Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) July 2016, pp. 259-265.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

A NOTE ON INTEGRAL TRANSFORMS ASSOCIATED WITH HUMBERT'S CONFLUENT HYPERGEOMETRIC FUNCTION

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Abstract. An expression in terms of the Humbert's confluent hypergeometric function of two variables is obtained for the integral transform involving the product of Bessel and Whittaker functions. Some derivations are given in the cases of some integral transforms corresponding to some special values of parameters and variables of Whittaker and Bessel functions.

1. Introduction

Many researchers (for example, [1],[2], [3], [7], [8], [9], [10], [12], [13], etc.) have studied a number of integral transforms involving a variety of special functions of mathematical physics. Such transforms play an important role in many diverse field of physics and engineering. As the integral transforms and special functions are indispensable in many branches of mathematics and applied mathematics, many researchers have studied their properties in many aspects, for example, Chun-Fang Li [6], Karimi et al. [14] and Belafhal and Hennani [4] introduced a new class of doughnut modified-Bessel-Gaussian vector beams with an amplitude of their transverse components given in terms of the modified Bessel functions. The propagation and the parametric characterization of laser beams including their beams quality have drawn a lot of attention (see [17],[18],[21]). A closed form expression in terms of the Humbert's confluent hypergeometric function of two variables Ψ_1 is derived for the integral transform involving the product of two Bessel functions.

Motivated by the above-mentioned work, in this paper, we establish a closed form of an integral transform involving the product of Bessel function J_μ and Whittaker function $M_{k,\nu}$ as follows:

$$I = \int_0^\infty x^{2s} e^{-\alpha x^2} J_\mu(\beta x) M_{k,\nu}(2\gamma x^2)\, dx, \tag{1}$$

whenever the improper integral converges.

2010 *Mathematics Subject Classification.* 33B15, 33C10, 33C15.

Key words and phrases. Bessel functions, Whittaker function, Humbert's confluent hypergeometric functions and Gamma function.

Submitted Sep. 21, 2015.

For specific values of $\alpha,\ s,\ \mu,\ k$ and ν, the above transform reduced to some integral transforms involving modified Bessel function, Laguerre polynomial, Hermite polynomial, exponential function, sine function and cosine function.

The Bessel function $J_\nu(z)$ of the first kind (and order ν), defined by (see [16], [19]):

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\ (z/2)^{\nu+2m}}{m!\ \Gamma(\nu+m+1)}\ (z \in C \setminus (-\infty, 0)). \tag{2}$$

It is well known that

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\ \cos z \tag{3}$$

and

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\ \sin z\ . \tag{4}$$

The Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ were introduced by Whittaker [22] (see also Whittaker and Watson [23]) in terms of confluent hypergeometric function ${}_1F_1$ (or Kummer's functions):

$$M_{k,\mu}(z) = \ z^{\mu+\frac{1}{2}}\ e^{-z/2}\ {}_1F_1\left(\frac{1}{2}+\mu-k\ ; 2\mu+1\ ; z\right). \tag{5}$$

and

$$W_{k,\mu}(z) = \ z^{\mu+\frac{1}{2}}\ e^{-z/2}\ U\left(\frac{1}{2}+\mu-k\ ; 2\mu+1\ ; z\right). \tag{6}$$

However the confluent hypergeometric function disappears when 2μ is an integer, so whittaker functions are often defined instead. The whittaker functions are related to the parabolic cylinder functions.

When $\mid arg(z) \mid < \frac{3\pi}{2}$ and 2μ is not an integer,

$$W_{k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)}\ M_{k,\mu(z)} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} M_{k,-\mu}(z). \tag{7}$$

When $\mid arg(-z) \mid < \frac{3\pi}{2}$ and 2μ is not an integer,

$$W_{-k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)}\ M_{-k,\mu(-z)} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu+k)} M_{-k,-\mu}(-z). \tag{8}$$

Here we recall the relation of Whittaker function with some other special functions which are given as follows :

$$M_{k,-k-\frac{1}{2}}(z) = \ e^{\frac{z}{2}}\ z^{-k}. \tag{9}$$

$$M_{0,\nu}(2z) = \ 2^{2\nu+\frac{1}{2}}\ \Gamma(1+\nu)\ \sqrt{z}\ I_\mu(z), \tag{10}$$

where $I_\mu(z)$ is Modified Bessel function (see [16], [19]).

$$M_{0,\frac{1}{2}}(2z) = \ 2\sinh z. \tag{11}$$

$$M_{\frac{p}{2}+\frac{1}{2}+q,\frac{p}{2}}(z) = \frac{m!}{(p+1)_q} e^{-\frac{z}{2}} z^{\frac{p}{2}+\frac{1}{2}} L_q^p(z), \tag{12}$$

where $L_q^p(z)$ is the generalized Laguerre polynomial (see [16], [19]).

$$M_{\frac{1}{4}+p,-\frac{1}{4}}(z^2) = (-1)^p \frac{p!}{2p!} e^{\frac{-z^2}{2}} \sqrt{z} H_{2p}(z), \tag{13}$$

$$M_{\frac{3}{4}+p,\frac{1}{4}}(z^2) = (-1)^p \frac{p!}{(2p+1)!} \frac{e^{\frac{-z^2}{2}} \sqrt{z}}{2} H_{2p+1}(z) . \tag{14}$$

where $H_p(z)$ is the generalized Hermite polynomial (see [16], [19]).

2. Main result

This section deals with an integral transform involving the product of Bessel and Whittaker functions, which is expressed in terms of Humbert's confluent hypergeometric function of two variables.

Theorem 2.1. The following transformation holds true:

$$\int_0^\infty x^{2s} e^{-\alpha x^2} J_\mu(\beta x) M_{k,\nu}(2\gamma x^2)\, dx = (\beta)^\mu (\gamma)^{\nu+\frac{1}{2}} \left(\frac{1}{2}\right)^{\mu-\nu+\frac{1}{2}}$$
$$\times \left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}+\nu+1} \frac{\Gamma(s+\nu+\frac{\mu}{2}+1)}{\Gamma(\mu+1)}$$
$$\times \Psi_1\left(s+\nu+\frac{\mu}{2}+1, \nu-k+\frac{1}{2}; \mu+1; 2\nu+1; \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)}\right), \tag{15}$$

where $\Re(\mu) > -1$, $\Re(s+\nu+\frac{\mu}{2}) > -1$, $\Re(\alpha+\gamma) > 2\gamma$ and Ψ_1 denotes one of the Humbert's confluent hypergeometric function of two variables defined as follows (see [15]):

$$\Psi_1(a, b; c, c'; w, z) = \sum_{k=0}^\infty \sum_{p=0}^\infty \frac{(a)_{k+p}(b)_k}{(c)_k(c')_p} \frac{w^k}{k!} \frac{z^p}{p!},$$

with $|w| < 1$, $|z| < \infty$.

Proof. In order to derive the result (15), we denote the left-hand side of (15) by I, expending J_μ and $M_{k,\nu}$ as a series with the help of (2) and (5) and then changing the order of summation and integration, which is guaranteed under the conditions, we arrive at

$$I = (2\gamma)^{\nu+\frac{1}{2}} \left(\frac{\beta}{2}\right)^\mu \sum_{m=0}^\infty \frac{\left(\frac{-\beta^2}{4}\right)^m}{m!\, \Gamma(1+m+\mu)} A_m, \tag{16}$$

where

$$A_m = \int_0^\infty x^{2(s+\nu+\frac{\mu}{2}+m+\frac{1}{2})} e^{-(\alpha+\gamma)x^2} {}_1F_1\left(\frac{1}{2}+\nu-k\ ; 2\nu+1\ ; 2\gamma x^2\right) dx . \tag{17}$$

Using the result ([11], p.815, Eq.7.522)

$$\int_0^\infty x^{\sigma-1}e^{-\mu x}\ {}_mF_n(\alpha_1,\alpha_2,\cdots,\alpha_m\ ;\beta_1,\beta_2,\cdots,\beta_n;\lambda x)$$

$$=\Gamma(\sigma)\ \mu^{-\sigma}\ {}_{m+1}F_n\left(\alpha_1,\alpha_2,\cdots,\alpha_m,\sigma\ ;\beta_1,\beta_2,\cdots,\beta_n;\frac{\lambda}{\mu}\right) \tag{18}$$

(with $m\le n,\ \Re(\sigma)>0,\ \Re(\mu)>0$, if $m<n;\ \Re(\mu)>\lambda$), in (17), we obtain

$$A_m=\frac{1}{2}\Gamma(s+\nu+\frac{\mu}{2}+m+1)(\alpha+\gamma)^{-(s+\nu+\frac{\mu}{2}+m+1)}$$

$$\times\ {}_2F_1\left(\nu-k+\frac{1}{2},s+\nu+\frac{\mu}{2}+m+1;2\nu+1;\frac{2\gamma}{\alpha+\gamma}\right) \tag{19}$$

Substituting (19) in (16), we obtain

$$I=(\gamma)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}-\nu}\left(\frac{\gamma}{\alpha+\gamma}\right)^{\nu}\left(\frac{\beta}{2}\right)^{\mu}\left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}+1}\sum_{m=0}^{\infty}\frac{\left[\frac{-\beta^2}{4(\alpha+\gamma)}\right]^m}{m!}$$

$$\times\ \frac{\Gamma(s+\nu+\frac{\mu}{2}+m+1)}{\Gamma(\mu+1)(\mu+1)_m}\ {}_2F_1\left(\nu-k+\frac{1}{2},s+\nu+\frac{\mu}{2}+m+1;2\nu+1;\frac{2\gamma}{\alpha+\gamma}\right). \tag{20}$$

Now expanding ${}_2F_1$ in its defining series and then arranging the resulting expression in terms of Humbert's confluent hypergeometric function of two variables Ψ_1, we get the required result. This completes the proof.

3. Special cases

In this section, we derive a known and some (presumably) new transforms involving exponential function, Modified Bessel function, Laguerre polynomial, Hermite polynomials and sine hyperbolic function.

Corollary 3.1. The following transformation holds true:

$$\int_0^\infty x^{2s-2k}\ e^{(\gamma-\alpha)x^2}\ J_\mu(\beta x)dx=(\beta)^\mu\ \left(\frac{1}{2}\right)^{\mu+k+1}\left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}-k+\frac{1}{2}}$$

$$\times\ \frac{\Gamma(s+\nu+\frac{\mu}{2}+1)}{\Gamma(\mu+1)}\ F^{1:\ 0:\ 0}_{0:\ 1;\ 0}\left[\begin{array}{ccc} s-k+\frac{\mu}{2}+\frac{1}{2}: & \_\,; & \_\,; \\ \_\,: & \mu+1; & \_\,; \end{array}\ \frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right], \tag{21}$$

where $\Re(\mu)>-1$, $\Re(s+\nu+\frac{\mu}{2})>-1$ and $F^{A:B;D}_{E:G;H}(x,y)$ is the Kampé de Fériet function [19].

This corollary can be established by taking $\nu=-k-\frac{1}{2}$ in (15) and then using the result (9).

Corollary 3.2. The following transformation holds true:

$$\int_0^\infty x^{2s+1}\ e^{-\alpha x^2}\ J_\mu(\beta x) I_\nu(\gamma x^2) dx = (\beta)^\mu\ (\gamma)^\nu\ \left(\frac{1}{2}\right)^{\mu+\nu+1}$$

$$\times\ \left(\frac{1}{\alpha+\gamma}\right)^{s+\nu+\frac{\mu}{2}+1} \frac{\Gamma(s+\nu+\frac{\mu}{2}+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}$$

$$\times\, \Psi_1\left(s+\nu+\frac{\mu}{2}+1, \nu+\frac{1}{2}; 2\nu+1; \mu+1; \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)}\right), \tag{22}$$

where $\Re(\mu) > -1$, $\Re(\nu) > -1$ and $\Re(s+\nu+\frac{\mu}{2}) > -1$.

This corollary can be established by replacing s by $s-\frac{1}{2}$, $k=0$ in (15) and then using the result (10). Also, it is noticed that the above transformation is the known result of Belafhal and Hennani [4].

Corollary 3.3. The following transformation holds true:

$$\int_0^\infty x^{2s}\ e^{-\alpha x^2}\ J_\mu(\beta x)\ \sinh(\gamma x^2) dx = (\beta)^\mu\ (\gamma)\ \left(\frac{1}{2}\right)^{\mu+\frac{1}{2}} \left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}+\frac{3}{2}}$$

$$\times\ \frac{\Gamma(s+\frac{\mu}{2}+\frac{3}{2})}{\Gamma(\mu+1)}\ \Psi_1\left(s+\frac{\mu}{2}+\frac{3}{2},\ 1;\ 2; \mu+1; \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)}\right), \tag{23}$$

where $\Re(\mu) > -1$ and $\Re(s+\frac{\mu}{2}) > -\frac{3}{2}$.

This corollary can be established by setting $k=0$, $\nu=\frac{1}{2}$ in (15) and then using the result (11).

Corollary 3.4. The following transformation holds true:

$$\int_0^\infty x^{2s+p+1}\ e^{-(\alpha+\gamma)x^2}\ J_\mu(\beta x)\ L_q^p(2\gamma x^2) dx = (\beta)^\mu\ \frac{(p+1)_q}{q!} \left(\frac{1}{2}\right)^{\mu+1}$$

$$\times\ \left(\frac{1}{\gamma}\right)^{\frac{p}{2}-\nu} \left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}+\nu+1} \frac{\Gamma(s+\frac{p}{2}+\frac{\mu}{2}+1)}{\Gamma(\mu+1)}$$

$$\times\ \Psi_1\left(s+\frac{p}{2}+\frac{\mu}{2}+1, -q\ ;\ p+1; \mu+1; \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)}\right), \tag{24}$$

where $\Re(\mu) > -1$, $\Re(s+\frac{p}{2}+\frac{\mu}{2}) > -1$ and $L_q^p(z)$ is the generalized Laguerre polynomial [16].

The above corollary can be established by setting $k=\frac{p}{2}+\frac{1}{2}+q$ (q is non negative integer), $\nu=\frac{p}{2}$ in (15) and then using the result (12).

Corollary 3.5. The following transformation holds true:

$$\int_0^\infty x^{2s+\frac{1}{2}}\ e^{-(\alpha+\gamma)x^2}\ J_\mu(\beta x)\ H_{2p}\sqrt{(2\gamma x^2)}\ dx = (-1)^{-p}\ \frac{2p!}{p!}\ (\beta)^\mu$$

$$\times\ \left(\frac{1}{2}\right)^{\mu+1} \left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}+\frac{3}{4}}\ \frac{\Gamma(s+\frac{\mu}{2}+\frac{3}{4})}{\Gamma(\mu+1)}$$

$$\times \Psi_1\left(s+\frac{\mu}{2}+\frac{3}{4},-p\ ;\ \frac{1}{2};\mu+1;\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right), \tag{25}$$

where $\Re(\mu)>-1$, $\Re(p)>-\frac{1}{2}$, $\Re(s+\frac{\mu}{2})>-\frac{3}{4}$ and $H_p(z)$ is the generalized Hermite polynomial [16].

The above corollary can be established by setting $k=\frac{1}{4}+p$, $\nu=-\frac{1}{4}$ in (15) and then using the result (13).

Corollary 3.6. The following transformation holds true:

$$\int_0^\infty x^{2s+\frac{1}{2}}\ e^{-(\alpha+\gamma)x^2}\ J_\mu(\beta x)\ H_{2p+1}\sqrt{(2\gamma x^2)}\ dx=(-1)^{-p}\ \frac{(2p+1)!}{p!}\ (\beta)^\mu\ \left(\frac{1}{2}\right)^{\mu-\frac{1}{2}}$$

$$\times\ \left(\frac{1}{\gamma}\right)^{-\frac{1}{2}}\left(\frac{1}{\alpha+\gamma}\right)^{s+\frac{\mu}{2}+\frac{5}{4}}\ \frac{\Gamma(s+\frac{\mu}{2}+\frac{5}{4})}{\Gamma(\mu+1)}$$

$$\times \Psi_1\left(s+\frac{\mu}{2}+\frac{5}{4},-p\ ;\ \frac{3}{2};\mu+1;\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right), \tag{26}$$

where $\Re(\mu)>-1$, $\Re(p)>-1$, $\Re(s+\frac{\mu}{2})>-\frac{5}{4}$.

The above corollary can be established by setting $k=\frac{3}{4}+p$, $\nu=\frac{1}{4}$ in (15) and then using the result (14).

4. Concluding Remark

We have derived the following close form expression of Belafhal and Hennani [4]:

$$I=\int_0^\infty x^{2s}e^{-\alpha x^2}J_\mu(\beta x)M_{k,\nu}(2\gamma x^2)\ dx,$$

from which we have deduced some important integral transforms for special values of the parameters. The results presented in this paper are (presumably) new, general in character and likely to find certain applications in the theory of special functions.

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