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ON A COUPLED SYSTEM OF VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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Abstract. Volterra-Stieltjes integral equations have been studied in the space of continuous functions in many papers for example, (see [2]-[8]). Our aim here is to study the existence of at least one solution for a coupled system of nonlinear integral equations of Volterra-Stieltejs type in the space of continuous functions defined on a closed bounded interval. The main tool utilized in our considerations is the technique associated with certain Schauder fixed point theorem.

1. Introduction and Preliminaries

Let $I = [0, T]$ be a fixed interval. Denote by $C(I) = C[0, T]$ the class of all continuous functions defined on I with the standard norm

$$\| x \| = \sup_{t \in I} | x(t) | .$$

Consider the nonlinear Riemann-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s)) \ d_s g(t, s), \ \ t \in I \tag{1}$$

where $g : I \times I \to R$ and the symbol d_s indicates the integration with respect to s. Equations of type (1) and some of their generalizations were considered in several papers by J. Banaś (see [4]). The properties of the Volterra-Stieljes integral operator were studies also by J. Banaś in [2]-[6]
Further facts concerning Stieltjes integrals and their properties (see Banaś [1]). The solvability of the coupled systems of integral equations in $C[0, T]$ was proved (see [12]-[14]).

In this paper, we generalize this result for the coupled system of Volterra-Stieltjes

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integral equations

$$x(t) = p_1(t) + \lambda_1 \int_0^t f_1(s, x(s), y(s)) \, d_s g_1(t, s), \ t \in I$$
$$y(t) = p_2(t) + \lambda_2 \int_0^t f_2(s, x(s), y(s)) \, d_s g_2(t, s), \ t \in I \tag{2}$$

in the Banach space $C(I)$, we study the existence of at least one solution for the coupled system (2).

2. Existence of solutions

In this section we study the existence of continuous solutions $x, \ y \in C(I)$ for the coupled system of nonlinear integral equations of Volterra-Stieltjes type (2). Now we formulate assumptions under which coupled system (2) will be considered. Namely, we shall assume that:

(i) $p_i \in C(I), \ \lambda_i \in R, \ i = 1, 2.$

(ii) $f_i : I \times R^2 \to R, \ (i = 1, 2)$ is continuous on $I, \ \forall x, y \in R^2, \ t \in I$ such that there exist continuous functions $k_i : I \to I$ and two positive constants b_i such that:
$$| f_i(t, x, y) | \leq k_i(t) + b_i(\max\{| x |, | y |\})$$
for $t \in I$ and $x, y \in R$.

(iii) $g_i : I \times I \to R, \ i = 1, 2$ and for all $t_1, \ t_2 \in I$ with $t_1 < t_2$, the functions $s \to g_i(t_2, s) - g_i(t_1, s)$ is nondecreasing on I.

(iv) $g_i(0, s) = 0$ for any $s \in I, \ i = 1, 2.$

(v) The functions $t \to g_i(t, t)$ and $t \to g_i(t, 0)$ are continuous on $I, \ i = 1, 2.$ Put
$\mu = \sup | g_i(t, t) | + \ \sup | g_i(t, 0) |$ on I.

Now, let X be the Banach space of all ordered pairs $(x, y), \ x, y \in C(I)$ with the norm
$$\|(x, y)\|_X \ = \max\{\|x\|_{C(I)}, \|y\|_{C(I)}\}$$
where
$$\|x\| = \sup_{t \in I} | x(t) |, \ \| y \| = \sup_{t \in I} | y(t) | .$$
It is clear that $(X, \|(x, y)\|_X)$ is a Banach space.

Theorem 1. Let the assumptions (i)-(v) be satisfied, then the coupled system (2) has at least one solution in X.

Proof: Define the operator T by putting
$$T(x, y)(t) = (T_1 x(t), T_2 y(t))$$
where
$$T_1 x(t) = p_1(t) + \lambda_1 \int_0^t f_1(s, u(s)) \, d_s g_1(t, s)$$
$$T_2 y(t) = p_2(t) + \lambda_2 \int_0^t f_2(s, u(s)) \, d_s g_2(t, s)$$

$$u = (x, y).$$

For every $u \in X,\ t \in I,\ f_i(., u(.))\ (i = 1, 2)$ is continuous on I. Observe that Assumptions (iii) and (iv) imply that the function $s \to g(t, s)$ is nondecreasing on the interval I, for any fixed $t \in I$. Indeed, putting $t_2 = t,\ t_1 = 0$ in (iii) and keeping in mind (iv), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t, s)$ is of bounded variation on I. Hence it follows that, $f_i(t, x(t), y(t))$ are Riemann-Stieltjes integrable on I with respect to $s \to g_i(t, s)$. Thus T_i make sense.

We will prove a few results concerning the continuity and compactness of these operators in the space of continuous functions.
We denoted $K := \max\{k_i(t) : t \in I,\ i = 1, 2\}$, and we define the set U by

$$U := \{u = (x, y) \mid (x, y) \in R^2 : \|(x, y)\|_X \leq r,\ r = \frac{\|p_i\| + \lambda K\mu}{1 - \lambda b_i\mu}\}$$

The remainder of the proof will be given in four steps.

Step 1: The operator T transforms X into X.
For $u = (x, y) \in U$, for all $\epsilon > 0,\ \delta > 0$ and for each $t_1, t_2 \in I,\ t_1 < t_2$ such that $\mid t_2 - t_1 \mid < \delta$, we have

$$\begin{aligned}
\mid T_1x(t_2) &- T_1x(t_1) \mid \leq \mid p_1(t_2) - p_1(t_1) \mid \\
&+ \mid \lambda_1 \int_0^{t_2} f_1(s, x(s), y(s))\ d_sg_1(t_2, s) - \lambda_1 \int_0^{t_1} f_1(s, x(s), y(s))\ d_sg_1(t_1, s) \mid \\
&\leq \mid p_1(t_2) - p_1(t_1) \mid \\
&+ \mid \lambda_1 \int_0^{t_2} f_1(s, x(s), y(s))\ d_sg_1(t_2, s) - \lambda_1 \int_0^{t_1} f_1(s, x(s), y(s))\ d_sg_1(t_2, s) \mid \\
&+ \mid \lambda_1 \int_0^{t_1} f_1(s, x(s), y(s))\ d_sg_1(t_2, s) - \lambda_1 \int_0^{t_1} f_1(s, x(s), y(s))\ d_sg_1(t_1, s) \mid \\
&\leq \mid p_1(t_2) - p_1(t_1) \mid + \mid \lambda_1 \int_{t_1}^{t_2} f_1(s, x(s), y(s))\ d_sg_1(t_2, s) \mid \\
&+ \mid \lambda_1 \int_0^{t_1} f_1(s, x(s), y(s))\ d_s(g_1(t_2, s) - g_1(t_1, s)) \mid \\
&\leq \mid p_1(t_2) - p_1(t_1) \mid + \mid \lambda_1 \mid \int_{t_1}^{t_2} \mid f_1(s, x(s), y(s)) \mid\ d_s(\bigvee_{z=0}^{s} g_1(t_2, z)) \\
&+ |\lambda_1| \int_0^{t_1} \mid f_1(s, x(s), y(s)) \mid\ d_s(\bigvee_{z=0}^{s} [g_1(t_2, z) - g_1(t_1, z)]) \\
&\leq \mid p_1(t_2) - p_1(t_1) \mid + \lambda \int_{t_1}^{t_2} (k_1(s) + b_1(\max\{\mid x(s) \mid, \mid y(s) \mid\}))\ d_s(\bigvee_{z=0}^{s} g_1(t_2, z)) \\
&+ \lambda \int_0^{t_1} (k_1(s) + b_1(\max\{\mid x(s) \mid, \mid y(s) \mid\}))\ d_s(\bigvee_{z=0}^{s} [g_1(t_2, z) - g_1(t_1, z)])
\end{aligned}$$

$$
\begin{aligned}
&\leq \mid p_1(t_2)-p_1(t_1)\mid +\lambda(K+rb_1)\int_{t_1}^{t_2} d_s(g_1(t_2,s)) \\
&+ \lambda(K+rb_1)\int_0^{t_1} d_s(g_1(t_2,s)-g_1(t_1,s)) \\
&\leq \mid p_1(t_2)-p_1(t_1)\mid +\lambda(K+rb_1)[g_1(t_2,t_2)-g_1(t_2,t_1)] \\
&+ \lambda(K+rb_1)\{[g_1(t_2,t_1)-g_1(t_1,t_1)]-[g_1(t_2,0)-g_1(t_1,0)]\} \\
&\leq \mid p_1(t_2)-p_1(t_1)\mid +\lambda(K+rb_1)\{[g_1(t_2,t_2)-g_1(t_1,t_1)] \\
&- [g_1(t_2,0)-g_1(t_1,0)]\} \\
&\leq \mid p_1(t_2)-p_1(t_1)\mid +\lambda(K+rb_1)[\mid g_1(t_2,t_2)-g_1(t_1,t_1)\mid \\
&+ \mid g_1(t_2,0)-g_1(t_1,0)\mid].
\end{aligned}
$$

where $\lambda := \max\{|\lambda_1|,|\lambda_2|\}$.
Hence

$$
\begin{aligned}
\mid T_1x(t_2)-T_1x(t_1)\mid &\leq \mid p_1(t_2)-p_1(t_1)\mid +\lambda(K+rb_1)[\mid g_1(t_2,t_2)-g_1(t_1,t_1)\mid \\
&+ \mid g_1(t_2,0)-g_1(t_1,0)\mid].
\end{aligned}
$$

Hence, from the continuity of the functions g_1 assumption (v), we deduce that T_1 maps $C(I)$ into $C(I)$.

As done above we can obtain

$$
\begin{aligned}
\mid T_2y(t_2)-T_2y(t_1)\mid &\leq \mid p_2(t_2)-p_2(t_1)\mid +\lambda(K+rb_2)[\mid g_2(t_2,t_2)-g_2(t_1,t_1)\mid \\
&+ \mid g_2(t_2,0)-g_2(t_1,0)\mid].
\end{aligned}
$$

Also, by our assumption (v), we see that T_2 maps $C(I)$ into $C(I)$.

Now, from the definition of the operator T we get

$$
\begin{aligned}
Tu(t_2)-Tu(t_1) &= T(x,y)(t_2)-T(x,y)(t_1) \\
&= (T_1x(t_2),T_2y(t_2))-(T_1x(t_1),T_2y(t_1)) \\
&= (T_1x(t_2)-T_1x(t_1),T_2y(t_2)-T_2y(t_1))
\end{aligned}
$$

Therefore, T maps X into X.
Also, note that the class of $\{Tu(t)\}$ is equi-continuous on I.

Step 2: The operator T map U into U.
for $(x,y)\in U$, we have

$$
\begin{aligned}
\mid T_1x(t)\mid &\leq \mid p_1(t)\mid + \mid \lambda_1\int_0^t f_1(s,x(s),y(s))\, d_sg_1(t,s)\mid \\
&\leq \mid p_1(t)\mid +|\lambda_1|\int_0^t \mid f_1(s,x(s),y(s))\mid\ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \|p_1\|+\lambda\int_0^t (k_1(s)+b_1(\max\{\mid x(s)\mid,\mid y(s)\mid\}))\ d_s(\bigvee_{z=0}^{s} g_1(t,z))
\end{aligned}
$$

$$\begin{aligned}
&\leq \|p_1\| + \lambda \int_0^t (k_1(s) + rb_1)\ d_s g_1(t,s)) \\
&\leq \|p_1\| + \lambda(K + rb_1) \int_0^t d_s g_1(t,s) \\
&\leq \|p_1\| + \lambda(K + rb_1)[g_1(t,t) - g_1(t,0)] \\
&\leq \|p_1\| + \lambda(K + rb_1)[\sup_t |g_1(t,t)| + \sup_t |g_1(t,0)|] \\
&\leq \|p_1\| + \lambda(K + rb_1)\mu
\end{aligned}$$

Hence

$$\|T_1 x\| \leq \|p_1\| + \lambda(K + rb_1)\mu < r.$$

By a similar way can deduce that

$$\|T_2 y\| \ \leq \ \|p_2\| + \lambda(K + rb_2)\mu < r.$$

Therefore,

$$\|Tu\| = \|T(x,y)\| = \|T_1 x, T_2 y\| = \max\{\|T_1 x\|, \|T_2 y\|\} \leq r.$$

Thus for every $u = (x,y) \in U$, we have $Tu \in U$ and hence $TU \subset U$, (i.e $T : U \to U$). This means that the functions of TU are uniformly bounded on I.

Step 3: The operator T is compact.
The compactness of the operator T is a consequence of the estimates of the quantities $|T_1 x(t_2) - T_1 x(t_1)|, |T_2 y(t_2) - T_2 y(t_1)|$ conducted in Step 1, assumption (v) and the Arzel?a-Ascoli theorem.

Step 4: The operator T is continuous.
Firstly, we prove that T_1 is continuous. Let $\epsilon^* > 0$, the continuity of f_i yields $\exists\ \delta = \delta(\epsilon^*)$ such that $|f_i(t,x,y) - f_i(t,u,y)| < \epsilon^*$ whenever $\|x - u\| \leq \delta$, thus if $\|x - u\| \leq \delta$, we arrive at:

$$\begin{aligned}
\mid T_1 x(t) - T_1 u(t) \mid &\leq \mid \lambda_1 \int_0^t f_1(s, x(s), y(s))\ d_s g_1(t,s) - \lambda_1 \int_0^t f_1(s, u(s), y(s))\ d_s g_1(t,s) \mid \\
&\leq |\lambda_1| \int_0^t \mid f_1(s, x(s), y(s)) - f_1(s, u(s), y(s)) \mid\ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \epsilon^* \lambda \int_0^t d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \epsilon^* \lambda \int_0^t d_s g_1(t,s) \\
&\leq \epsilon^* \lambda\ [g_1(t,t) - g_1(t,0)] \\
&\leq \epsilon^* \lambda\ [\mid g_1(t,t) \mid + \mid g_1(t,0) \mid] \\
&\leq \epsilon^* \lambda\ [\sup_{t \in I} \mid g_1(t,t) \mid + \sup_{t \in I} \mid g_1(t,0) \mid] \leq \epsilon
\end{aligned}$$

where $\epsilon := \epsilon^* \lambda \mu$.
Therefore,

$$\mid T_1 x(t) - T_1 u(t) \mid \leq \epsilon.$$

This means that the operator T_1 is continuous.
By a similar way as done above we can prove that for any $y,\ v \in C[0,T]$ and $\| y-v \|< \delta$, we have

$$| T_2y(t) - T_2v(t) | \leq \epsilon.$$

Hence T_2 is continuous operator.
The operators T_i $(i = 1,2)$ is continuous operator it imply that T is continuous operator.
Since all conditions of Schauder fixed point theorem are satisfied, then T has at least one fixed point $u = (x,y) \in U$, which completes the proof.

In what follows, we provide an example illustrating the above obtained results.

Example : Consider the functions $g_i : I \times I \to R$ defined by the formula

$$\begin{aligned} g_1(t,s) &= \begin{cases} t\ln\frac{t+s}{t}, & \text{for } t \in (0,1],\ s \in I, \\ 0, & \text{for } t = 0,\ s \in I. \end{cases} \\ g_2(t,s) &= t(t+s-1),\ t \in I. \end{aligned}$$

It can be easily seen that the functions $g_1(t,s)$ and $g_2(t,s)$ satisfies assumptions (iii)-(v) given in Theorem 1, and $g_1(t,s)$ is function of bounded variation but it is not continuous on I. In this case, the coupled system of Volterra-Stieltjes integral equations (2) has the form

$$\begin{aligned} x(t) &= p_1(t) + \lambda_1 \int_0^t \frac{t}{t+s} f_1(s,x(s),y(s))\ ds,\ t \in I \\ y(t) &= p_2(t) + \lambda_2 \int_0^t t f_2(s,x(s),y(s))\ ds,\ t \in I. \end{aligned} \tag{3}$$

Also, consider the functions $f_i : I \times R^2 \to R$ defined by the formula

$$f_1(t,x,y) = t + x + y,$$

$$f_2(t,x,y) = t + x^2 - y^2.$$

Now, it can be easily seen that the functions f_1 and f_2 satisfies assumptions (ii) given in Theorem 1:

$$\begin{aligned} | f_1(t,x,y) | &\leq | t+x+y | \\ &\leq | t | + | x | + | y | \\ &\leq T + 2\max\{| x |, | y |\} \end{aligned}$$

And

$$\begin{aligned} | f_2(t,x,y) | &\leq | t + x^2 - y^2 | \\ &\leq | t | + | x^2 - y^2 | \\ &\leq T + | (x-y)(x+y) | \\ &\leq T + 2\max\{| x |, | y |\} \end{aligned}$$

Hence, $k_i(t) = T$, and $b_i = 2$.
Therefore, the functions f_i satisfies the assumption

$$| f_i(t, x, y) | \leq k_i(t) + b_i(\max\{| x |, | y |\}).$$

Therefore, the coupled system (3) has at least one solution $x, y \in C[0, T]$.

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