*Electronic Journal of Mathematical Analysis and Applications*
Vol. 5(1) Jan. 2017, pp. 88-96.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

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# VALUE-SHARING AND UNIQUENESS OF ENTIRE FUNCTIONS

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Abstract. In this paper, we study the uniqueness of entire functions sharing a nonzero value and obtain some results improving the results obtained by Harina P. Waghamore and Tanuja A[[5]].

## 1. Introduction

In the present paper, meromorphic functions are always regarded as meromorphic in the entire complex plane. We use the standard notation of the Nevanlinna value-distribution theory, such as $T(r,f), N(r,f), \overline{N}(r,f), m(r,f)$ etc., as explained in Hayman [[6]], Yang [[8]], and Yi and Yang [[9]]. We denote by $S(r,f)$ any function such that $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set $r$ of finite linear measure.

Let $a$ be a finite complex number and let $k$ be a positive integer. By $E_{k)}(a,f)$, we denote the set of zeros of $f-a$ with multiplicities at most $k$, where each zero is counted according to its multiplicity. Also let $\overline{E}_{k)}(a,f)$ be the set of zeros of $f-a$ whose multiplicities are not greater than $k$ and each zero is counted only once. In addition, by $N_{(k}\left(r,\frac{1}{f-a}\right)\left(or\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)\right)$, we denote the counting function with respect to the set $E_{k)}(a,f)(or\overline{E}_{k)}(a,f))$.

We set

$$N_k\left(r,\frac{1}{f-a}\right) = \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$$

and define

$$\Theta(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}\left(r,\frac{1}{f-a}\right)}{T(r,f)}$$

and

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)},$$

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a finite complex number. We say that $f$ and $g$ share the value $a$ CM(counting multiplicities) if $f$ and $g$ have the same $a$-points with the same multiplicities. We also say that

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2010 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
*Key words and phrases.* Differential equations, stability, existence, uniqueness.
Submitted April 29, 2016.

$f$ and $g$ share the value $a$ IM(ignoring multiplicities) if we do not consider the multiplicities. We denote by $\overline{N}_L\left(r, \frac{1}{f-a}\right)$ the counting function for $a$-points of both $f$ and $g$ at which $f$ has larger multiplicity than $g$ (in the case where the multiplicities are not counted). Similarly, we have the notation $\overline{N}_L(r, \frac{1}{g-a})$. Further, by $N_0(r, \frac{1}{F'})$, we denote the counting function of those zeros of $F'$ that are not zeros of $F(F-1)$.

Recently, R. S. Dyavanal [[2]] proved the following theorems.

**Theorem A([[2]])**. Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where $c_1$, $c_2$ and $c$ are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

**Theorem B([[2]])**. Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying $(n-2)s \geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \ \ f = \frac{(n+2)(1-h^{n+1})h}{(n+1)(1-h^{n+2})}$$

where $h$ is a non-constant meromorphic function.

**Theorem C([[2]])**. Let $f$ and $g$ be two transcendental entire functions, whose zeros are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying $(n-2)s \geq 7$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

**Theorem D([[2]])**. Let $f$ and $g$ be two transcendental entire functions, whose zeros are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying $(n-2)s \geq 5$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.

In 2014, Harina P. Waghamore and Tanuja A.[[5]] ask whether there exists a corresponding unicity theorem for $[f^n P(f)]^{(k)}$ where $P(f)$ is a polynomial. In this paper, they gave a positive answer to above question by proving the following Theorems.

**Theorem E**. Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let $n$, $k$, $m$ be three positive integers with $s(n+m) > 4k+12$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 CM, then either $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = (n+m, ...n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

**Corollary 1**. Let $f$ and $g$ be two non-constant entire functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let $n$, $k$, $m$ be three positive integers with $s(n+m) > 2k+6$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 CM, then the conclusions of Theorem E hold.

**Theorem F**. Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let

$P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let $n$, $k$, $m$ be three positive integers with $s(n+m) > 9k+16$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 IM, then either $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = (n+m, ...n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

**Corollary 2**. Let $f$ and $g$ be two non-constant entire functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let $n$, $k$, $m$ be three positive integers with $s(n+m) > 5k+9$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 IM, then the conclusions of Theorem F hold.

In the present paper, we always use $L(z)$ to denote an arbitrary polynomial of degree $n$, i.e.,

$$L(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 = a_n (z - c_1)^{l_1} (z - c_2)^{l_2} ... (z - c_s)^{l_s} \quad (1)$$

where $a_i, i = 0, 1, ..., n, a_n \neq 0$, and $c_j, j = 1, 2, ..., s$, are finite complex number constants; $c_1, c_2, ..., c_s$ are all distinct zeros of $L(z), l_1, l_2, ..., l_s$. $s, n$ are all positive integers satisfying the equality

$$l_1 + l_2 + ... + l_s = n \text{ and } l = max\{l_1, l_2 ... l_s\} \quad (2)$$

In this paper, we study the existence of solutions for $[L(f)]^{(k)}$ and the corresponding uniqueness theorems. Thus, we obtain the following results as a generalization of the theorems presented above:

**Theorem 1.1.** Let $f(z)$ and $g(z)$ be two non constant entire functions and let $n, k$ and $l$ be three positive integers such that $4l > 3n+2k+8$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share 1 CM, then either $f = b_1 e^{bz} + c$ and $g = b_2 e^{-bz} + c$ or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $b_1, b_2$ and $b$ are three constants such that $(-1)^k (b_1 b_2)^n (nb)^{2k} = 1$ and $R(w_1, w_2) = L(w_1) - L(w_2)$.

**Remark 1.** Put $l = n$ in theorem 1.1, we get $n > 2k+4$.

**Theorem 1.2.** Let $f(z)$ and $g(z)$ be two non constant entire functions and let $n, k$ and $l$ be three positive integers such that $7l > 6n+5k+7$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share 1 IM, then either $f = b_1 e^{bz} + c$, and $g = b_2 e^{-bz} + c$ or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$. where $b_1, b_2$ and $b$ are three constants such that $(-1)^k (b_1 b_2)^n (nb)^{2k} = 1$ and $R(w_1, w_2) = L(w_1) - L(w_2)$.

**Remark 2.** Put $l = n$ in theorem 1.2, we get $n > 5k+7$.

**Remark 3.** If $L(f) \equiv L(g)$, then we get

$$a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f \equiv a_n g^n + a_{n-1} g^{n-1} + ... + a_1 g.$$

Let $h = \frac{f}{g}$. If $h$ is a constant, then we substitute $f = gh$ in this equation and obtain $a_n g^n (h^n - 1) + a_{n-1} g^{n-1} (h^{n-1} - 1) + ... + a_1 g (h - 1) \equiv 0$. This yields $h^d = 1, d = (n, ..., n-i, ...1)$, and $a_{n-i} \neq 0$ for some $i = 0, 1, ..., n-1$. Thus $f \equiv tg$ for a constant $t$ such that $t^d = 1$. If $h$ is not a constant, then by virtue of the equation presented above, we know that $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = L(w_1) - L(w_2)$.

## 2. Some Lemmas

**Lemma 2.1**([[6]]) Let $f$ be a non-constant meromorphic function, let $k$ be a positive integer, and let $c$ be a non-zero finite complex number. Then

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

**Lemma 2.2**([[6]]) Let $f(z)$ be a nonconstant meromorphic function and let $a_1(z)$ and $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f), i = 1, 2$. Then

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-a_1}) + \overline{N}(r,\frac{1}{f-a_2}) + S(r,f).$$

**Lemma 2.3**([[9]]) Let $a_n(\neq 0), a_{n-1}...a_0$ be constants and let $f$ be a nonconstant meromorphic function. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_0) = nT(r,f)$$

**Lemma 2.4**([[4]]) Let $f$ and $g$ be two transcendental entire functions, and let $k$ be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and

$$\Delta = [\Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g)] > 3$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

**Lemma 2.5**([[7]]) Let $f$ and $g$ be two transcendental entire functions, and let $k$ be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and

$$\Delta = [\Theta(0,f) + \Theta(0,g) + 2\delta_{k+1}(0,f) + 3\delta_{k+1}(0,g)] > 6$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

**Lemma 2.6**([[5]]) Let $f(z)$ be a nonconstant entire function and let $k(\geq 2)$ be a positive integer. If $ff^{(k)} \neq 0$, then $f = e^{az+b}$, where $a$ and $b$ are constants.

## 3. Proofs of the Theorems.

**Proof of Theorem 1.1.** Let $L(z)$ and $l$ be given by (1.1) and (1.2), respectively. Without loss of generality, we can assume that $a_n = 1$, $l = l_1$ and $c = c_1$. This yields

$$\begin{aligned} \Theta(0, L(f)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{L(f)})}{T(r, L(f))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{\sum_{j=1}^{s} \overline{N}(r, \frac{1}{f-c_j})}{nT(r,f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n} \end{aligned} \tag{3}$$

similarly, we get

$$\Theta(0, L(g)) \geq \frac{l-1}{n} \tag{4}$$

Moreover, we have

$$\begin{aligned}\delta_{k+1}(0,L(f)) &= 1-\limsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{L(f)})}{T(r,L(f))}\\ &\geq 1-limsup_{r\to\infty}\frac{\sum_{j=1}^{s}N_{k+1}(r,\frac{1}{(f-c_j)^{l_1}})+N_{k+1}(r,\frac{1}{(f-c)^{l}})}{nT(r,f)}\\ &\geq 1-\limsup_{r\to\infty}\frac{(s-1)T(r,f)+(k+1)T(r,f)+S(r,f)}{nT(r,f)}\\ &\geq 1-\frac{s+k}{n}\geq\frac{l-k-1}{n}\end{aligned}\tag{5}$$

and similarly

$$\delta_{k+1}(0,L(g))\geq\frac{l-k-1}{n}\tag{6}$$

Since $4l>3n+3k+8$, from (3.1) to (3.4), we get
$\Delta=[\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)]>3$ we conclude that $h(z)\equiv 0$, i.e.,

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}=\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}-2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}$$

Solving this equation, we obtain

$$\frac{1}{f^{(k)}-1}=\frac{bg^{(k)}+a-b}{g^{(k)}-1}.$$

we can write the above equation as

$$\frac{1}{L(f)^{(k)}-1}=\frac{bL(g)^{(k)}+a-b}{L(g)^{(k)}-1}.\tag{7}$$

Further, we consider the following three cases:
**Case I.** If $b\neq 0$ and $a=b$, then it follows from (3.9) that

$$\frac{1}{L(f)^{(k)}-1}=\frac{bL(g)^{(k)}}{L(g)^{(k)}-1}.\tag{8}$$

**1.1.** If $b\neq -1$, then it follows from (3.9) that $[L(f)^{(k)}][L(g)^{(k)}]\equiv 1$, i.e.,

$$[(f-c)^{l}(f-c)^{l_2}...(f-c_s)^{l_s}]^{(k)}[(g-c)^{l}(g-c)^{l_2}...(g-c_s)^{l_s}]^{(k)}=1\tag{9}$$

**1.1.1.** If $s=1$, then we can rewrite (3.11) as follows:

$$[(f-c)^{n}]^{(k)}[(g-c)^{n}]^{(k)}=1.$$

and $4l>3n+2k+4$, $l=n$, we conclude that $n>2k+4$. Hence, $f-c\neq 0$ and $g-c\neq 0$. Thus, according to Lemma 2.4 we find

$$f=b_1e^{bz}+c,\ \ g=b_2e^{-bz}+c,$$

where $b_1,b_2$ and $b$ are three constants such that $(-1)^k(b_1b_2)^n(nb)^{2k}=1$.
**1.1.2.** For $s\geq 2$, we note that $4l>3n+2k+4$. Hence, $l>2k+4$. Suppose that $z_0$ is an $l$-fold zero of $f-c$. We know that $z_0$ must be an $(l-k)$-fold zero of $[(f-c)^{l}(f-c)^{l_2}...(f-c_s)^{l_s}]^{(k)}$. Note that it follows from (3.9) that $g$ is an entire function. This is a contradiction. Hence, $f-c\neq 0$ and $g-c\neq 0$. Thus, we get $f=e^{\alpha(z)}+c$, where $\alpha(z)$ is a non constant entire function. Therefore,

$$[f^{i}]^{(k)}=[(e^{\alpha}+c)^{i}]^{(k)}=p_i(\alpha',\alpha'',...,\alpha^{(k)})e^{i\alpha},\ \ i=1,2,...,n,\tag{10}$$

where $p_i, i = 1, 2, ..., n$, are differential polynomials in $\alpha', \alpha'', ..., \alpha^{(k)}$. Clearly, if $p_i \not\equiv 0$ and $T(r, p_i) = S(r, f), i = 1, 2, ..., n$, then it follows from (3.11) and (3.12) that

$$N(r, \frac{1}{p_n e^{(n-1)\alpha} + ... + p_1}) = S(r, f).$$

In view of Lemmas 2.2 and 2.3 and the fact that $f = e^{\alpha} + c$, we get

$$\begin{aligned}(n-1)T(r, f-c) &= T(r, p_n e^{(n-1)\alpha} + ... + p_1) + S(r, f)\\ &\leq \overline{N}(r, \frac{1}{p_n e^{(n-1)\alpha} + ... + p_1}) + \overline{N}(r, \frac{1}{p_n e^{(n-1)\alpha} + ... + p_2 e^{\alpha}})\\ &\leq \overline{N}(r, \frac{1}{p_n e^{(n-2)\alpha} + ... + p_2}) + S(r, f)\\ &\leq (n-2)T(r, f-c) + S(r, f),\end{aligned}$$

which is a contradiction.

**1.2.** If $a = b \neq -1$, then relation (3.10) can be rewritten as

$$L(g)^{(k)} = \frac{-1}{b} . \frac{1}{L(f)^{(k)} - (1+b)/b}. \tag{11}$$

From (3.13), we get

$$\overline{N}(r, \frac{1}{L(f)^{(k)} - (1+b)/b}) = \overline{N}(r, g) = S(r, f). \tag{12}$$

By relation (3.14) and Lemma 2.1, we obtain

$$\begin{aligned}nT(r, f) &= T(r, L(f)) + O(1)\\ &\leq N_{k+1}(r, \frac{1}{L(f)}) + N(r, \frac{1}{L(f)^{(k)} - (1+b)/b}) + S(r, f)\\ &\leq N_{k+1}(r, \frac{1}{(f-c)^l}) + N_{k+1}(r, \frac{1}{(f-c_2)^{l_2} ... (f-c_s)^{l_s}}) + S(r, f)\\ &\leq (k+s)T(r, f) \leq (k+n-l+1)T(r, f) + S(r, f),\end{aligned}$$

which is a contradiction because $4l > 3n + 2k + 4$.

**Case II.** $b \neq 0$ and $a \neq b$. We discuss the following sub cases:

**2.1.** Suppose that $b = -1$. Then $a \neq 0$ and relation (3.9) can be rewritten as

$$L(f)^{(k)} = \frac{a}{a + 1 - L(g)^{(k)}}. \tag{13}$$

It follows from (3.15) that

$$\overline{N}(r, \frac{1}{a + 1 - L(g)^{(k)}}) = \overline{N}(r, f) = S(r, g). \tag{14}$$

In view of (3.16) and Lemma 2.1 and 2.4, we find

$$nT(r, g) = T(r, L(g)) + O(1) \leq N_{k+1}(r, \frac{1}{L(g)}) + S(r, g).$$

Further, by using the argument as in Case 1.2, we arrive at a contradiction.

**2.2.** suppose that $b \neq -1$. then relation (3.9) be rewritten as

$$L(f)^{(k)} - \frac{b+1}{b} = \frac{-a}{b^2} . \frac{1}{L(g)^{(k)} + (a-b)/b}. \tag{15}$$

It follows from (3.17) that

$$\overline{N}(r, \frac{1}{L(g)^{(k)} - (b+1)/b}) = \overline{N}(r, g). \tag{16}$$

By using (3.18) and Lemmas 2.1 and 2.4, we arrive at a contradiction in exactly the same way as in Case 1.2.

**Case III.** $b = 0$ and $a \neq 0$. Then relation (3.9) can be rewritten as

$$L(g)^{(k)} = aL(f)^{(k)} + (1-a), \tag{17}$$

$$L(g) = aL(f) + (1-a)p_1(z), \tag{18}$$

where $p_1$ is a polynomial with $degp_1 \leq k$. If $a \neq 1$, then $(1-a)p_1 \not\equiv 0$. Together with (3.20) and Lemma 2.2, this yields

$$\begin{aligned} nT(r,g) = T(r, L(g)) + O(1) &\leq \overline{N}(r, \frac{1}{L(g)}) + \overline{N}(r, \frac{1}{L(f)}) + S(r,g) \\ &\leq \sum_{i=1}^{s} \overline{N}(r, \frac{1}{g - c_i}) + \sum_{j=1}^{s} \overline{N}(r, \frac{1}{f - c_j}) + S(r,g) \\ &\leq s[T(r,f) + T(r,g)] + S(r,g). \end{aligned} \tag{19}$$

Since $n = l + l_2 + ... + l_s$, we get $n - l = l_2 + ... + l_s \geq s - 1$, i.e., $n - l \geq s - 1, n - s \geq l - 1$. In view of the inequality $4l > 3n + 2k + 4$, we conclude that

$$l - 1 > 3(n-l) + 2k + 4 > 3(s-1) + 2k + 3$$

and hence,

$$n - s \geq l - 1 > 3(s-l) + 2k + 3,$$

i.e., $n - s > 3(s-1) + 2k + 3$. Therefore,

$$s < \frac{n - 2k}{4}$$

and thus,

$$nT(r,g) < \frac{n-2k}{4}[T(r,g) + T(r,f)] + S(r,g). \tag{20}$$

On the other hand, it follows from (3.20) and Lemma 2.3 that

$$T(r,g) = T(r,f) + S(r,g).$$

Substituting this relation in (3.24), we conclude that

$$\frac{3n + 4k}{4} T(r,g) < S(r,g),$$

which is a contradiction.

Thus $a = 1$ and therefore, it follows from (3.20) that $L(f) = L(g)$.

Further, we consider the case where $f$ and $g$ are polynomials. Suppose that $f - c$ and $g - c$ have $u$ and $v$ pairwise distinct zeros, respectively. Then $f - c$ and $g - c$ admit the representations

$$f - c = k_1(z - a_1)^{n_1}(z - a_2)^{n_2}...(z - a_u)^{n_u},$$

$$g - c = k_2(z - b_1)^{m_1}(z - b_2)^{m_2}...(z - b_v)^{m_v},$$

and hence,

$$[f - c]^l = k_1^l(z - a_1)^{l_{n_1}}(z - a_2)^{l_{n_2}}...(z - a_u)^{l_{n_u}}, \tag{21}$$

$$[g - c]^l = k_2^l(z - b_1)^{l_{m_1}}(z - b_2)^{l_{m_2}}...(z - b_v)^{l_{m_v}}, \tag{22}$$

where $k_1$ and $k_2$ are nonzero constants, $n_i l > 2k+4$, $m_j l > 2k+4$, and $n_i, m_j, j = 1, 2, ..., u, j = 1, 2, ..., v$, are positive integers. Differentiating (3.20), we get

$$L(g)^{(k+1)} = aL(f)^{(k+1)}. \tag{23}$$

It follows from (3.23)(3.24) and (3.25) that

$$\begin{aligned}(z-a_1)^{l_{n_1}-k-1}(z-a_2)^{l_{n_2}-k-1}...(z-a_u)^{l_{n_u}-k-1}\xi_1(z)\\ = (z-b_1)^{l_{m_1}-k-1}(z-b_2)^{l_{m_2}-k-1}...(z-b_v)^{l_{m_v}-k-1}\xi_2(z),\end{aligned} \tag{24}$$

where $\xi_1$ and $\xi_2$ are polynomials, $deg\xi_1 = (n-l)\Sigma_{i=1}^{u} n_i + (u-1)(k+1)$, and $deg\xi_2 = (n-l)\Sigma_{j=1}^{v} m_j + (v-1)(k+1)$. Thus, in view of the fact that $4l > 3n+2k+4$, we find $3l-2n > (n-l)+2k+4 > 2k+4$. Then $(3l-2n)n_i > 2k+4$, $(3l-2n)m_j > 2k+4$, $i = 1, 2, ..., u, j = 1, 2, ..., v$. Hence,

$$\begin{aligned}&\Sigma_{i=1}^{u}[n_i l - (k+1)] - \Sigma_{i=1}^{v} n_i(n-l) = \Sigma_{i=1}^{u}[n_i(3l-2n) - (k+1)]\\ &> u(k+3) > (u-1)(k+1),\end{aligned}$$

i.e.,

$$\Sigma_{i=1}^{u}[n_i l - (k+1)] > (n-l)\Sigma_{i=1}^{v} n_i + (u-1)(k+1).$$

Similarly,

$$\Sigma_{j=1}^{v}[m_j l - (k+1)] > (n-l)\Sigma_{j=1}^{v} m_j + (v-1)(k+1).$$

Thus, by using (3.26), we show that there exists $z_0$ such that $L(f(z_0)) = L(g(z_0)) = 0$, where the multiplicity of $z_0$ is greater than $2k+4$. Together with (3.20), this yields $p_1(z) = 0$, which also proves the claim.
Therefore, it follows from (3.19) and (3.20) that $a = 1$ and, therefore, $L(f) \equiv L(g)$. This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.**

Let $f(z)$ and $l$ be given by (1.1) and (1.2), respectively. Without loss of generality, we can assume that $a_n = 1$, $l = l_1$ and $c = c_1$. This yields

$$\begin{aligned}\Theta(0, L(f)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{L(f)})}{T(r, L(f))}\\ &\geq 1 - \limsup_{r\to\infty} \frac{\sum_{j=1}^{s} \overline{N}(r, \frac{1}{f-c_j})}{nT(r, f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n}\end{aligned} \tag{25}$$

similarly, we get

$$\Theta(0, L(g)) \geq \frac{l-1}{n} \tag{26}$$

Moreover, we have

$$\delta_{k+1}(0, L(f)) \geq \frac{l-k-1}{n} \tag{27}$$

$$\delta_{k+1}(0, L(g)) \geq \frac{l-k-1}{n} \tag{28}$$

Since $(4k+14)l > (4k+13)n + 9k + 12$, from (3.23) to (3.36), we get

$$\Delta = [\Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G)] > 6$$

Proceeding as in the proof of the theorem 1.1, we get Theorem 1.2.

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