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CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH CERTAIN INTEGRAL OPERATOR

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Abstract. In this paper, we introduce two classes of analytic functions associated with certain integral operator and investigate convolution properties, coefficient estimates and inclusion properties of these classes.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}(\gamma)$ and $\mathcal{K}(\gamma)$ $(0 \leq \gamma < 1)$ denote the subclasses of $\mathcal{A}$ that consists, respectively, of starlike of order γ and convex of order γ in $\mathbb{U}$. It is well-known that $\mathcal{S}(\gamma) \subset \mathcal{S}(0) = \mathcal{S}$ and $\mathcal{K}(\gamma) = \mathcal{K}(0) = \mathcal{K}$ (see [10]).

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence, (cf., e.g., [8] and [9]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

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Making use of the principal of subordination between analytic functions, we introduce the subclasses $\mathcal{S}[A,B]$ and $\mathcal{K}[A,B]$ of the class $\mathcal{A}$ for $-1 \le B < A \le 1$ (see [1], [4], [5], [6] and [11]) which are defined by

$$\mathcal{S}[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}) \right\}, \tag{4}$$

and

$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : \frac{\left(zf'(z)\right)'}{f'(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}) \right\}. \tag{5}$$

We note that

$$\mathcal{S}[1-2\gamma,-1] = \mathcal{S}(\gamma), \ \mathcal{K}[1-2\gamma,-1] = \mathcal{K}(\gamma) \ (0 \le \gamma < 1).$$

Jung et al. [7] introduced the integral operator $Q_\beta^\alpha : \mathcal{A} \to \mathcal{A}$ as follows:

$$Q_\beta^\alpha f(z) = \begin{cases} \binom{\alpha+\beta}{\beta} \dfrac{\alpha}{z^\beta} \int_0^z \left(1-\dfrac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt & (\alpha > 0; \beta > -1), \\ f(z) & (\alpha = 0; \beta > -1). \end{cases} \tag{6}$$

For $f \in \mathcal{A}$ given by (1), then from (6), we deduce that

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{k=2}^{\infty} \frac{\Gamma(\beta+k)}{\Gamma(\alpha+\beta+k)} a_k z^k \ \ (\alpha \ge 0; \beta > -1). \tag{7}$$

It is easily verified from the definition (7) that (see [7])

$$z\left(Q_\beta^\alpha f(z)\right)' = (\alpha+\beta)\, Q_\beta^{\alpha-1} f(z) - (\alpha+\beta-1)\, Q_\beta^\alpha f(z). \tag{8}$$

Next, by using the integral operator Q_β^α, we introduce the following classes of analytic functions for $\alpha \ge 0; \beta > -1$ and $-1 \le B < A \le 1$:

$$\mathcal{S}_\beta^\alpha[A,B] = \left\{ f \in \mathcal{A} : Q_\beta^\alpha f(z) \in \mathcal{S}[A,B] \right\}, \tag{9}$$

and

$$\mathcal{K}_\beta^\alpha[A,B] = \left\{ f \in \mathcal{A} : Q_\beta^\alpha f(z) \in \mathcal{K}[A,B] \right\}. \tag{10}$$

We also note that

$$f(z) \in \mathcal{K}_\beta^\alpha[A,B] \Leftrightarrow zf'(z) \in \mathcal{S}_\beta^\alpha[A,B]. \tag{11}$$

In this paper, we investigate convolution properties for functions belongs to the classes $\mathcal{S}_\beta^\alpha[A,B]$ and $\mathcal{K}_\beta^\alpha[A,B]$ associated with the integral operator Q_β^α. Using convolution properties, we find the necessary and sufficient conditions, coefficient estimates and inclusion properties for these classes.

2. Convolution Properties

Unless otherwise mentioned, we assume throughout this paper that $0 < \theta < 2\pi$, $-1 \le B < A \le 1$, $\alpha \ge 0$ and $\beta > -1$.

Lemma 1 [2]**.** The function f defined by (1) is in the class $\mathcal{S}[A,B]$ if and only if

$$\frac{1}{z}\left[f(z) * \frac{z - \frac{e^{-i\theta}+A}{A-B} z^2}{(1-z)^2} \right] \ne 0 \ \ (z \in \mathbb{U}). \tag{12}$$

Lemma 2 [2]**.** The function $f(z)$ defined by (1) is in the class $\mathcal{K}[A,B]$ if and only if

$$\frac{1}{z}\left[f(z)*\frac{z-\frac{2\mathrm{e}^{-i\theta}+A+B}{A-B}z^2}{(1-z)^3}\right]\neq 0 \quad (z\in\mathbb{U}). \tag{13}$$

Theorem 1. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{S}_\beta^\alpha[A,B]$ is that

$$1-\sum_{k=2}^{\infty}\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}a_kz^{k-1}\neq 0 \ \ (z\in\mathbb{U}). \tag{14}$$

Proof. From Lemma 1, we find that $f\in\mathcal{S}_\beta^\alpha[A,B]$ if and only if

$$\frac{1}{z}\left[Q_\beta^\alpha f(z)*\frac{z-\frac{\mathrm{e}^{-i\theta}+A}{A-B}z^2}{(1-z)^2}\right]\neq 0 \ \ (z\in\mathbb{U}). \tag{15}$$

From (8), the left hand side of (15) may be written as

$$\begin{aligned}&\frac{1}{z}\left[Q_\beta^\alpha f(z)*\left(\frac{z}{(1-z)^2}-\frac{\mathrm{e}^{-i\theta}+A}{A-B}\frac{z^2}{(1-z)^2}\right)\right]\\ =\ &\frac{1}{z}\left[z\left(Q_\beta^\alpha f(z)\right)'-\frac{\mathrm{e}^{-i\theta}+A}{A-B}\left\{z\left(Q_\beta^\alpha f(z)\right)'-Q_\beta^\alpha f(z)\right\}\right]\\ =\ &1-\sum_{k=2}^{\infty}\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}a_kz^{k-1}.\end{aligned}$$

Thus, the proof of The Theorem 1 is completed. □

Theorem 2. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{K}_\beta^\alpha[A,B]$ is that

$$1-\sum_{k=2}^{\infty}k\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}a_kz^{k-1}\neq 0 \quad (z\in\mathbb{U}). \tag{16}$$

Proof. From Lemma 1, we find that $f\in\mathcal{K}_\beta^\alpha[A,B]$ if and only if

$$\frac{1}{z}\left[Q_\beta^\alpha f(z)*\frac{z-\frac{2\mathrm{e}^{-i\theta}+A+B}{A-B}z^2}{(1-z)^3}\right]\neq 0 \qquad (z\in\mathbb{U}). \tag{17}$$

From (8), the left hand side of (17) may be written as

$$\begin{aligned}&\frac{1}{z}\left[Q_\beta^\alpha f(z)*\left(\frac{z}{(1-z)^3}-\frac{2\mathrm{e}^{-i\theta}+A+B}{A-B}\frac{z}{(1-z)^3}\right)\right]\\ =\ &\frac{1}{z}\left[\frac{z}{2}\left(zQ_\beta^\alpha f(z)\right)''-\frac{2\mathrm{e}^{-i\theta}+A+B}{2(A-B)}z^2\left(Q_\beta^\alpha f(z)\right)''\right]\\ =\ &1-\sum_{k=2}^{\infty}k\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}a_kz^{k-1},\end{aligned}$$

this proves Theorem 2. □

3. Coefficient Estimates and Inclusion Properties

In this section, we determine coefficient estimates and inclusion properties for a function of the form (1) to be in the classes $\mathcal{S}_{\beta}^{\alpha}[A,B]$ and $\mathcal{K}_{\beta}^{\alpha}[A,B]$.

Theorem 3. If the function f defined by (1) belongs to the class $\mathcal{S}_{\beta}^{\alpha}[A,B]$, then

$$\sum_{k=2}^{\infty}(k-1+A-kB)\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}|a_k|\leq A-B. \tag{18}$$

Proof. Since

$$\left|1-\sum_{k=2}^{\infty}\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}a_k z^{k-1}\right|$$
$$>1-\sum_{k=2}^{\infty}\left|\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\right|\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}|a_k|,$$

and

$$\left|\frac{(k-1)\mathrm{e}^{-i\theta}-A+kB}{A-B}\right|=\frac{\left|(k-1)\mathrm{e}^{-i\theta}-A+kB\right|}{A-B}\leq\frac{k-1+A-kB}{A-B},$$

the result follows from Theorem 1. □

Similarly, we can prove the following theorem.

Theorem 4. If the function f defined by (1) belongs to the class $\mathcal{K}_{\beta}^{\alpha}[A,B]$, then

$$\sum_{k=2}^{\infty}k(k-1+A-kB)\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta+k)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k)}|a_k|\leq A-B. \tag{19}$$

We will discuss two inclusion relations for the classes $\mathcal{S}_{\beta}^{\alpha}[A,B]$ and $\mathcal{K}_{\beta}^{\alpha}[A,B]$. To prove these results we shall require the following lemma:

Lemma 3 [3]**.** Let h be convex (univalent) in $\mathbb{U}$, with $\Re\{\gamma h(z)+\eta\}>0$ for all $z\in\mathbb{U}$. If p is analytic in $\mathbb{U}$, with $p(0)=h(0)$, then

$$p(z)+\frac{zp'(z)}{\gamma p(z)+\eta}\prec h(z)\Rightarrow p(z)\prec h(z).$$

Theorem 5. Suppose that

$$\Re\left\{\frac{z}{1+Bz}\right\}>-\frac{\alpha+\beta}{A-B}\quad(z\in\mathbb{U}). \tag{20}$$

If $f\in\mathcal{S}_{\beta}^{\alpha-1}[A,B]$ with $\alpha>1$ and $Q_{\beta}^{\alpha}f(z)\neq 0$ for all $z\in\mathbb{U}$, then $f\in\mathcal{S}_{\beta}^{\alpha}[A,B]$.

Proof. Suppose that $f\in\mathcal{S}_{\beta}^{\alpha-1}[A,B]$, and let define the function

$$p(z)=\frac{z\left(Q_{\beta}^{\alpha}f(z)\right)'}{Q_{\beta}^{\alpha}f(z)}\quad(z\in\mathbb{U}). \tag{21}$$

Then p is analytic in $\mathbb{U}$ with $p(0)=1$, and using the relation (8), from (21) we obtain

$$p(z)+\alpha+\beta-1=(\alpha+\beta)\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}. \tag{22}$$

Taking the logarithmic differentiation on both sides of (22) and then using (21), we deduce that

$$p(z) + \frac{zp^{'}(z)}{p(z)+\alpha+\beta-1} \prec \frac{1+Az}{1+Bz}. \tag{23}$$

From (20), we see that $\Re\left\{\frac{1+Az}{1+Bz}+\alpha+\beta-1\right\} > 0, z \in \mathbb{U}$. Since the function $\frac{1+Az}{1+Bz}$ is convex (univalent) in $\mathbb{U}$, according to Lemma 3 the subordination (23) implies $p(z) \prec \frac{1+Az}{1+Bz}$, which proves that $f \in \mathcal{S}_{\beta}^{\alpha}[A,B]$. □

From the duality formula (11), the above theorem yields the following inclusion:

Theorem 6. Suppose that (20) holds.If $f \in \mathcal{K}_{\beta}^{\alpha-1}[A,B]$ with $\alpha > 1$ and$Q_{\beta}^{\alpha}f(z) \neq 0$ for all $z \in \mathbb{U}$, then $f \in \mathcal{K}_{\beta}^{\alpha}[A,B]$.

Proof. Applying (11) and Theorem 5, we observe that

$$\begin{aligned} f \in \mathcal{K}_{\beta}^{\alpha-1}[A,B] \quad &\Longleftrightarrow zf^{'} \in \mathcal{S}_{\beta}^{\alpha-1}[A,B] \quad (\text{ from } 11) \\ &\Longrightarrow zf^{'} \in \mathcal{S}_{\beta}^{\alpha}[A,B] \quad (\text{ by Theorem } 5\ ) \\ &\Longleftrightarrow f \in \mathcal{K}_{\beta}^{\alpha}[A,B]. \end{aligned}$$

which evidently proves Theorem 6. □

References

[1] O. P. Ahuja, Families of analytic functions related to Ruscheweyh derivatives and subordinate to convex functions, Yokohama Math. J., 41(1993), 39-50.
[2] M. K. Aouf and T. M. Seoudy, Classes of analytic functions related to the Dziok–Srivastava operator, Integral Transforms and Special Functions, 22(2011), no. 6, 423–430.
[3] P. J. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, Second order differential inequalities in the complex plane, J. Math. Anal. Appl., 65 (1978), 289-305.
[4] R. M. Goel and B. S. Mehrok,On the coefficients of a subclass of starlike functions, Indian J. Pure Appl. Math., 12(1981), 634-647.
[5] W. Janowski, Some extremal problems for certain families of analytic functions, Bull. Polish Acad. Sci., 21(1973),17-25.
[6] W. Janowski, Some extremal problems for certain families of analytic functions, Ann. Polon. Math., 28(1973),297-326.
[7] T. B. Jung, Y. C. Kim and H. M. Srivastava,The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176(1993), 138-147.
[8] S. S. Miller and P. T. Mocanu, Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.
[9] S. S. Miller and P. T Mocanu, Subordinates of differential superordinations, Complex Variables 48 (2003), no. 10, 815–826.
[10] M. S. Robertson, On the theory of univalent functions, Ann. Math., 37 (1936), 374- 408.
[11] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, Canad. J. Math., 1(1985), 48-61.

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