

ON A SAKAGUCHI TYPE CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH QUASI-SUBORDINATION IN THE SPACE OF MODIFIED SIGMOID FUNCTIONS

S.O. OLATUNJI, E.J. DANSU AND A. ABIDEMI

ABSTRACT. In this work, the authors introduce a new class $\mathcal{G}_q^\lambda(\Phi, s, b)$ of analytic functions associated with quasi-subordination in the space of modified sigmoid functions. The coefficient estimates including the relevant connection to the famous classical Fekete-Szegő inequality of functions belonging to the class were determined. Also, the improved results for the associated classes involving subordination and majorization were briefly discussed.

1. INTRODUCTION

Special function is an information process that is inspired by the way nervous system such as the brain processes information. It composed of large number of highly interconnected processing elements (neurons) working together to solve a specific task. This function has been overshadowed by other fields like real analysis, functional analysis, algebra, topology, differential equations and so on because it works in the same way the brain does. It can be learnt by examples and cannot be programmed to solve a specific task. The function is divided into three namely, ramp function, threshold function and the sigmoid function.

The most popular among all is the sigmoid function because of its gradient descent learning algorithm. Sigmoid function can be evaluated in different ways, most especially by truncated series expansion (for details see [2], [4], [5] and [6]).

The sigmoid function of the form

$$h(z) = \frac{1}{1 + e^{-z}} \quad (1)$$

is differentiable and has the following properties:

- (i) It outputs real numbers between 0 and 1.
- (ii) It maps a very large input domain to a small range of outputs.
- (iii) It never loses information because it is an injective function.
- (iv) It increases monotonically.

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With all the aforementioned properties, sigmoid function play a major role in geometric functions theory.

Let us denote by Γ the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in E) \quad (2)$$

which are analytic in the open unit disc $E = \{z : z \in E \text{ and } |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Further, denote by S the class of analytic normalized and univalent functions in E .

Recall that S^* and CV denote the class of starlike and convex functions of the form (2) which their geometric condition satisfies $Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$ and $Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0$. The two classes have been studied repeatedly by many researchers and their results are too numerous to discuss.

For two analytic functions, f and g , such that $f(0) = g(0)$, we say that f is subordinate to g in E and write $f(z) \prec g(z)$, $z \in E$, if there exists a Schwartz function $w(z)$ with $w(0) = 0$ and $|w(z)| \leq |z|$, $z \in E$ such that $f(z) = g(w(z))$, ($z \in E$). Furthermore, if the function g is univalent in E , then we have the following equivalence;

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(E) \subset g(E) \quad (3)$$

The definition can be found in [7].

Furthermore, f is said to be quasi-subordinate to g in E and written as $f(z) \prec_q g(z)$, $z \in E$, if there exists an analytic function $\varphi(z) \leq 1$ ($z \in E$) such that $\frac{f(z)}{\varphi(z)}$ is analytic in E and

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in E), \quad (4)$$

that is, there exists a Schwartz function $w(z)$ such that $f(z) = \varphi(z)g(w(z))$, $z \in E$. This concept of quasi-subordination is given by [8].

Also, it is observed that if $\varphi(z) = 1$, ($z \in E$), then the quasi-subordination \prec_q becomes the well-known subordination \prec , and for the Schwartz function $w(z) = z$ ($z \in E$), the quasi-subordination \prec_q becomes the majorization " \ll ". In this case,

$$f(z) \prec_q g(z) \Rightarrow f(z) = \varphi(z)g(z) \Rightarrow f(z) \ll g(z), \quad z \in E \quad (5)$$

and this concept of majorization was revealed by [9].

Recently, Frasin[1] introduced and studied coefficient inequalities for certain classes of Sakaguchi type functions, $f \in \Gamma$ where $s, b \in \mathbb{C}$ with $s \neq b$ and for some α , ($0 \leq \alpha < 1$) satisfies

$$Re \left\{ \frac{(s-b)zf'(z)}{f(sz) - f(bz)} \right\} > \alpha, \quad z \in E \quad (6)$$

Obradovic[10] introduced a class of functions $f \in \Gamma$ for which $0 < \lambda < 1$ satisfies the inequality that

$$Re \left\{ f'(z) \left(\frac{z}{f(z)} \right)^{1+\lambda} \right\} > 0, \quad z \in E \quad (7)$$

and referred such functions as functions of non-Bazilevic type.

We also denote by P , the class of functions ϕ analytic in E such that $\phi(0) = 1$ and $Re(\phi(z)) > 0$, $z \in E$.

Ma and Minda [3] defined a class of starlike functions by using the method of subordination, and studied a class $S^*(\phi)$ which is defined by

$$S^*(\phi) = \left\{ h \in \Gamma : \frac{zh'(z)}{h(z)} \prec \phi(z), z \in E \right\} \quad (8)$$

where $\phi \in P$ and $\phi(E)$ is symmetrical about the real axis and $\phi'(0) > 0$. Such function is called Ma and Minda starlike function with respect to ϕ .

A sharp bound of the functional $|a_3 - \mu a_2^2|$ for univalent functions $f \in \Gamma$ of the form (2) with real μ ($0 \leq \mu \leq 1$) was obtained by Fekete and Szegő [16] approach. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \Gamma$ with any complex μ is generally known as the classical Fekete-Szegő inequality. Researchers like [13], [14] and [15] just to mention but few have studied several subclasses of functions making use of Fekete-Szegő problem and the very interesting results obtained can be found in many literature.

In this work, we determine the coefficient estimates including a Fekete-Szegő inequality of the class $\mathcal{G}_q^\lambda(\Phi, s, b)$, $\mathcal{G}^\lambda(\Phi, s, b)$ and the class involving majorization. Our results are new in this direction and they give birth to many corollaries.

Following equation (6) and motivated by the earlier works in [17], we define and introduce a new class of functions by invoking quasi-subordination. For the purpose of our results, the following lemmas and definition are employed.

Lemma 1. *Let the Schwartz function $w(z)$ be given by*

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \dots, (z \in E) \quad (9)$$

then

$$|w_1| \leq 1, |w_2 - tw_1^2| \leq 1 + (|t| - 1)|w_1|^2 \leq \max\{1, |t|\} \quad (10)$$

where $t \in \mathbb{C}$ [11].

Lemma 2. *Let h be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \quad (11)$$

then $\Phi(z) \in P$, $|z| < 1$ where $\Phi(z)$ is a modified sigmoid function [2].

Lemma 3. *Let*

$$\Phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \quad (12)$$

then $|\Phi_{n,m}(z)| < 2$ [2].

Lemma 4. *Let $\Phi(z) \in P$ and it is starlike, then f is a normalized univalent function of the form (1) [2].*

Taking $m = 1$, Fadipe-Joseph et al. [2] remarked the following:

Remark 5. *Let*

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (13)$$

where $c_n = \frac{(-1)^n}{2n!}$, then $|c_n| \leq 2$, $n = 1, 2, 3, \dots$. This result is sharp for each n .

Definition 1. Let $\Phi(z) \in P$ be univalent, for $s \neq t$, $s, t \in \mathbb{C}$ and $\lambda \geq 0$, a function $f \in \Gamma$ is said to be in the class $\mathcal{G}_q^\lambda(\Phi, s, b)$ if

$$\left[f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda - 1 \right] \prec_q (\Phi(z) - 1), \quad z \in E \quad (14)$$

where powers are considered to be having only principal values.

From the definition in (14), it follows that $f \in \mathcal{G}_q^\lambda(\Phi, s, b)$ if and only if there exists an analytic function $\varphi(z)$ with $|\varphi(z)| \leq 1$, ($z \in E$) such that

$$\frac{f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda - 1}{\varphi(z)} \prec (\Phi(z) - 1) \quad (z \in E). \quad (15)$$

If in the subordination condition (15), $\varphi(z) = 1$, ($z \in E$), then the class $\mathcal{G}_q^\lambda(\Phi, s, b)$ is denoted by $\mathcal{G}^\lambda(\Phi, s, b)$ and the functions therein satisfy the condition that

$$f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda \prec \Phi(z), \quad z \in E. \quad (16)$$

2. MAIN RESULTS

Let $f \in \Gamma$ be of the form (2), then for $s \neq t$, $s, t \in \mathbb{C}$, $k \in \mathbb{N}$

$$f(sz) - f(bz) = (s-b)z + (s^2 - b^2)a_2 z^2 + (s^3 - b^3)a_3 z^3 + (s^4 - b^4)a_4 z^4 \dots \quad (17)$$

Hence, for $\lambda \geq 0$, we get

$$\left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda = [1 + (s+b)a_2 z + (s^2 + sb + b^2)a_3 z^2 + (s^3 + s^2 b + sb^2 + b^3)a_4 z^3 + \dots]^{-\lambda} \quad (18)$$

which gives

$$1 - \lambda(s+b)a_2 z + \lambda \left[\frac{\lambda+1}{2}(s+b)^2 a_2^2 - (s^2 + sb + b^2)a_3 \right] z^2 + \dots \quad (19)$$

Theorem 1. *Let $f \in \Gamma$ of the form (2) belong to the class $\mathcal{G}_q^\lambda(\Phi, s, b)$, then*

$$|a_2| \leq \frac{1}{2|2 - \lambda(s+b)|} \quad (20)$$

and for some $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{1}{2|3 - \lambda(s^2 + sb + b^2)|} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s+b)]^2} - \frac{\lambda \left[1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right]}{2 - \lambda(s+b)} \right) \right| (s+b) \right\} \quad (21)$$

and the result is sharp.

Proof. Let $f \in \mathcal{G}_q^\lambda(\Phi, s, b)$, then from (14), we have

$$f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda - 1 = \varphi(z)[\Phi(z) - 1], \quad z \in E \tag{22}$$

where the function $\Phi(z)$ is a modified sigmoid function given by

$$\Phi(z) = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{z^6}{64} + \frac{779z^7}{20160} - \dots, \tag{23}$$

the analytic function, $\varphi(z)$ in E , of the form

$$\varphi(z) = d_0 + d_1z + d_2z^2 + \dots \tag{24}$$

and Schwartz function $w(z)$ in (9). Therefore, the right-hand side of (22) gives

$$\begin{aligned} \varphi(z)[\Phi(w(z)) - 1] &= (d_0 + d_1z + d_2z^2 + \dots) \left[\frac{w_1}{2}z + \frac{w_2}{2}z^2 + \left(\frac{w_3}{2} - \frac{w_1^3}{24} \right)z^3 + \dots \right] \\ &= \frac{d_0w_1}{2}z + \left(\frac{w_2d_0}{2} + \frac{w_1d_1}{2} \right)z^2 + \dots \end{aligned} \tag{25}$$

Using the series expansion of $f'(z)$ from (2), and the expansion given by (19), we get

$$f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda - 1 = [2 - \lambda(s+b)]a_2z + \left[[3 - \lambda(s^2 + sb + b^2)]a_3 - \lambda \left(2 - \frac{1+\lambda}{2}(s+b) \right) (s+b)a_2^2 \right]z^2 + \dots \tag{26}$$

From the expansions (25) and (26), on equating the coefficients of z and z^2 in (22), we find that

$$[2 - \lambda(s+b)]a_2 = \frac{d_0w_1}{2} \tag{27}$$

$$[3 - \lambda(s^2 + sb + b^2)]a_3 - \lambda \left[2 - \frac{1+\lambda}{2}(s+b) \right] (s+b)a_2^2 = \frac{w_2d_0}{2} + \frac{w_1d_1}{2} \tag{28}$$

Now, (27) gives

$$a_2 = \frac{d_0w_1}{2[2 - \lambda(s+b)]} \tag{29}$$

which in view of (28) yields

$$[3 - \lambda(s^2 + sb + b^2)]a_3 = \frac{\lambda[4 - (1 + \lambda)(s+b)](s+b)}{8[2 - \lambda(s+b)]^2}d_0^2w_1^2 + \frac{1}{2}(w_2d_0 + w_1d_1) \tag{30}$$

and therefore

$$a_3 = \frac{1}{2[3 - \lambda(s^2 + sb + b^2)]} \left[d_1w_1 + d_0 \left(w_2 + \frac{d_0\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right) (s+b)}{4[2 - \lambda(s+b)]} w_1^2 \right) \right]. \tag{31}$$

For some $\mu \in \mathbb{C}$, we obtain from (29) and (31);

$$a_3 - \mu a_2^2 = \frac{1}{2[3 - \lambda(s^2 + sb + b^2)]} \left[d_1w_1 + w_2d_0 - \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s+b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right) (s+b)}{[2 - \lambda(s+b)]} \right) d_0^2w_1^2 \right]. \tag{32}$$

Since $\varphi(z)$ given by (24) is analytic and bounded in E , therefore, on using [7](p 172), we have for some y ($|y| \leq 1$):

$$|d_0| \leq 1 \quad \text{and} \quad d_1 = (1 - d_0^2)y. \tag{33}$$

On putting the value of d_1 from (33) into (32), we get

$$a_3 - \mu a_2^2 = \frac{1}{2[3 - \lambda(s^2 + sb + b^2)]} \left[yw_1 + w_2 d_0 - \left[\frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right) (s + b)}{2 - \lambda(s + b)} \right) w_1^2 + yw_1 \right] d_0^2 \right]. \quad (34)$$

If $d_0 = 0$ in (34), we at once get

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2 \left| 3 - \lambda(s^2 + sb + b^2) \right|} \quad (35)$$

But if $d_0 \neq 0$, let us then suppose that

$$F(d_0) := yw_1 + w_2 d_0 - \left[\frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right) (s + b)}{2 - \lambda(s + b)} \right) w_1^2 + yw_1 \right] d_0^2 \quad (36)$$

which is a polynomial in d_0 and hence analytic in $|d_0| \leq 1$, and maximum $|F(d_0)|$ is attained at $d_0 = e^{i\theta}$, ($0 \leq \theta < 2\pi$). We find that

$$\max_{0 \leq \theta < 2\pi} |F(e^{i\theta})| = |F(1)| \quad (37)$$

and

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2 \left| 3 - \lambda(s^2 + bs + b^2) \right|} \left| w_2 - \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right) (s + b)}{[2 - \lambda(s + b)]} \right) w_1^2 \right| \quad (38)$$

which on using Lemma 1 shows that

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2 \left| 3 - \lambda(s^2 + bs + b^2) \right|} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)} \right) (s + b)}{[2 - \lambda(s + b)]} \right) \right| \right\} \quad (39)$$

and this last above inequality together with (38) thus establishes the result in Theorem 1. This completes the proof. \square

For the case when $s = 1$;

Corollary 1. *Let $f \in \Gamma$ of the form (2) belong to the class $\mathcal{G}_q^\lambda(\Phi, 1, b)$, then*

$$\left| a_2 \right| \leq \frac{1}{2 \left| 2 - \lambda(1 + b) \right|} \quad (40)$$

and for some $\mu \in \mathbb{C}$,

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2 \left| 3 - \lambda(1 + b + b^2) \right|} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{2\mu[3 - \lambda(1 + b + b^2)]}{[2 - \lambda(1 + b)]^2} - \frac{\lambda \left(1 + \frac{2-(1+b)}{2-\lambda(1+b)} \right) (1 + b)}{[2 - \lambda(1 + b)]} \right) \right| \right\} \quad (41)$$

and the result is sharp.

Putting $b = -1$ in Corollary 1, we obtain;

Corollary 2. *Let $f \in \Gamma$ of the form (2) belong to the class $\mathcal{G}_q^\lambda(\Phi, 1, -1)$, then*

$$\left| a_2 \right| \leq \frac{1}{4} \quad (42)$$

and for some $\mu \in \mathbb{C}$,

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2 \left| 3 - \lambda \right|} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{2\mu(3 - \lambda)}{4} \right) \right| \right\} \quad (43)$$

and the result is sharp.

Setting $\lambda = 0$ in Corollary 2, we have

Corollary 3. Let $f \in \Gamma$ of the form (2) belong to the class $\mathcal{G}_q^\lambda(\Phi, 1, -1)$, then

$$|a_2| \leq \frac{1}{4} \quad (44)$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{6} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{3\mu}{2} \right) \right| \right\} \quad (45)$$

and the result is sharp.

Setting $\lambda = 1$ in Corollary 2, it gives

Corollary 4. Let $f \in \Gamma$ of the form (2) belong to the class $\mathcal{G}_q^\lambda(\Phi, 1, -1)$, then

$$|a_2| \leq \frac{1}{4} \quad (46)$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \max \left\{ 1, \left| \frac{1}{4} (\mu) \right| \right\} \quad (47)$$

and the result is sharp.

Theorem 2. Let $f \in \Gamma$ of the form (2) belong to the class $\mathcal{G}^\lambda(\Phi, s, b)$, then

$$|a_2| \leq \frac{1}{2 |2 - \lambda(s+b)|} \quad (48)$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{2 |3 - \lambda(s^2 + sb + b^2)|} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s+b)]^2} - \frac{\lambda \left(1 + \frac{2 - (s+b)}{2 - \lambda(s+b)} \right) (s+b)}{[2 - \lambda(s+b)]} \right) \right| \right\} \quad (49)$$

and the result is sharp.

Proof. Let $f \in \mathcal{G}^\lambda(\Phi, s, b)$. Similar to the proof of Theorem 1, if $\varphi(z) = 1$, then (24) evidently implies that $d_0 = 1$ and $d_n = 0$, $n \in \mathbb{N}$. Hence, in view of (29), (32) and Lemma 1, we obtain the desired result of Theorem 2. \square

The next result is devoted to the majorization, and the result pertaining to it is contained in the following.

Theorem 3. Let $s \neq b$, $s, b \in \mathbb{C}$. If a function $f \in \Gamma$ of the form (2) satisfies

$$\left[f'(z) \left(\frac{(s-b)z}{f(sz) - f(bz)} \right)^\lambda - 1 \right] \ll [\Phi(z) - 1], \quad z \in E \quad (50)$$

then

$$|a_2| \leq \frac{1}{2 |2 - \lambda(s+b)|} \quad (51)$$

and for some $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{2 |3 - \lambda(s^2 + sb + b^2)|} \max \left\{ 1, \left| \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s+b)]^2} - \frac{\lambda \left(1 + \frac{2 - (s+b)}{2 - \lambda(s+b)} \right) (s+b)}{[2 - \lambda(s+b)]} \right) \right| \right\} \quad (52)$$

and the result is sharp.

Proof. Following the proof of Theorem 1, if $w(z) \equiv z$ in (9) so that $w_1 = 1$ and $w_n = 0$, $n = 2, 3, \dots$, then in view of (29) and (32), we get

$$|a_2| \leq \frac{1}{2|2 - \lambda(s + b)|} \quad (53)$$

and

$$a_3 - \mu a_2^2 = \frac{1}{2[3 - \lambda(s^2 + sb + b^2)]} \left[d_1 - \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right) (s + b)}{[2 - \lambda(s + b)]} \right) d_0^2 \right]. \quad (54)$$

On putting the value of d_1 from (33) in (54), it is seen that

$$a_3 - \mu a_2^2 = \frac{1}{2[3 - \lambda(s^2 + sb + b^2)]} \left\{ y - \left[\frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right) (s + b)}{[2 - \lambda(s + b)]} \right) + y \right] d_0^2 \right\}. \quad (55)$$

If $d_0 = 0$ in (55), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{2|3 - \lambda(s^2 + sb + b^2)|} \quad (56)$$

and if $d_0 \neq 0$, let

$$G(d_0) := y - \left[\frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right) (s + b)}{[2 - \lambda(s + b)]} \right) + y \right] d_0^2 \quad (57)$$

which being a polynomial in d_0 is analytic in $|d_0| \leq 1$, and maximum of $|G(d_0)|$ is attained at $d_0 = e^{i\theta}$ ($0 \leq \theta < 2\pi$). We thus find that

$$\max_{0 \leq \theta < 2\pi} |G(e^{i\theta})| = |G(1)| \quad (58)$$

and consequently,

$$|a_3 - \mu a_2^2| \leq \frac{1}{2|3 - \lambda(s^2 + sb + b^2)|} \left| \frac{1}{4} \left(\frac{2\mu[3 - \lambda(s^2 + sb + b^2)]}{[2 - \lambda(s + b)]^2} - \frac{\lambda \left(1 + \frac{2-(s+b)}{2-\lambda(s+b)}\right) (s + b)}{[2 - \lambda(s + b)]} \right) \right| \quad (59)$$

which together with (56) establishes the desired result of Theorem 3. \square

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SUNDAY O. OLATUNJI

DEPARTMENT OF MATHEMATICAL SCIENCES, FEDERAL UNIVERSITY OF TECHNOLOGY, AKURE

E-mail address: olatunjiso@futa.edu.ng, olatfem_80@yahoo.com

EMMANUEL J. DANSU

DEPARTMENT OF MATHEMATICAL SCIENCES, FEDERAL UNIVERSITY OF TECHNOLOGY, AKURE

E-mail address: ejdansu@futa.edu.ng

AFEEZ ABIDEMI

DEPARTMENT OF MATHEMATICAL SCIENCES, FEDERAL UNIVERSITY OF TECHNOLOGY, AKURE

E-mail address: aabidemi@futa.edu.ng