (f''(x))^{rq} G^r(h(\xi);\lambda,\alpha,r)} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2} \\
&\left. \times \sqrt[rq]{m(f''(b))^{rq} G^r(h(1-\xi);\lambda,\alpha,r) + (f''(x))^{rq} G^r(h(\xi);\lambda,\alpha,r)} \right\}.
\end{aligned} \tag{2.37}
$$

Corollary 2.44. *In Corollary 2.43 for $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we get the following inequality for generalized relative semi-(m, s)-Breckner-preinvex mappings:*

$$
\begin{aligned}
&|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| \leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),m\theta(a))} \\
&\times \left\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2} \right. \\
&\times \sqrt[rq]{m(f''(a))^{rq} G^r((1-\xi)^s;\lambda,\alpha,r) + (f''(x))^{rq} D^r(\lambda,\alpha,s,r)} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2} \\
&\left. \times \sqrt[rq]{m(f''(b))^{rq} G^r((1-\xi)^s;\lambda,\alpha,r) + (f''(x))^{rq} D^r(\lambda,\alpha,s,r)} \right\},
\end{aligned} \tag{2.38}
$$

where

$$
\begin{aligned}
D(\lambda,\alpha,s,r) &:= \frac{r\lambda^{\frac{s+r(\alpha+2)}{r\alpha}}}{s+2r} - \frac{r\lambda^{\frac{s+r(\alpha+2)}{r\alpha}}}{s+r(\alpha+2)} \\
&+ \frac{r}{s+2r}\left(1-\lambda^{\frac{s+r(\alpha+2)}{r\alpha}}\right) - \frac{r\lambda}{s+2r}\left(1-\lambda^{\frac{s+2r}{r\alpha}}\right).
\end{aligned}
$$

Corollary 2.45. *In Corollary 2.43 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, $r \neq \frac{s}{2}$ and $r \neq \frac{s}{\alpha+2}$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings:*

$$
\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),m\theta(a))} \\
&\times \Bigg\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2} \\
&\times \sqrt[rq]{m(f''(a))^{rq}G^r((1-\xi)^s;\lambda,\alpha,r) + (f''(x))^{rq}E^r(\lambda,\alpha,s,r)} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2} \\
&\times \sqrt[rq]{m(f''(b))^{rq}G^r((1-\xi)^s;\lambda,\alpha,r) + (f''(x))^{rq}E^r(\lambda,\alpha,s,r)} \Bigg\},
\end{aligned}
\tag{2.39}
$$

where

$$
\begin{aligned}
E(\lambda,\alpha,s,r) &:= \frac{r\lambda^{\frac{r(\alpha+2)-s}{r\alpha}}}{2r-s} - \frac{r\lambda^{\frac{r(\alpha+2)-s}{r\alpha}}}{r(\alpha+2)-s} \\
&+ \frac{r}{2r-s}\left(1-\lambda^{\frac{r(\alpha+2)-s}{r\alpha}}\right) - \frac{r\lambda}{2r-s}\left(1-\lambda^{\frac{2r-s}{r\alpha}}\right).
\end{aligned}
$$

Corollary 2.46. *In Theorem 2.37 for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0,1]$ for all $t \in [0,1]$, we get the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$
\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),m\theta(a))} \sqrt[q]{G(\xi(1-\xi);\lambda,\alpha,r)} \\
&\times \Bigg\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2} \sqrt[rq]{m(f''(a))^{rq} + (f''(x))^{rq}} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2} \sqrt[rq]{m(f''(b))^{rq} + (f''(x))^{rq}} \Bigg\}.
\end{aligned}
\tag{2.40}
$$

Corollary 2.47. *In Corollary 2.43 for* $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, *we get the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\begin{aligned}
|I_{f,\Lambda,\theta,m}(x;\lambda,\alpha,a,b)| &\leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\Lambda(\theta(b),m\theta(a))} \\
&\times \Bigg\{ |\Lambda(\theta(x),m\theta(a))|^{\alpha+2} \\
&\times \sqrt[rq]{m(f''(a))^{rq}G^r\left(\frac{\sqrt{1-\xi}}{2\sqrt{\xi}};\lambda,\alpha,r\right) + (f''(x))^{rq}G^r\left(\frac{\sqrt{\xi}}{2\sqrt{1-\xi}};\lambda,\alpha,r\right)} \\
&+ |\Lambda(\theta(x),m\theta(b))|^{\alpha+2} \\
&\times \sqrt[rq]{m(f''(b))^{rq}G^r\left(\frac{\sqrt{1-\xi}}{2\sqrt{\xi}};\lambda,\alpha,r\right) + (f''(x))^{rq}G^r\left(\frac{\sqrt{\xi}}{2\sqrt{1-\xi}};\lambda,\alpha,r\right)} \Bigg\}.
\end{aligned} \tag{2.41}$$

Remark 2.48. For $\alpha = 1$, by our Theorems 2.26 and 2.37, we can get some new special Hermite-Hadamard, Ostrowski and Simpson type inequalities for classical integrals associated with generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings.

Remark 2.49. Also, applying our Theorems 2.26 and 2.37, for different values of $\lambda \in (0,1)$ and if $f''(x) \leq K$, for all $x \in I$, we can get some new special Hermite-Hadamard, Ostrowski and Simpson type inequalities for fractional integrals associated with generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings.

3. Applications to special means

Definition 3.1. [2] A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

Let consider some special means for arbitrary positive real numbers $\alpha \neq \beta$ as follows: The arithmetic mean $A := A(\alpha, \beta)$; The geometric mean $G := G(\alpha, \beta)$; The harmonic mean $H := H(\alpha, \beta)$; The power mean $P_r := P_r(\alpha, \beta)$; The identric mean $I := I(\alpha, \beta)$; The logarithmic mean $L := L(\alpha, \beta)$; The generalized log-mean $L_p := L_p(\alpha, \beta)$; The weighted p-power mean $M = M_p$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $\overline{M} := M(\theta(a), \theta(b)) : [\theta(a), \theta(a) + \Lambda(\theta(b), \theta(a))] \times [\theta(a), \theta(a) + \Lambda(\theta(b), \theta(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, therefore one can obtain various inequalities using the results of Section 2 for these means as follows: Replace $\Lambda(\theta(y), \mathbf{m}(t)\theta(x))$ with $\Lambda(\theta(y), \theta(x))$ where $\mathbf{m}(t) \equiv 1$, for all $t \in [0,1]$ and setting $\Lambda(\theta(y), \theta(x)) = M(\theta(x), \theta(y))$ for all $x, y \in I$, in (2.20) and (2.31), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} \left|I_{f,\Lambda,\theta}(x;\lambda,\alpha,a,b)\right| &\leq \frac{\sqrt[p]{A(\alpha,\lambda,p)}}{\overline{M}} \\ &\times \left\{ M^{\alpha+2}(\theta(a),\theta(x)) \sqrt[rq]{(f''(a))^{rq} I^r(h_1(\xi);r) + (f''(x))^{rq} I^r(h_2(\xi);r)} \right. \\ &\left. + M^{\alpha+2}(\theta(b),\theta(x)) \sqrt[rq]{(f''(b))^{rq} I^r(h_1(\xi);r) + (f''(x))^{rq} I^r(h_2(\xi);r)} \right\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \left|I_{f,\Lambda,\theta}(x;\lambda,\alpha,a,b)\right| &\leq \frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\overline{M}} \\ &\times \left\{ M^{\alpha+2}(\theta(a),\theta(x)) \right. \\ &\times \sqrt[rq]{(f''(a))^{rq} G^r(h_1(\xi);\lambda,\alpha,r) + (f''(x))^{rq} G^r(h_2(\xi);\lambda,\alpha,r)} \\ &+ M^{\alpha+2}(\theta(b),\theta(x)) \\ &\left. \times \sqrt[rq]{(f''(b))^{rq} G^r(h_1(\xi);\lambda,\alpha,r) + (f''(x))^{rq} G^r(h_2(\xi);\lambda,\alpha,r)} \right\}. \end{aligned} \tag{3.2}$$

Letting $M(\theta(x),\theta(y)) := A, G, H, P_r, I, L, L_p, M_p, \forall\, x, y \in I$ in (3.1) and (3.2), we get the inequalities involving means for a particular choices of $(f''(x))^q$ that are generalized relative semi-1-$(r;h_1,h_2)$-preinvex mappings.

Remark 3.2. Also, applying our Theorems 2.26 and 2.37 for appropriate choices of functions h_1 and h_2 (see Remark 2.6) such that $(f''(x))^q$ to be generalized relative semi-1-$(r;h_1,h_2)$-preinvex mappings (for examples $f(x) = x^\alpha$, where $\alpha > 1, \forall\, x > 0$; $f(x) = \frac{1}{x}$, $\forall\, x > 0$ etc.), we can deduce some new inequalities using above special means. The details are left to the interested reader.

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