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ON SOME SOLVABLE SYSTEMS OF DIFFERENCE EQUATIONS WITH SOLUTIONS ASSOCIATED TO FIBONACCI NUMBERS

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Abstract. This paper deal with form, the periodicity and the stability of the solutions of the systems of difference equations

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}},\ y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}},\ n,\ k \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions $x_{-k},\ x_{-k+1}, \ldots,\ x_0,\ y_{-k},\ y_{-k+1}, \ldots,\ y_0$ are non zero real numbers.

1. Introduction

Real phenomena in Biology, Economics, Physics, and so forth are usually modeled by differential and difference equations (where the former is use for continuous case and the latter is use for discrete case).

Recently, there have been a growing interest in the study of solving difference equations, see for example, [2]-[8], [13]-[16], and the references cited therein. In these researches, authors are interested in investigating the form, periodicity, boundedness, (local and global) stability, and asymptotic behavior of solutions of various difference equations of nonlinear types.

In [17], D. T. Tollu, Y. Yazlik, and N. Taskara investigated the dynamics of the solutions of the two difference equations

$$x_{n+1} = \frac{1}{\pm 1 + x_n}, \quad n \in \mathbb{N}_0,$$

which were then extended by J. B. Bacani and J. F. T. Rabago to the two difference equations

$$x_{n+1} = \frac{q}{\pm p + x_n^{\nu}}, \quad n \in \mathbb{N}_0,$$

with $p, q \in \mathbb{R}^+$ and $\nu \in \mathbb{N}$ in [1]. Following the results found in the latter paper, Y. Halim and M. Bayram studied in [11] the solutions, stability character, and asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha}{\pm\beta + \gamma x_{n-k}}, \quad n \in \mathbb{N}_0.$$

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The results found in [1], [11], and [17] are intriguing mainly because the form of solutions obtained for these difference equations contained expressions for Fibonacci and Horadam numbers.

Motivated by these fascinating results, we shall determine the form and investigate the periodicity and global character of the solutions of the systems

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}}, \quad y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}}, \quad n \in \mathbb{N}_0. \tag{1.1}$$

where $k \in \mathbb{N}_0$ and the initial conditions are nonzero real numbers.

Now, we turn on the organization of the rest of the paper. In the succeeding section (Section 2) we shall give a brief introduction of Fibonacci numbers and discuss an overview of some basic tools in the analysis of difference equations. In Section 3, we shall study the dynamics of the system

$$x_{n+1} = \frac{1}{1 + y_{n-k}}, \quad y_{n+1} = \frac{1}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0. \tag{1.2}$$

More precisely, we shall give the form of its solutions and investigate the asymptotic stability of its positive solutions. For the other systems which are in the form as in (1.1), the form of their respective solutions are obtained through simple substitutions in (1.2) (see Section 4).

2. Preliminaries

A well-known recurrence sequence of order two is the widely studied Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$ recursively defined by the recurrence relation

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1). \tag{2.1}$$

As a result of the definition (2.1), it is conventional to define $F_0 = 0$. Various problems involving Fibonacci numbers have been formulated and extensively studied by many authors. Different generalizations and extensions of Fibonacci sequence have also been introduced and thoroughly investigated (see for example [12]). An interesting property of this integer sequence is that the ratio of its successive terms converges to the well-known golden mean (or the golden ratio) $\phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$. For more fascinating properties of Fibonacci numbers we refer the readers to [15].

Now, in the rest of this section we shall present some basic notations and results on the study of nonlinear difference equation which will be useful in our investigation, for more details, see for example [9].

Let f and g be two continuously differentiable functions:

$$f: I^{k+1} \times J^{k+1} \longrightarrow I, \quad g: I^{k+1} \times J^{k+1} \longrightarrow J, \quad I, J \subseteq \mathbb{R}$$

and for $n, k \in \mathbb{N}_0$, consider the system of difference equations

$$\begin{cases} x_{n+1} = & f(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}) \\ y_{n+1} = & g(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}) \end{cases} \tag{2.2}$$

where $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$ and $(y_{-k}, y_{-k+1}, \ldots, y_0) \in J^{k+1}$.

Define the map

$$H: I^{k+1} \times J^{k+1} \longrightarrow I^{k+1} \times J^{k+1}$$

by

$$H(W) = (f_0(W), f_1(W), \ldots, f_k(W), g_0(W), g_1(W), \ldots, g_k(W))$$

where

$$W = (u_0, u_1, \ldots, u_k, v_0, v_1, \ldots, v_k)^T,$$
$$f_0(W) = f(W), f_1(W) = u_0, \ldots, f_k(W) = u_{k-1},$$
$$g_0(W) = g(W), g_1(W) = v_0, \ldots, g_k(W) = v_{k-1}.$$

Let

$$W_n = [x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}]^T,$$

then, we can easily see that system (2.2) is equivalent to the following system written in vector form

$$W_{n+1} = H(W_n), \ n = 0, 1, \ldots, \tag{2.3}$$

that is

$$\left\{ \begin{array}{rcl} x_{n+1} & = & f\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right) \\ x_n & = & x_n \\ & \vdots & \\ x_{n-k+1} & = & x_{n-k+1} \\ y_{n+1} & = & g\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1} \ldots, , y_{n-k}\right) \\ y_n & = & y_n \\ & \vdots & \\ y_{n-k+1} & = & y_{n-k+1} \end{array} \right. .$$

Definition 1 (Equilibrium point). *An equilibrium point $(\overline{x}, \overline{y}) \in I \times J$ of system (2.2) is a solution of the system*

$$\left\{ \begin{array}{l} x = f\left(x, x, \ldots, x, y, y, \ldots, y\right), \\ y = g\left(x, x, \ldots, x, y, y, \ldots, y\right). \end{array} \right.$$

Furthermore, an equilibrium point $\overline{W} \in I^{k+1} \times J^{k+1}$ of system (2.3) is a solution of the system

$$W = H(W).$$

Definition 2 (Stability). *Let $\overline{W}$ be an equilibrium point of system (2.3) and $\| \,.\, \|$ be any norm (e.g. the Euclidean norm).*

(1) *The equilibrium point $\overline{W}$ is called stable (or locally stable) if for every $\epsilon > 0$ exist δ such that $\|W_0 - \overline{W}\| < \delta$ implies $\|W_n - \overline{W}\| < \epsilon$ for $n \geq 0$.*

(2) *The equilibrium point $\overline{W}$ is called asymptotically stable (or locally asymptotically stable) if it is stable and there exist $\gamma > 0$ such that $\|W_0 - \overline{W}\| < \gamma$ implies*
$$\|W_n - \overline{W}\| \to 0, \, n \to +\infty.$$

(3) *The equilibrium point $\overline{W}$ is said to be global attractor (respectively global attractor with basin of attraction a set $G \subseteq I^{k+1} \times J^{k+1}$, if for every W_0 (respectively for every $W_0 \in G$)*
$$\|W_n - \overline{W}\| \to 0, \, n \to +\infty.$$

(4) *The equilibrium point $\overline{W}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every W_0 (respectively for every $W_0 \in G$),*
$$\|W_n - \overline{W}\| \to 0, \, n \to +\infty.$$

(5) *The equilibrium point $\overline{W}$ is called unstable if it is not stable.*

Remark 1. *Clearly,* $(\overline{x},\overline{y}) \in I \times J$ *is an equilibrium point for system* (2.2) *if and only if* $\overline{W} = (\overline{x},\overline{x},\dots,\overline{x},\overline{y},\overline{y},\dots,\overline{y}) \in I^{k+1} \times J^{k+1}$ *is an equilibrium point of system* (2.3).

From here on, by the stability of the equilibrium points of system (2.2), we mean the stability of the corresponding equilibrium points of the equivalent system (2.3).

The linearized system, associated to system (2.3), about the equilibrium point

$$\overline{W} = (\overline{x},\overline{x},\cdots,\overline{x},\overline{y},\overline{y},\cdots,\overline{y})$$

is given by

$$W_{n+1} = AW_n,\ n = 0, 1, \dots$$

where A is the Jacobian matrix of the map H at the equilibrium point $\overline{W}$ given by

$$A = \begin{bmatrix} \frac{\partial f_0}{\partial u_0}(\overline{W}) & \frac{\partial f_0}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_0}{\partial u_k}(\overline{W}) & \frac{\partial f_0}{\partial v_0}(\overline{W}) & \frac{\partial f_0}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_0}{\partial v_k}(\overline{W}) \\ \frac{\partial f_1}{\partial u_0}(\overline{W}) & \frac{\partial f_1}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_1}{\partial u_k}(\overline{W}) & \frac{\partial f_1}{\partial v_0}(\overline{W}) & \frac{\partial f_1}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_1}{\partial v_k}(\overline{W}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial u_0}(\overline{W}) & \frac{\partial f_k}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_k}{\partial u_k}(\overline{W}) & \frac{\partial f_k}{\partial v_0}(\overline{W}) & \frac{\partial f_k}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_k}{\partial v_k}(\overline{W}) \\ \frac{\partial g_0}{\partial u_0}(\overline{W}) & \frac{\partial g_0}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_0}{\partial u_k}(\overline{W}) & \frac{\partial g_0}{\partial v_0}(\overline{W}) & \frac{\partial g_0}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_0}{\partial v_k}(\overline{W}) \\ \frac{\partial g_1}{\partial u_0}(\overline{W}) & \frac{\partial g_1}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_1}{\partial u_k}(\overline{W}) & \frac{\partial g_1}{\partial v_0}(\overline{W}) & \frac{\partial g_1}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_1}{\partial v_k}(\overline{W}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_k}{\partial u_0}(\overline{W}) & \frac{\partial g_k}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_k}{\partial u_k}(\overline{W}) & \frac{\partial g_k}{\partial v_0}(\overline{W}) & \frac{\partial g_k}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_k}{\partial v_k}(\overline{W}) \end{bmatrix}$$

Theorem 1 (Linearized stability)**.** *If all the eigenvalues of the Jacobian matrix A lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{W}$ of system* (2.3) *is asymptotically stable. On the other hand, if at least one eigenvalue of the Jacobian matrix A have absolute value greater than one, then the equilibrium point $\overline{W}$ of system* (2.3) *is unstable.*

Now, we are in the position to investigate the form and behavior of solutions of the system (1.2) and this is the content of the next section.

3. The system $x_{n+1} = \frac{1}{1+y_{n-k}}$, $y_{n+1} = \frac{1}{1+x_{n-k}}$

In this section, we give the explicit form of solutions of the system of difference equations

$$x_{n+1} = \frac{1}{1+y_{n-k}}, \quad y_{n+1} = \frac{1}{1+x_{n-k}} \tag{3.1}$$

where the initial values are arbitrary nonzero real numbers with the restriction that $x_{-k}, y_{-k}, \dots, x_0, y_0 \notin \{-\frac{F_{2n}}{F_{2n-1}}\}_{n=0}^{\infty} \bigcup \{-\frac{F_{2n+1}}{F_{2n}}\}_{n=0}^{\infty}$.

3.1. Form of the solutions.

The following theorem describes the form of the solutions of system (3.1).

Theorem 2. *Let* $\{x_n, y_n\}_{n\geq -k}$ *be a solution of* (3.1)*. Then, for* $n = 0, 1, \dots,$

$$x_{2(k+1)n+i} = \frac{F_{2n+1} + F_{2n} y_{i-(k+1)}}{F_{2n+2} + F_{2n+1} y_{i-(k+1)}}, \qquad i = 1, 2, ..., k+1,$$
$$y_{2(k+1)n+i} = \frac{F_{2n+1} + F_{2n} x_{i-(k+1)}}{F_{2n+2} + F_{2n+1} x_{i-(k+1)}}, \qquad i = 1, 2, ..., k+1,$$
$$x_{2(k+1)n+i} = \frac{F_{2n+2} + F_{2n+1} x_{i-(2k+2)}}{F_{2n+3} + F_{2n+2} x_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2,$$
$$y_{2(k+1)n+i} = \frac{F_{2n+2} + F_{2n+1} y_{i-(2k+2)}}{F_{2n+3} + F_{2n+2} y_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2.$$

Proof. From (3.1) we have

$$x_1 = \frac{1}{1+y_{-k}}, \ x_2 = \frac{1}{1+y_{-k+1}}, \dots, x_{k+1} = \frac{1}{1+y_0},$$
$$y_1 = \frac{1}{1+x_{-k}}, \ y_2 = \frac{1}{1+x_{-k+1}}, \dots, y_{k+1} = \frac{1}{1+x_0},$$

and

$$x_{k+2} = \frac{1+x_{-k}}{2+x_{-k}}, \ x_{k+3} = \frac{1+x_{-k+1}}{2+x_{-k+1}}, \dots, x_{2k+2} = \frac{1+x_0}{2+x_0},$$
$$y_{k+2} = \frac{1+y_{-k}}{2+y_{-k}}, \ y_{k+3} = \frac{1+y_{-k+1}}{2+y_{-k+1}}, \dots, y_{2k+2} = \frac{1+y_0}{2+y_0}.$$

So, the result hold for $n = 0$. Suppose now that $n \geq 1$ and that our assumption holds for $n-1$. That is,

$$x_{2(k+1)(n-1)+i} = \frac{F_{2n-1} + F_{2n-2} y_{i-(k+1)}}{F_{2n} + F_{2n-1} y_{i-(k+1)}}, \ i = 1, 2, \dots, k+1, \tag{3.2}$$
$$y_{2(k+1)(n-1)+i} = \frac{F_{2n-1} + F_{2n-2} x_{i-(k+1)}}{F_{2n} + F_{2n-1} x_{i-(k+1)}}, \ i = 1, 2, \dots, k+1, \tag{3.3}$$
$$x_{2(k+1)(n-1)+i} = \frac{F_{2n} + F_{2n-1} x_{i-(2k+2)}}{F_{2n+1} + F_{2n} x_{i-(2k+2)}}, \ i = k+2, k+3, \dots, 2k+2, \tag{3.4}$$
$$y_{2(k+1)(n-1)+i} = \frac{F_{2n} + F_{2n-1} y_{i-(2k+2)}}{F_{2n+1} + F_{2n} y_{i-(2k+2)}}, \ i = k+2, k+3, \dots, 2k+2. \tag{3.5}$$

For $i = 1, \dots, k+1$, it follows from (3.1), (3.2), and (3.3) that

$$\begin{aligned} x_{2(k+1)n+i} &= \frac{1}{1 + y_{2(k+1)n-(1+k)+i}} \\ &= \frac{1}{1 + \frac{1}{1 + x_{2(k+1)(n-1)+i}}} \\ &= \frac{\frac{(F_{2n}+F_{2n-1})+(F_{2n-1}+F_{2n-2})y_{i-(k+1)}}{F_{2n}+F_{2n-1}y_{i-(k+1)}}}{\frac{2F_{2n}+F_{2n-1}+2F_{2n-1}y_{i-(k+1)}+F_{2n-2}y_{i-(k+1)}}{F_{2n}+F_{2n-1}y_{i-(k+1)}}} \\ &= \frac{F_{2n+1} + F_{2n} y_{i-(k+1)}}{F_{2n+1} + F_{2n} + (F_{2n-1} + F_{2n}) y_{i-(k+1)}} \\ &= \frac{F_{2n+1} + F_{2n} y_{i-(k+1)}}{F_{2n+2} + F_{2n+1} y_{i-(k+1)}}, \end{aligned}$$

and

$$\begin{aligned}
y_{2(k+1)n+i} &= \frac{1}{1+x_{2(k+1)n-(1+k)+i}} \\
&= \frac{1}{1+\frac{1}{1+y_{2(k+1)(n-1)+i}}} \\
&= \frac{\frac{(F_{2n}+F_{2n-1})+(F_{2n-1}+F_{2n-2})x_{i-(k+1)}}{F_{2n}+F_{2n-1}x_{i-(k+1)}}}{\frac{2F_{2n}+F_{2n-1}+2F_{2n-1}x_{i-(k+1)}+F_{2n-2}x_{i-(k+1)}}{F_{2n}+F_{2n-1}x_{i-(k+1)}}} \\
&= \frac{F_{2n+1}+F_{2n}x_{i-(k+1)}}{F_{2n+1}+F_{2n}+(F_{2n-1}+F_{2n})x_{i-(k+1)}} \\
&= \frac{F_{2n+1}+F_{2n}x_{i-(k+1)}}{F_{2n+2}+F_{2n+1}x_{i-(k+1)}}.
\end{aligned}$$

Similarly, for $i = k+2, k+3, \dots, 2k+2$, from(3.1), (3.4), and (3.5), we get

$$\begin{aligned}
x_{2(k+1)n+i} &= \frac{1}{1+y_{2(k+1)n-(1+k)+i}} \\
&= \frac{1}{1+\frac{1}{1+x_{2(k+1)(n-1)+i}}} \\
&= \frac{\frac{F_{2n+1}+F_{2n}+F_{2n}x_{i-(2k+2)}+F_{2n-1}x_{i-(2k+2)}}{F_{2n+1}+F_{2n}x_{i-(2k+2)}}}{\frac{2F_{2n+1}+F_{2n}+2F_{2n}x_{i-(2k+2)}+F_{2n-1}x_{i-(2k+2)}}{F_{2n+1}+F_{2n}x_{i-(2k+2)}}} \\
&= \frac{F_{2n+2}+F_{2n+1}x_{i-(2k+2)}}{F_{2n+1}+F_{2n+2}+F_{2n}x_{i-(2k+2)}+F_{2n+1}x_{i-(2k+2)}} \\
&= \frac{F_{2n+2}+F_{2n+1}x_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}x_{i-2(k+2)}},
\end{aligned}$$

and

$$\begin{aligned}
y_{2(k+1)n+i} &= \frac{1}{1+x_{2(k+1)n-(1+k)+i}} \\
&= \frac{1}{1+\frac{1}{1+y_{2(k+1)(n-1)+i}}} \\
&= \frac{\frac{F_{2n+1}+F_{2n}+F_{2n}y_{i-(2k+2)}+F_{2n-1}y_{i-(2k+2)}}{F_{2n+1}+F_{2n}y_{i-(2k+2)}}}{\frac{2F_{2n+1}+F_{2n}+2F_{2n}y_{i-(2k+2)}+F_{2n-1}y_{i-(2k+2)}}{F_{2n+1}+F_{2n}y_{i-(2k+2)}}} \\
&= \frac{F_{2n+2}+F_{2n+1}y_{i-(2k+2)}}{F_{2n+1}+F_{2n+2}+F_{2n}y_{i-(2k+2)}+F_{2n+1}y_{i-(2k+2)}} \\
&= \frac{F_{2n+2}+F_{2n+1}y_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}y_{i-2(k+2)}}.
\end{aligned}$$

□

3.2. **Global stability of positive solutions.** In this section we study the asymptotic behavior of positive solutions of the system (3.1).

Let $I = J = (0, +\infty)$ and consider the functions

$$f:\ I^{k+1} \times J^{k+1} \longrightarrow I,\ g:\ I^{k+1} \times J^{k+1} \longrightarrow J$$

defined by

$$f(u_0, u_1, \ldots, u_k, v_0, v_1, \ldots, v_k) = \frac{1}{1+v_k},$$
$$g(u_0, u_1, \ldots, u_k, v_0, v_1, \ldots, v_k) = \frac{1}{1+u_k}.$$

Corollary 1. *System* (3.1) *has a unique equilibrium point in* $I \times J$*, namely*

$$E := \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right).$$

Proof. Clearly the system

$$\overline{x} = \frac{1}{1+\overline{y}},\ \overline{y} = \frac{1}{1+\overline{x}},$$

has a unique solution in $I \times J$ which is

$$(\overline{x}, \overline{y}) = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right).$$

□

Theorem 3. *The equilibrium point* E *is locally asymptotically stable.*

Proof. The the linearized system about the equilibrium point

$$\overline{W} = \left(\frac{-1+\sqrt{5}}{2}, \ldots, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \ldots, \frac{-1+\sqrt{5}}{2}\right) \in I^{k+1} \times J^{k+1}$$

is given by

$$X_{n+1} = AX_n,\ X_n = (x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k})^T \tag{3.6}$$

and

$$A = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$$

where B, C are $(k+1) \times (k+1)$ matrix and given by

$$B = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 & \frac{-3+\sqrt{5}}{2} \\ \vdots & \ddots & \ddots & \dots & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{bmatrix}.$$

Let

$$P(\lambda) = \det(A - \lambda I_{2k+2}) = \det \begin{bmatrix} B - \lambda I_{k+1} & C \\ C & B - \lambda I_{k+1} \end{bmatrix}.$$

So, after some elementary calculations, we get

$$P(\lambda) = \lambda^{2k+2} - \left(\frac{-3+\sqrt{5}}{2}\right)^2.$$

Now, consider the two functions defined by

$$a(\lambda) = \lambda^{2k+2}, \quad b(\lambda) = \left(\frac{-3+\sqrt{5}}{2}\right)^2.$$

We have

$$|b(\lambda)| < |a(\lambda)|, \forall \lambda : |\lambda| = 1$$

Thus, by Rouche's theorem, all zeros of $P(\lambda) = a(\lambda) - b(\lambda) = 0$ lie in $|\lambda| < 1$. So, by theorem (1), we get that E is locally asymptotically stable. □

Theorem 4. *The equilibrium point E is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n \geq -k}$ be a solution of (3.1). By theorem (3) we need only to prove that E is global attractor, that is

$$\lim_{n\to\infty} (x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}) = \overline{E},$$

or equivalently

$$\lim_{n\to\infty} (x_n, y_n) = E.$$

To do this, we prove that for $i = 1, \dots, 2k+1$ we have

$$\lim_{n\to+\infty} x_{2(k+1)n+i} = \lim_{n\to+\infty} y_{2(k+1)n+i} = \frac{-1+\sqrt{5}}{2}.$$

For $i = 1, \dots, k+1$, it follows from theorem (2) that

$$\begin{aligned} \lim_{n\to+\infty} x_{2(k+1)n+i} &= \lim_{n\to+\infty} \frac{F_{2n+1} + F_{2n} y_{i-(k+1)}}{F_{2n+2} + F_{2n+1} y_{i-(k+1)}} \\ &= \lim_{n\to+\infty} \frac{1 + \frac{F_{2n}}{F_{2n+1}} y_{i-(k+1)}}{\frac{F_{2n+2}}{F_{2n+1}} + y_{i-(k+1)}}. \end{aligned} \tag{3.7}$$

and

$$\lim_{n\to+\infty} y_{2(k+1)n+i} = \lim_{n\to+\infty} \frac{F_{2n+1}+F_{2n}x_{i-(k+1)}}{F_{2n+2}+F_{2n+1}x_{i-(k+1)}}$$
$$= \lim_{n\to+\infty} \frac{1+\frac{F_{2n}}{F_{2n+1}}x_{i-(k+1)}}{\frac{F_{2n+2}}{F_{2n+1}}+x_{i-(k+1)}}. \tag{3.8}$$

Using Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ n \in \mathbb{N}_0 \tag{3.9}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, we get

$$\lim_{n\to+\infty} \frac{F_{2n}}{F_{2n+1}} = \lim_{n\to+\infty} \frac{\alpha^{2n} \times \frac{1-\left(\frac{\beta}{\alpha}\right)^{2n}}{\alpha-\beta}}{\alpha^{2n+1} \times \frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}{\alpha-\beta}} = \frac{1}{\alpha}, \tag{3.10}$$

similarly we get

$$\lim_{n\to+\infty} \frac{F_{2n+2}}{F_{2n+1}} = \alpha. \tag{3.11}$$

Thus, from (3.7)-(3.11), we get

$$\lim_{n\to+\infty} x_{2(k+1)n+i} = \frac{1+\frac{1}{\alpha}y_{i-(k+1)}}{\alpha+y_{i-(k+1)}} = \frac{1}{\alpha} = \frac{-1+\sqrt{5}}{2},$$

$$\lim_{n\to+\infty} y_{2(k+1)n+i} = \frac{1+\frac{1}{\alpha}x_{i-(k+1)}}{\alpha+x_{i-(k+1)}} = \frac{1}{\alpha} = \frac{-1+\sqrt{5}}{2}.$$

By the same arguments, we get, for $i = k+2, k+2, \ldots, 2k+1$:

$$\lim_{n\to+\infty} x_{2(k+1)n+i} = \lim_{n\to+\infty} y_{2(k+1)n+i} = \frac{-1+\sqrt{5}}{2}.$$

□

Remark 2. *If $x_{i_0-(k+1)} = \overline{x} = \frac{-1+\sqrt{5}}{2}$ (respectively $y_{i_0-(k+1)} = \overline{y} = \frac{-1+\sqrt{5}}{2}$) for some $1 \le i_0 \le k+1$, then for $n = 0, 1, \ldots,$*

$$y_{(k+1)n+i_0} = \frac{-1+\sqrt{5}}{2}\ (\textit{respectively}\ x_{(k+1)n+i_0} = \frac{-1+\sqrt{5}}{2}).$$

Using the fact that

$$\overline{x} = \overline{y} = \frac{1}{1+\overline{x}} = \frac{1}{1+\overline{y}}$$

and theorem (2), we get

$$y_{(k+1)n+i_0} = \frac{F_{2n+1}+F_{2n}\overline{x}}{F_{2n+2}+F_{2n+1}\overline{x}} = \frac{F_{2n+1}+\frac{F_{2n}}{1+\overline{x}}}{F_{2n+2}+F_{2n+1}\overline{x}} = \frac{\frac{F_{2n+1}+F_{2n}+\overline{x}F_{2n+1}}{1+\overline{x}}}{F_{2n+2}+F_{2n+1}\overline{x}}$$
$$= \frac{\frac{F_{2n+2}+\overline{x}F_{2n+1}}{1+\overline{x}}}{F_{2n+2}+F_{2n+1}\overline{x}} = \frac{1}{1+\overline{x}} = \overline{x},$$

and

$$x_{(k+1)n+i_0} = \frac{F_{2n+1} + F_{2n}\overline{y}}{F_{2n+2} + F_{2n+1}\overline{y}} = \frac{F_{2n+1} + \frac{F_{2n}}{1+\overline{y}}}{F_{2n+2} + F_{2n+1}\overline{y}} = \frac{\frac{F_{2n+1}+F_{2n}+\overline{y}F_{2n+1}}{1+\overline{y}}}{F_{2n+2} + F_{2n+1}\overline{y}}$$
$$= \frac{\frac{F_{2n+2}+\overline{y}F_{2n+1}}{1+\overline{y}}}{F_{2n+2} + F_{2n+1}\overline{y}} = \frac{1}{1+\overline{y}} = \overline{y}.$$

Similarly, if $x_{i_0-(2k+2)} = \overline{x} = \frac{-1+\sqrt{5}}{2}$ *(respectively* $y_{i_0-(2k+2)} = \overline{y} = \frac{-1+\sqrt{5}}{2}$*) for some* $k+2 \le i_0 \le 2k+2$*, then for* $n = 0, 1, \ldots,$

$$y_{(k+1)n+i_0} = \frac{-1+\sqrt{5}}{2} \text{ (respectively } x_{(k+1)n+i_0} = \frac{-1+\sqrt{5}}{2}).$$

Example 1. *For confirming results of this section, we consider the following numerical example.*

Let $k = 5$ *in Eq.*(3.1)*, then we obtain the system*

$$x_{n+1} = \frac{1}{1+y_{n-5}},\ y_{n+1} = \frac{1}{1+x_{n-5}}. \tag{3.12}$$

Assume $x_{-5} = 1$, $x_{-4} = 1.6$, $x_{-3} = 3.4$, $x_{-2} = 6.1$, $x_{-1} = 2$, $x_0 = 1.3$, $y_{-5} = 0.7$, $y_{-4} = 4.2$, $y_{-3} = 0.3$, $y_{-2} = 2.4$, $y_{-1} = 0.2$ *and* $y_0 = 5$ *(see Fig.* (1)*).*

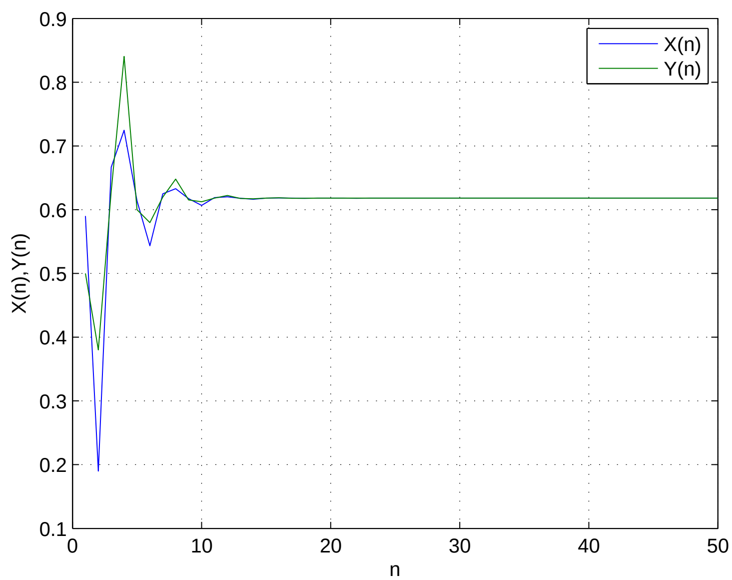

FIGURE 1. This figure shows that the solution of the Eq.(3.12) is global attractor, that is $\lim_{n\to\infty}(x_n, y_n) = E$.

3.3. **Other systems.** The other systems which are of the form as in (1.1) can recovered from (3.1) by appropriate substitution and this is seen in the proof of the following corollaries:

Corollary 2. *Let $\{x_n, y_n\}_{n\geq -k}$ be a solution of*

$$x_{n+1} = \frac{1}{-1+y_{n-k}}, \quad y_{n+1} = \frac{1}{-1+x_{n-k}}. \tag{3.13}$$

where the initial values are arbitrary real numbers with $x_{-k}, y_{-k}, \ldots, x_0, y_0 \notin \left\{\frac{F_{2n}}{F_{2n-1}}\right\}_{n=0}^{\infty} \bigcup \left\{\frac{F_{2n+1}}{F_{2n}}\right\}_{n=0}^{\infty}$. Then, for $n = 1, 2, \ldots,$ we have

$$\begin{aligned}
x_{2(k+1)n+i} &= \tfrac{F_{2n+1}-F_{2n}y_{i-(k+1)}}{F_{2n+2}-F_{2n+1}y_{i-(k+1)}}, \quad & i = 1, 2, \ldots, k+1,\\
x_{2(k+1)n+i} &= \tfrac{F_{2n+2}-F_{2n+1}x_{i-(2k+2)}}{F_{2n+3}-F_{2n+2}x_{i-(2k+2)}}, \quad & i = k+2, k+3, \ldots, 2k+2,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+1}-F_{2n}x_{i-(k+1)}}{F_{2n+2}-F_{2n+1}x_{i-(k+1)}}, \quad & i = 1, 2, \ldots, k+1,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+2}-F_{2n+1}y_{i-(2k+2)}}{F_{2n+3}-F_{2n+2}y_{i-(2k+2)}}, \quad & i = k+2, k+3, \ldots, 2k+2.
\end{aligned}$$

Proof. It follows from theorem (2) by replacing (x_n, y_n) by $(-x_n, -y_n)$. □

Corollary 3. *Let $\{x_n, y_n\}_{n\geq -k}$ be a solution of*

$$x_{n+1} = \frac{1}{1-y_{n-k}}, \quad y_{n+1} = \frac{1}{-1-x_{n-k}}. \tag{3.14}$$

where the initial values are arbitrary real numbers such that $y_{-k}, \ldots, y_0 \notin \left\{\frac{F_{2n}}{F_{2n-1}}\right\}_{n=0}^{\infty} \bigcup \left\{\frac{F_{2n+1}}{F_{2n}}\right\}_{n=0}^{\infty}$, and $x_{-k}, \ldots,, x_0 \notin \left\{-\frac{F_{2n}}{F_{2n-1}}\right\}_{n=0}^{\infty} \bigcup \left\{-\frac{F_{2n+1}}{F_{2n}}\right\}_{n=0}^{\infty}$. Then, for $n = 0, 1, \ldots,$ we have

$$\begin{aligned}
x_{2(k+1)n+i} &= \tfrac{F_{2n+1}-F_{2n}y_{i-(k+1)}}{F_{2n+2}-F_{2n+1}y_{i-(k+1)}}, \quad & i = 1, 2, \ldots, k+1,\\
x_{2(k+1)n+i} &= \tfrac{F_{2n+2}+F_{2n+1}x_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}x_{i-(2k+2)}}, \quad & i = k+2, k+3, \ldots, 2k+2,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+1}+F_{2n}x_{i-(k+1)}}{F_{2n+2}+F_{2n+1}x_{i-(k+1)}}, \quad & i = 1, 2, \ldots, k+1,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+2}-F_{2n+1}y_{i-(2k+2)}}{F_{2n+3}-F_{2n+2}y_{i-(2k+2)}}, \quad & i = k+2, k+3, \ldots, 2k+2,
\end{aligned}$$

Proof. It follows from theorem (2) by replacing y_n by $-y_n$. □

Corollary 4. *Let $\{x_n, y_n\}_{n\geq -k}$ be a solution of*

$$x_{n+1} = \frac{1}{-1-y_{n-k}}, \quad y_{n+1} = \frac{1}{1-x_{n-k}}. \tag{3.15}$$

where the initial values are arbitrary real numbers such that $x_{-k}, \ldots, x_0 \notin \left\{\frac{F_{2n}}{F_{2n-1}} \cdots\right\}_{n=0}^{\infty} \bigcup \left\{\frac{F_{2n+1}}{F_{2n}}\right\}_{n=0}^{\infty}$ and $y_{-k}, \ldots, y_0 \notin \left\{-\frac{F_{2n}}{F_{2n-1}}\right\}_{n=0}^{\infty} \bigcup \left\{-\frac{F_{2n+1}}{F_{2n}}\right\}_{n=0}^{\infty}$. Then, for $n = 0, 1, \ldots,$ we have

$$\begin{aligned}
x_{2(k+1)n+i} &= \tfrac{F_{2n+1}+F_{2n}y_{i-(k+1)}}{F_{2n+2}+F_{2n+1}y_{i-(k+1)}}, \quad & i = 1, 2, \ldots, k+1,\\
x_{2(k+1)n+i} &= \tfrac{F_{2n+2}-F_{2n+1}x_{i-(2k+2)}}{F_{2n+3}-F_{2n+2}x_{i-(2k+2)}}, \quad & i = k+2, k+3, \ldots, 2k+2,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+1}-F_{2n}x_{i-(k+1)}}{F_{2n+2}-F_{2n+1}x_{i-(k+1)}}, \quad & i = 1, 2, \ldots, k+1,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+2}+F_{2n+1}y_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}y_{i-(2k+2)}}, \quad & i = k+2, k+3, \ldots, 2k+2,
\end{aligned}$$

Proof. It follows from theorem (2) by replacing x_n by $-x_n$. □

4. Conclusion

We have studied in this work the system of difference equations (3.1) and gave the explicit form of its solutions in terms of Fibonacci numbers. We have also investigated the asymptotic behavior of positive solutions of the system (3.1) and showed that the solutions of equations in (3.1) actually converge to E. We expect that more fascinating results shall be obtained by many researchers in relation to Fibonacci numbers and related sequences in future papers.

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