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THE TOPOLOGICAL GROUPS OF TRIPLE ALMOST LACUNARY χ^3 SEQUENCE SPACES DEFINED BY A ORLICZ FUNCTION

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Abstract. In this paper we introduce a new concept for almost lacunary in topological groups of χ^3 sequence spaces strong $P-$ convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of almost lacunary in topological groups of χ^3 sequence spaces and also some inclusion theorems are discussed.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in *Apostol [1]* and double sequence spaces is found in *Hardy [5]*, *Subramanian et al. [10-12]*, and many others. Later on investigated by some initial work on triple sequence spaces is found in *Sahiner et al. [9]* , *Esi et al. [2-4]*, *Subramanian et al. [13-15]*, *Prakash et al. [16-19]*, *Deepmala et al. [21]*, *Mishra et al. [22-24]* and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m, n, k = 1, 2, 3, ...) .$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

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The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y) = sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \quad (1)$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\, \Gamma^3$. Let $\phi = \{finite\ sequences\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq}$ for all $m,n,k \in \mathbb{N}$,

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{bmatrix}$$

with 1 in the $(m,n,k)^{th}$ position and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The triple gai sequences will be denoted by χ^3.

2. Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P - limx = 0$) (i.e) $((m+n+k)!\,|x_{mnk}|)^{1/m+n+k} \to 0$ as $m,n,k \to \infty$. We shall write more briefly as $P - convergent\ \ to\ \ 0$.
By X , we will denote an abelian topological Hausdorff group, written additively which satisfies the first axiom of countability.

2.1. **Definition.** A Orlicz function was introduced by *Nakano [20]*. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0$, $y \ge 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

2.2. **Definition.** A triple sequence $x = (x_{mnk}) \in X$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$P - lim_{p,q,u\to\infty} sup_{r,s,t\ge 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} ((m+n+k)!\,|x_{mnk}|)^{1/m+n+k} \to 0.$$

that is, the average value of $(x_{mnk}) \in X$ taken over any rectangle
$\{(m,n,k) : r \le m \le r+p-1, s \le n \le s+q-1, t \le k \le t+u-1\}$ tends to 0 as

both p, q and u to ∞, and this $P-$ convergence is uniform in r, s and t. Let denote the set of sequences with this property as $\left[\widehat{\chi^3}\right](X)$.

2.3. **Definition.** The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by $I_{i,\ell,j} = \{(m, n, k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}, q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.

2.4. **Definition.** Let f be an Orlicz function and $P = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space: $\chi^3_f \left[AC_{\theta_{i,\ell,j}}, P\right](X) =$

$\left\{P - lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left[f\left((m+n+k)! \left|x_{m+r,n+s,k+t}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0, \right\}$, uniformly in r, s and t.

We shall denote $\chi^3_f \left[AC_{\theta_{i,\ell,j}}, P\right](X)$ as $\chi^3 \left[AC_{\theta_{i,\ell,j}}, P\right](X)$ respectively when $p_{mnk} = 1$ for all m, n and k If x is in $\chi^3 \left[AC_{\theta_{i,\ell,j}}, P\right](X)$, we shall say that x is almost lacunary χ^3 strongly $P-$convergent with respect to the Orlicz function f. Also note if $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\chi^3_f \left[AC_{\theta_{i,\ell,j}}, P\right](X) = \chi^3 \left[AC_{\theta_{i,\ell,j}}\right](X)$ which are defined as follows: $\chi^3 \left[AC_{\theta_{i,\ell,j}}\right](X) =$

$\left\{P - lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left[f\left((m+n+k)! \left|x_{m+r,n+s,k+t}\right|\right)^{1/m+n+k}\right] = 0, \right\}$, uniformly in r, s and t.

Again note if $p_{mnk} = 1$ for all m, n and k then $\chi^3_f \left[AC_{\theta_{i,\ell,j}}, P\right](X) = \chi^3_f \left[AC_{\theta_{i,\ell,j}}\right](X)$. we define $\chi^3_f \left[AC_{\theta_{i,\ell,j}}, P\right](X) =$

$\left\{P - lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m \in I_{k,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left[f\left((m+n+k)! \left|x_{m+r,n+s,k+t}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0, \right\}$, uniformly in r, s and t.

2.5. **Definition.** Let f be an Orlicz function $P = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space: $\chi^3_f [P](X) =$

$\left\{P - lim_{p,q,u \to \infty} \frac{1}{pqu} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{u} \left[f\left((m+n+k)! \left|x_{m+r,n+s,k+t}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0\right\}$, uniformly in r, s and t.

If we take $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\chi^3_f [P](X) = \chi^3(X)$.

2.6. **Definition.** Let $\theta_{i,\ell,j}$ be a triple lacunary sequence; the triple number sequence x is $\widehat{S_{\theta i,\ell,j}} - P-$ convergent to 0 then $P - lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} max_{r,s,t} \left|\left\{(m, n, k) \in I_{i,\ell,j} : f\left((m+n+k)! \left|x_{m+r,n+s,k+t} - 0\right|\right)^{1/m+n+k}\right\}\right| = 0$.

In this case we write $\widehat{S_{\theta i,\ell,j}} - lim \left(f(m+n+k)! \left|x_{m+r,n+s,k+t} - 0\right|\right)^{1/m+n+k} = 0$.

3. Main Results

3.1. **Theorem.** If f be any Orlicz function and a bounded factorable positive triple number sequence p_{mnk} then $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, P\right](X)$ is linear space
Proof: The proof is easy. Theorefore omit the proof.

3.2. **Theorem.** For any Orlicz function f, we have $\chi^3\left[AC_{\theta_{i,\ell,j}}\right](X) \subset \chi^3_f\left[AC_{\theta_{i,\ell,j}}\right](X)$
Proof: Let $x \in \chi^3\left[AC_{\theta_{i,\ell,j}}\right](X)$ so that for each r, s and u
$\chi^3\left[AC_{\theta_{i,\ell,j}}\right](X) =$
$\left\{lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]=0\right\}.$
Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \le t \le \delta$. We obtain the following,
$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon)+\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}\ and\ |x_{m+r,n+s,k+u}-0|>\delta} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]$
$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon)+\frac{1}{h_{i\ell j}}K\delta^{-1}f(2)\,h_{i\ell j}\,\chi^3\left[AC_{\theta_{i,\ell,j}}\right](X).$
Hence i, ℓ and j goes to infinity, for each r, s and u we are granted $x \in \chi^3_f\left[AC_{\theta_{i,\ell,j}}\right](X)$.

3.3. **Theorem.** Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence with $liminf_i q_i > 1,\ liminf_\ell \overline{q_\ell} > 1\ and\ liminf_j q_j > 1$ then for any Orlicz function f, $\chi^3_f(P)(X) \subset \chi^3_f\left(AC_{\theta_{i,\ell,j}}, P\right)(X)$
Proof: Suppose $liminf_i q_i > 1,\ liminf_\ell \overline{q_\ell} > 1$ and $liminf_j q_j > 1$ then there exists $\delta > 0$ such that $q_i > 1+\delta,\ \overline{q_\ell} > 1+\delta$ and $q_j > 1+\delta$ This implies $\frac{h_i}{m_i} \ge \frac{\delta}{1+\delta},\ \frac{h_\ell}{n_\ell} \ge \frac{\delta}{1+\delta}$ and $\frac{h_j}{k_j} \ge \frac{\delta}{1+\delta}$ Then for $x \in \chi^3_f(P)(X)$, we can write for each r, s and u.

$$B_{i,\ell,j} = \frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} =$$

$$\frac{1}{h_{i\ell j}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$$
$$\frac{1}{h_{i\ell j}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{i-1}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$$
$$\frac{1}{h_{i\ell j}}\sum_{m=m_{i-1}+1}^{m_i}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$$
$$\frac{1}{h_{i\ell j}}\sum_{k=k_j+1}^{k_j}\sum_{n=n_{\ell-1}+1}^{n_\ell}\sum_{m=1}^{m_{k-1}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$= \frac{m_i n_\ell k_j}{h_{i\ell j}}\left(\frac{1}{m_i n_\ell k_j}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$$
$$\frac{m_{k-1}n_{\ell-1}k_{j-1}}{h_{i\ell j}}\left(\frac{1}{m_{i-1}n_{\ell-1}kj-1}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$
$$- \quad \frac{k_{j-1}}{h_{i\ell j}}\left(\frac{1}{k_{j-1}}\sum_{m=m_{i-1}+1}^{m_i}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$
$$- \quad \frac{n_{\ell-1}}{h_{i\ell j}}\left(\frac{1}{n_{\ell-1}}\sum_{m=m_{k-1}+1}^{m_k}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$$
$$\frac{m_{k-1}}{h_{i\ell j}}\left(\frac{1}{m_{k-1}}\sum_{k=1}^{k_j}\sum_{n=n_{\ell-1}+1}^{n_\ell}\sum_{m=1}^{m_{k-1}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right).$$

Since $x \in \chi^3_f(P)(X)$ the last three terms tend to zero uniformly in m, n, k in the sense, thus, for each r, s and u

$$B_{i,\ell,j} = \frac{m_i n_\ell k_j}{h_{i\ell j}}\left(\frac{1}{m_i n_\ell k_j}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$$
$$\frac{m_{i-1}n_{\ell-1}k_{j-1}}{h_{i\ell j}}\left(\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} f\left[\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) +$$

$O(1)$.
Since $h_{i\ell j} = m_i n_\ell k_j - m_{i-1} n_{\ell-1} k_{j-1}$ we are granted for each r, s and u the following

$$\frac{m_i n_\ell k j}{h_{i\ell j}} \le \frac{1+\delta}{\delta} \text{ and } \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \le \frac{1}{\delta}.$$

The terms
$\left(\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$ and
$\left(\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$
are both gai sequences for all r, s and u. Thus $B_{i\ell j}$ is a gai sequence for each r, s and u. Hence $x \in \chi_f^3 \left(AC_{\theta_{i,\ell,j}}, P \right)(X)$.

3.4. **Theorem.** Let $\theta_{i,\ell,j} = \{m, n, k\}$ be a triple lacunary sequence with $limsup_\eta q_\eta < \infty$ and $limsup_i \overline{q_i} < \infty$ then for any Orlicz function f, $\chi_f^3 \left(AC_{\theta_{i,\ell,j}}, P \right)(X) \subset \chi_f^3 (p)(X)$.

Proof: Since $limsup_i q_i < \infty$ and $limsup_i \overline{q_i} < \infty$ there exists $H > 0$ such that $q_i < H$, $\overline{q_\ell} < H$ and $q_j < H$ for all i, ℓ and j. Let $x \in \chi_f^3 \left(AC_{\theta_{i,\ell,j}}, P \right)(X)$. Also there exist $i_0 > 0, \ell_0 > 0$ and $j_0 > 0$ such that for every $a \ge i_0$ $b \ge \ell_0$ and $c \ge j_0$ and r, s and u.

$$A'_{abc} = \frac{1}{h_{abc}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} f \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \to 0 \, as \, m, n, k \to \infty.$$

Let $G' = max \left\{ A'_{a,b,c} : 1 \le a \le i_0, \ 1 \le b \le \ell_0 \ and \ 1 \le c \le j_0 \right\}$ and p, q and t be such that $m_{i-1} < p \le m_i$, $n_{\ell-1} < q \le n_\ell$ and $m_{j-1} < t \le m_j$. Thus we obtain the following:

$$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$
$$\le \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$
$$\le \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$$
$$\left(\sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$
$$= \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} A'_{a,b,c} + \frac{1}{m_{k-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} h_{a,b,c} A'_{a,b,c}$$
$$\le \frac{G'}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} + \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le J)} h_{a,b,c} A'_{a,b,c}$$
$$\le \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \frac{1}{m_{i-1} n_{\ell-1} j_{j-1}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} h_{a,b,c} A'_{a,b,c}$$
$$\le \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \left(sup_{a \ge i_0 \bigcup b \ge \ell_0 \bigcup c \ge j_0} A'_{a,b,c} \right) \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} h_{a,b,c}$$
$$\le \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \frac{\epsilon}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} h_{a,b,c}$$
$$\le \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \epsilon H^3.$$

Since m_i, n_ℓ and k_j both approaches infinity as both p, q and t approaches infinity, it follows that

$$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} \left[\left((m+n+k)! \left| x_{m+r,n+s,k+u} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} = 0, \, uniformly \, in \, r, s \, and \, u.$$

Hence $x \in \chi_f^3 (P)(X)$.

3.5. **Theorem.** Let $\theta_{i,\ell,j}$ be a triple lacunary sequence then
(i) $(x_{mnk}) \in X \xrightarrow{P} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$
(ii)$\left(AC_{\theta_{i,\ell,j}}\right)$ is a proper subset of $\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$
(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \in X \xrightarrow{P} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$ then $(x_{mnk}) \in X \xrightarrow{P} \chi^3 \left(AC_{\theta_{i,\ell,j}}\right)(X)$
(iv) $\chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X) \bigcap \Lambda^3 = \chi^3 \left[AC_{\theta_{i,,\ell,j}}\right](X) \bigcap \Lambda^3$.

Proof: (i) Since for all r, s and u
$\left|\left\{(m,n,k) \in I_{i,\ell,j} : ((m+n+k)! \left|x_{m+r,n+s,k+u} - 0\right|)^{1/m+n+k}\right\} = 0\right| \leq$
$\sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j} \ and \ |x_{m+r,n+s,k+u}|=0} ((m+n+k)! \left|x_{m+r,n+s,k+u} - 0\right|)^{1/m+n+k} \leq$

$\sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} ((m+n)! \left|x_{m+r,n+s,k+u} - 0\right|)^{1/m+n+k}$, for all r, s and u
$P{-}lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} ((m+n+k)! \left|x_{m+r,n+s,k+u} - 0\right|)^{1/m+n+k} = 0$
This implies that for all r, s and u

$$P-lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left|\left\{(m,n,k) \in I_{i,\ell,j} : ((m+n+k)! \left|x_{m+r,n+s,k+u} - 0\right|)^{1/m+n+k} = 0\right\}\right| = 0.$$

(ii)let $x = (x_{mnk}) \in X$ be defined as follows:

$$(x_{mnk}) = \begin{bmatrix} 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots \\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots \\ . \\ . \\ . \\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots \\ . \\ . \\ . \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots \\ . \\ . \\ . \end{bmatrix};$$

Here x is an trible sequence and for all r, s and u
$P{-}lim_{i,\ell,j} \frac{1}{h_{k,\ell,j}} \left|\left\{(m,n,k) \in I_{i,\ell,j} : ((m+n+k)! \left|x_{m+r,n+s,k+u} - 0\right|)^{1/m+n+k} = 0\right\}\right| =$

$P - lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left(\frac{(m+n+k)! \left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k} = 0.$

Therefore $(x_{mnk}) \in X \xrightarrow{P} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$. Also
$P{-}lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} ((m+n+k)! \left|x_{m+r,n+s,k+u}\right|)^{1/m+n+k} =$

$P{-}\frac{1}{2} \left(lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left(\frac{(m+n+k)! \left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k} \left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k} \left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k} + 1\right) =$

$\frac{1}{2}$.

Therefore $(x_{mnk}) \in X \overset{P}{\nrightarrow} \chi^3 \left(AC_{\theta_{i,\ell,j}}\right)(X)$.

(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \in X \overset{P}{\to} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$ then $(x_{mnk}) \in X \overset{P}{\to} \chi^3 \left(AC_{\theta_{i,\ell,j}}\right)(X)$.

Suppose $x \in \Lambda^3$ then for all r, s and u, $((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k} \leq M$ for all m, n, k. Also for given $\epsilon > 0$ and i, ℓ and j large for all r, s and u we obtain the following:

$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} ((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k} =$

$\frac{1}{h_{i\ell j}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{k,\ell,j} \; and \; |x_{m+r,n+s,k+u}| \geq 0} ((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k} +$

$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \; and \; |x_{m+r,n+s,k+u}| \leq 0} ((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k}$

$\leq \frac{M}{h_{i\ell j}} \left| \left\{ (m,n,k) \in I_{i,\ell,j} : ((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k} \right\} = 0 \right| + \epsilon.$

Therefore $x \in \Lambda^3$ and $(x_{mnk}) \in X \overset{P}{\to} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$ then $(x_{mnk}) \in X \overset{P}{\to} \chi^3 \left(AC_{\theta_{i,\ell,j}}\right)(X)$.

(iv) $\chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X) \bigcap \Lambda^3 = \chi^3 \left[AC_{\theta_{i,\ell,j}}\right](X) \bigcap \Lambda^3$. follows from (i),(ii) and (iii).

3.6. **Theorem.** If f be any Orlicz function then $\chi^3_f \left[AC_{\theta_{i,\ell,j}}\right](X) \notin \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$

Proof: Let $x \in \chi^3_f \left[AC_{\theta_{i,\ell,j}}\right](X)$, for all r, s and u.

Therefore we have

$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f\left[((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k}\right] \geq$

$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \; and \; |x_{m+r,n+s,k+u}| = 0}$

$f\left[((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k}\right] >$

$\frac{1}{h_{i\ell j}} f(0) \left| \left\{ (m,n,k) \in I_{i,\ell,j} : ((m+n+k)!\,|x_{m+r,n+s,k+u} - 0|)^{1/m+n+k} \right\} = 0 \right|.$

Hence $x \in X \notin \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}}\right)(X)$.

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