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COMPARATIVE GROWTH ANALYSIS OF MEROMORPHIC FUNCTIONS ON THE BASIS OF THEIR RELATIVE L^*-ORDERS

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using their relative L^*-order and relative L^*-lower order.

1. Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [3] and [9]. In the sequel we use the following notation : $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

The following definition is well known:

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r},$$

When f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [7] defined it in the following way:

Definition 2. [7] *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,*

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon} \text{ for } r \ge r(\varepsilon) \text{ and}$$

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uniformly for $k\,(\geq 1)$ *.*
If further $L\,(r)$ *is differentiable, the above condition is equivalent to*

$$\lim_{r\to\infty}\frac{rL'\,(r)}{L\,(r)}=0\ .$$

Somasundaram and Thamizharasi [8] introduced the notions of L-order and L-lower order for entire functions. The more generalised concept for L-order and L-lower order for entire and meromorphic functions are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 3. [8] *The* L^**-order* $\rho_f^{L^*}$ *and the* L^**-lower order* $\lambda_f^{L^*}$ *of an entire function* f *are defined as*

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log^{[2]}M_f\,(r)}{\log\left[re^{L(r)}\right]}\ \textit{and}\ \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log^{[2]}M_f\,(r)}{\log\left[re^{L(r)}\right]},$$

When f *is meromorphic, one can easily verify that*

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log T_f\,(r)}{\log\left[re^{L(r)}\right]}\ \textit{and}\ \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log T_f\,(r)}{\log\left[re^{L(r)}\right]}.$$

For an entire function g, the Nevanlinna's characteristic function $T_g\,(r)$ is defined as $T_g\,(r)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\left|g(re^{i\theta})\right|d\theta$ where $\log^+x=\max\left(0,\log x\right)$ for $x>0$.

If g is non-constant then $T_g\,(r)$ is strictly increasing and continuous and its inverse $T_g^{-1}:(T_g\,(0)\,,\infty)\to(0,\infty)$ exists and is such that $\lim\limits_{s\to\infty}T_g^{-1}\,(s)=\infty$.

Lahiri and Banerjee [6] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 4. [6] *Let* f *be meromorphic and* g *be entire. The relative order of* f *with respect to* g *denoted by* $\rho_g\,(f)$ *is defined as*

$$\begin{aligned}\rho_g\,(f)&=\inf\left\{\mu>0:T_f\,(r)<T_g\,(r^\mu)\ \textit{for all large}\ r\right\}\\&=\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [6] if $g\,(z)=\exp z$.
Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g\,(f)$ in the following manner :

$$\lambda_g\,(f)=\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log r}.$$

In the line of Somasundaram and Thamizharasi [8] and Lahiri and Banerjee [6] one may define the relative L^*-order and relative L^*-lower order of a meromorphic function f with respect to an entire function g in the following manner:

Definition 5. *The relative* L^**-order* $\rho_g^{L^*}\,(f)$ *and the relative* L^**-lower order* $\lambda_g^{L^*}\,(f)$ *of a meromorphic function* f *with respect to an entire function* g *are defined by*

$$\rho_g^{L^*}\,(f)=\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log\left[re^{L(r)}\right]}\ \textit{and}\ \lambda_g^{L^*}\,(f)=\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log\left[re^{L(r)}\right]}.$$

In this paper we study some growth properties of composition of entire and meromorphic functions with respect to their relative L^*-orders and relative L^*-lower orders as compared to the corresponding left and right factors.

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] *Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f\left(\exp\left(r^{\mu}\right)\right) .$$

Lemma 2. [5] *Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) > T_g\left(\exp\left(r^{\mu}\right)\right),$$

where $0 < \mu < \rho_g$.

Lemma 3. [2] *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_f\left(\exp\left(r^{\mu}\right)\right) .$$

Lemma 4. [2] *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_g\left(\exp\left(r^{\mu}\right)\right) .$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function and g, h be any two entire functions such that*

$$(i)\ \liminf_{r\to\infty} \frac{\log T_h^{-1}(r)}{\left(\log\left[re^{L(r)}\right]\right)^{\alpha}} = A,\ \text{a real number } > 0$$

and

$$(ii)\ \liminf_{r\to\infty} \frac{\log T_h^{-1} T_f\left(\exp r^{\mu}\right)}{\left(\log T_h^{-1}(r)\right)^{\beta+1}} = B,\ \text{a real number } > 0$$

for any α, β, μ satisfying $0 < \alpha < 1,\ \beta > 0,\ \alpha(\beta+1) > 1$ and $0 < \mu < \rho_g \leq \infty$. Then

$$\rho_h(fog) = \infty.$$

Proof. From (i) we have for all sufficiently large values of r that

$$\log T_h^{-1}(r) \geq (A-\varepsilon)\left(\log re^{L(r)}\right)^{\alpha} \tag{1}$$

and from (ii) we obtain for all sufficiently large values of r that

$$\log T_h^{-1} T_f\left(\exp r^{\mu}\right) \geq (B-\varepsilon)\left(\log T_h^{-1}(r)\right)^{\beta+1} . \tag{2}$$

Also $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, (1) and (2) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log T_h^{-1} T_{f \circ g}(r) &\geq \log T_h^{-1} T_f\left(\exp\left(r^\mu\right)\right) \\
i.e.,\ \log T_h^{-1} T_{f \circ g}(r) &\geq (B-\varepsilon)\left(\log T_h^{-1}(r)\right)^{\beta+1} \\
i.e.,\ \log T_h^{-1} T_{f \circ g}(r) &\geq (B-\varepsilon)\left[(A-\varepsilon)\left(\log\left[re^{L(r)}\right]\right)^\alpha\right]^{\beta+1} \\
i.e.,\ \log T_h^{-1} T_{f \circ g}(r) &\geq (B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left[re^{L(r)}\right]\right)^{\alpha(\beta+1)} \\
i.e.,\ \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} &\geq \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left[re^{L(r)}\right]\right)^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]}
\end{aligned}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \liminf_{r\to\infty} \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left[re^{L(r)}\right]\right)^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha(\beta+1)>1$, it follows from above that

$$\rho_h^{L^*}(fog) = \infty$$

which proves the theorem. □

In the line of Theorem 1 and with the help of Lemma 2, one may state the following theorem without its proof :

Theorem 2. *Let f be a meromorphic function with non zero lower order and g be an entire function with non zero finite order. For another entire function h, also suppose that*

$$(i)\ \liminf_{r\to\infty} \frac{\log T_h^{-1}(r)}{\left(\log\left[re^{L(r)}\right]\right)^\alpha} = A,\ \textit{a real number } >0$$

and

$$(ii)\ \liminf_{r\to\infty} \frac{\log T_h^{-1} T_g\left(\exp r^\mu\right)}{\left(\log T_h^{-1}(r)\right)^{\beta+1}} = B,\ \textit{a real number } >0$$

for any α, β, μ satisfying $0<\alpha<1,\ \beta>0,\ \alpha(\beta+1)>1$ and $0<\mu<\rho_g$. Then

$$\rho_h^{L^*}(fog) = \infty\ .$$

Theorem 3. *Let f be a meromorphic function and g, h be any two entire functions such that*

$$(i)\ \liminf_{r\to\infty} \frac{\log T_h^{-1}\left(\exp\left(r^\mu\right)\right)}{\left(\log^{[2]} r\right)^\alpha} = A,\ \textit{a real number } >0$$

and

$$(ii)\ \liminf_{r\to\infty} \frac{\log\left[\frac{\log T_h^{-1}\left(T_f\left(\exp r^\mu\right)\right)}{\log T_h^{-1}\left(\exp r^\mu\right)}\right]}{\left[\log T_h^{-1}\left(\exp r^\mu\right)\right]^\beta} = B,\ \textit{a real number } >0$$

for any α, β satisfying $\alpha>1,\ 0<\beta<1,\ \alpha\beta>1$ and $0<\mu<\rho_g\leq\infty$. Then

$$\rho_h^{L^*}(fog) = \infty\ .$$

Proof. From (i) we have for all sufficiently large values of r that

$$\log T_h^{-1}\left(\exp\left(r^{\mu}\right)\right) \geq \left(\left(A-\varepsilon\right)\log^{[2]} r\right)^{\alpha} \tag{3}$$

and from (ii) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log\left[\frac{\log T_h^{-1}\left(T_f(\exp r^{\mu})\right)}{\log T_h^{-1}\left(\exp r^{\mu}\right)}\right] &\geq (B-\varepsilon)\left[\log T_h^{-1}\left(\exp r^{\mu}\right)\right]^{\beta} \\ i.e.,\ \frac{\log T_h^{-1}\left(T_f(\exp r^{\mu})\right)}{\log T_h^{-1}\left(\exp r^{\mu}\right)} &\geq \exp\left[(B-\varepsilon)\left[\log T_h^{-1}\left(\exp r^{\mu}\right)\right]^{\beta}\right]. \end{aligned} \tag{4}$$

Also $T_h^{-1}(r)$ is increasing function of r, it follows from Lemma 1, (3) and (4) for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \frac{\log T_h^{-1}T_f\left(\exp\left(r^{\mu}\right)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \frac{\log T_h^{-1}T_f\left(\exp\left(r^{\mu}\right)\right)}{\log T_h^{-1}\left(\exp\left(r^{\mu}\right)\right)} \cdot \frac{\log T_h^{-1}\left(\exp\left(r^{\mu}\right)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \exp\left[(B-\varepsilon)\left[\log T_h^{-1}\left(\exp r^{\mu}\right)\right]^{\beta}\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha}}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]} r\right)^{\alpha\beta}\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha}}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq$$
$$\exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]} r\right)^{\alpha\beta-1}\log^{[2]} r\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha}}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq$$
$$(\log r)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]} r\right)^{\alpha\beta-1}} \cdot \frac{(A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha}}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq$$
$$\liminf_{r\to\infty}(\log r)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]} r\right)^{\alpha\beta-1}} \cdot \frac{(A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha}}{\log\left[re^{L(r)}\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha > 1$, $\alpha\beta > 1$, the theorem follows from above. □

Theorem 4. *Let f be a meromorphic function with non zero lower order and g be an entire function with non zero finite order. Further suppose that*

$$(i)\ \liminf_{r\to\infty}\frac{\log T_h^{-1}\left(\exp\left(r^{\mu}\right)\right)}{\left(\log^{[2]} r\right)^{\alpha}}=A,\ \text{a real number } >0$$

and

$$(ii)\ \liminf_{r\to\infty}\frac{\log\left[\frac{\log T_h^{-1}\left(T_g\left(\exp r^{\mu}\right)\right)}{\log T_h^{-1}\left(\exp r^{\mu}\right)}\right]}{\left[\log T_h^{-1}\left(\exp r^{\mu}\right)\right]^{\beta}}=B,\ \text{a real number } >0$$

for any α,β satisfying $\alpha>1,\ 0<\beta<1,\ \alpha\beta>1$ and $0<\mu<\rho_g$ where h is also an entire function. Then

$$\rho_h^{L^*}\left(fog\right)=\infty\ .$$

We omit the proof of Theorem 4 as it can be carried out in the line of Theorem 3 and with the help of Lemma 2.

Theorem 5. *Let f be a meromorphic function and g, h be any two entire functions such that g is of non zero finite order and $\lambda_h^{L^*}(f)>0$. Then*

$$\rho_h^{L^*}\left(fog\right)=\infty\ .$$

Proof. Suppose $0<\mu<\rho_g\leq\infty$.
As $T_h^{-1}(r)$ is an increasing function of r, we get from Lemma 1, for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log T_h^{-1}T_{f\circ g}(r) &\geq \log T_h^{-1}T_f\left(\exp\left(r^{\mu}\right)\right)\\
i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left[r^{\mu}e^{L\left(r^{\mu}\right)}\right]\\
i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} &\geq \frac{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[r^{\mu}+L^{\left(r^{\mu}\right)}\right]}{\log\left[re^{L(r)}\right]}\\
i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} &\geq \liminf_{r\to\infty}\frac{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[r^{\mu}+L^{\left(r^{\mu}\right)}\right]}{\log r+L\left(r^{\mu}\right)}
\end{aligned}$$

$$i.e.,\ \rho_h^{L^*}\left(fog\right)=\infty\ .$$

Thus the theorem follows. □

In the line of Theorem 5 one can easily prove the following theorem:

Theorem 6. *Let f be a meromorphic function of non zero lower order. Also suppose that g and h be any two entire functions such that g is of non zero order and $\lambda_h^{L^*}(g)>0$. Then*

$$\rho_h^{L^*}\left(fog\right)=\infty\ .$$

Theorem 7. *Let f be a meromorphic function and g,h be any two entire functions such that g is of non zero finite order and $\lambda_h^{L^*}(f)>0$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)}=\infty.$$

Proof. In view of Theorem 5, we obtain that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} &\geq \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\cdot \\ &\quad \liminf_{r\to\infty}\frac{\log\left[re^{L(r)}\right]}{\log T_h^{-1}T_f(r)}\end{aligned}$$

$$\begin{aligned} i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} &\geq \rho_h^{L^*}(fog)\cdot\frac{1}{\rho_h^{L^*}(f)} \\ i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} &= \infty.\end{aligned}$$

Thus the theorem follows. □

Theorem 8. *Let f be a meromorphic function with non zero lower order and g, h be any two entire function such that g is of non zero order and $\lambda_h^{L^*}(g) > 0$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)} = \infty.$$

Proof of Theorem 8 is omitted as it can be carried out in the line of Theorem 7 and in view of Theorem 6.

Theorem 9. *Let f be a meromorphic function and h be an entire function such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Also let g be an entire function with non zero order. Then for every positive constant A and every real number α,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\left\{\log T_h^{-1}T_f(r^A)\right\}^{1+\alpha}} = \infty\ .$$

Proof. If α be such that $1+\alpha \leq 0$, then the theorem is trivial. So we suppose that $1+\alpha > 0$.

Since $T_h^{-1}(r)$ is an increasing function of r, we get from Lemma 1 for a sequence of values of r tending to infinity that

$$\begin{aligned}\log T_h^{-1}T_{f\circ g}(r) &\geq \log T_h^{-1}T_f\left(\exp\left(r^\mu\right)\right) \\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[r^\mu + L\left(r^\mu\right)\right]\ . \qquad (5)\end{aligned}$$

where we choose $0 < \mu < \rho_g \leq \infty$.

Again from the definition of $\rho_h^{L^*}(f)$, it follows for all sufficiently large values of r that

$$\begin{aligned}\log T_h^{-1}T_f(r^A) &\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left(A\log r + L\left(r\right)\right) \\ i.e.,\ \left\{\log T_h^{-1}T_f(r^A)\right\}^{1+\alpha} &\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)^{1+\alpha}\left(A\log r + L\left(r\right)\right)^{1+\alpha}\ . \qquad (6)\end{aligned}$$

Now from (5) and (6), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\left\{\log T_h^{-1}T_f(r^A)\right\}^{1+\alpha}} \geqslant \frac{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[r^\mu + L\left(r^\mu\right)\right]}{\left(\rho_h^{L^*}(f)+\varepsilon\right)^{1+\alpha}\left(A\log r + L\left(r\right)\right)^{1+\alpha}}.$$

Since $\frac{r^\mu}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, the theorem follows from above. □

In the line of Theorem 9 and with the help of Lemma 2 one may state the following theorem without its proof :

Theorem 10. *Let f be a meromorphic function with non zero finite lower order and g be an entire function with non zero finite order. Also let h be an entire function such that $\rho_h^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(g) > 0$. Then for every positive constant A and every real number α,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\left\{\log T_h^{-1} T_f(r^A)\right\}^{1+\alpha}} = \infty \ .$$

Theorem 11. *Let f be a meromorphic function and g be an entire function with non zero order. Also let h be an entire function such that $0 < \lambda_h^{L^*}(f)$ and $\rho_h^{L^*}(g) < \infty$. Then for every positive constant A and every real number α,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\left\{\log T_h^{-1} T_g(r^A)\right\}^{1+\alpha}} = \infty \ .$$

Theorem 12. *Let f be a meromorphic function with non zero finite lower order and g be an entire function with non zero finite order. Also let h be an entire function such that $0 < \lambda_h^{L^*}(g) \le \rho_h^{L^*}(g) < \infty$. Then for every positive constant A and every real number α,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\left\{\log T_h^{-1} T_g(r^A)\right\}^{1+\alpha}} = \infty \ .$$

We omit the proof of Theorem 11 and Theorem 12 as those can be carried out in the line of Theorem 9 and Theorem 10 respectively.

Theorem 13. *Let f be a meromorphic function with non zero finite order and lower order. Also let g, h be two entire functions such that $0 < \lambda_h^{L^*}(f) \le \rho_h^{L^*}(f) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f(\exp(r^\mu))} = 0 \ \textit{if}\ \mu > \lambda_g \ .$$

Proof. If $1+\alpha \le 0$, then the theorem is obvious. We consider $1+\alpha > 0$.
Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 3 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &< \log T_h^{-1} T_f(\exp(r^\mu)) \\ \textit{i.e., } \log T_h^{-1} T_{f\circ g}(r) &< \left(\rho_h^{L^*}(f) + \varepsilon\right)[r^\mu + L(r^\mu)] \ . \end{aligned} \tag{7}$$

Again for all sufficiently large values of r, we get that

$$\log T_h^{-1} T_f(\exp(r^\mu)) \ge \left(\lambda_h^{L^*}(f) - \varepsilon\right)[r^\mu + L(r^\mu)] \ . \tag{8}$$

Hence for a sequence of values of r tending to infinity, we obtain from (7) and (8) that

$$\frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f(\exp(r^\mu))} \le \frac{(\rho_h(f) + \varepsilon)^{1+\alpha}[r^\mu + L(r^\mu)]^{1+\alpha}}{[r^\mu + L(r^\mu)]} \ , \tag{9}$$

So from (9), we obtain that

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f(\exp(r^\mu))} = 0 \ .$$

This proves the theorem. □

Theorem 14. *Let f be a meromorphic function with non zero finite order and lower order. Also let g, h be any two entire functions such that $\rho_h^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(g) > 0$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_g\left(\exp\left(r^{\mu}\right)\right)} = 0 \text{ if } \mu > \lambda_g .$$

The proof is omitted as it can be carried out in the line of Theorem 13.

Theorem 15. *Let f be a meromorphic function with finite order and g be an entire function with non zero finite lower order. Also let h be another entire function such that $\lambda_h^{L^*}(f) > 0$ and $\rho_h^{L^*}(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)} = 0 \text{ if } \mu > \lambda_g .$$

Theorem 16. *Let f be a meromorphic function with finite order and g be an entire function with non zero finite lower order. Also let h be another entire function such that $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_g\left(\exp\left(r^{\mu}\right)\right)} = 0 \text{ if } \mu > \lambda_g .$$

We omit the proof of Theorem 15 and Theorem 16 as those can be carried out in the line of Theorem 13 and Theorem 14 respectively with the help of Lemma 4.

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