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LACUNARY $\mathcal{I}$-CONVERGENT AND LACUNARY $\mathcal{I}$-BOUNDED SEQUENCE SPACES DEFINED BY AN ORLICZ FUNCTION

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Abstract. A lacunary sequence is an increasing sequence $\theta = (k_r)$ such that $k_r - k_{r-1} \to \infty$ as $r \to \infty$. In this paper, we define the spaces of lacunary ideal convergent and lacunary ideal bounded sequences with respect to an Orlicz function. We establish some inclusion relations of the resulting sequence spaces.

1. **Preliminaries, background and notation**

Let ω denote the space of *all real or complex valued sequences* and $\mathbb{N}$, $\mathbb{C}$ stand for the set of *natural numbers, complex numbers.*

Let $X \neq \emptyset$. A class $\mathcal{I} \subseteq 2^X$ is called an *ideal* if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$) ([29]). An ideal is called *non-trivial* if $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be *admissible* if $\mathcal{I}$ contains every finite subset of X.

The notion of ideal convergence was first introduced by Kostyrko et al [25] in the following way. Let $\mathcal{I}$ be a non-trivial ideal in $\mathbb{N}$. A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to be $\mathcal{I}$*-convergent* to $l \in \mathbb{R}$ if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$ belongs to $\mathcal{I}$. Also, Kostyrko et al [27] studied the concept of $\mathcal{I}$-convergence in metric spaces. Later the idea of $\mathcal{I}$-convergence was extended to an arbitrary topological space by Lahiri and Das [31]. Note that if $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$, then the ideal convergence coincides with the usual convergence.

A sequence $x = (x_n)$ is said to be $\mathcal{I}$*-bounded* if there exists an $K > 0$ such that $\{n \in \mathbb{N} : |x_n| > K\} \in \mathcal{I}$.

For more details about $\mathcal{I}$-convergence, we refer to [6, 7, 15, 16, 17, 26, 30, 36, 41].

An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Since M is a convex function and $M(0) = 0$, $M(\alpha x) \leq \alpha M(x)$ for all $\alpha \in (0, 1)$.

M is said to satisfy Δ_2-condition for all $x \in [0, \infty)$ if there exists a constant $K > 0$ such that $M(Lx) \leq KLM(x)$, where $L > 1$ (see [28]).

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Throughout this paper, $p = (p_i)$ will be a sequence of positive real numbers such that $0 < h = \inf p_i \leq p_i \leq H = \sup p_i < \infty$. Also, the inequalities for every $i = 1, 2, ...$

$$|a_i + b_i|^{p_i} \leq D\left\{|a_i|^{p_i} + |b_i|^{p_i}\right\} \tag{1}$$

and

$$|a|^{p_i} \leq \max\{1, |a|^H\}$$

will be used, where $a, a_i, b_i \in \mathbb{C}$ and $D = \max\left\{1, 2^{H-1}\right\}$.

Let $A = (a_{ij})$ be an infinite matrix of real or complex numbers a_{ij}, where $i, j \in \mathbb{N}$. We write $Ax = (A_i(x))$ if $A_i(x) = \sum\limits_{j=1}^{\infty} a_{ij}x_j$ converges for each $i \in \mathbb{N}$.

Lindenstrauss and Tzafriri [32] used the idea of Orlicz function to construct the following sequence space:

$$\ell_M = \left\{x \in \omega : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

The space ℓ_M becomes a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1\right\},$$

which is called an *Orlicz sequence space*.

By using Orlicz function, the following sequence spaces were defined in [34]:

$$\ell_M(p) = \left\{x \in \omega : \sum_{i=1}^{\infty} \left[M\left(\frac{|x_i|}{\rho}\right)\right]^{p_i} < \infty \text{ for some } \rho > 0\right\},$$

$$W(M,p) = \left\{x \in \omega : \frac{1}{n}\sum_{i=1}^{n} \left[M\left(\frac{|x_i - l|}{\rho}\right)\right]^{p_i} \to 0 \text{ as } n \to \infty \text{ for some } \rho > 0 \text{ and } l > 0\right\},$$

$$W_0(M,p) = \left\{x \in \omega : \frac{1}{n}\sum_{i=1}^{n} \left[M\left(\frac{|x_i|}{\rho}\right)\right]^{p_i} \to 0 \text{ as } n \to \infty \text{ for some } \rho > 0\right\}$$

and

$$W_\infty(M,p) = \left\{x \in \omega : \sup_n \frac{1}{n}\sum_{i=1}^{n} \left[M\left(\frac{|x_i|}{\rho}\right)\right]^{p_i} < \infty \text{ for some } \rho > 0\right\},$$

where $p = (p_i)$ is any sequence of positive real numbers.

Later on Orlicz sequence spaces were investigated by Esi [8], Bhardwaj and Singh [2], Esi and Et [10], Et [12], Tripathy et al [42], Bektaş and Altın [1], Şahiner and Gürdal [40] and many others.

Recently, the authors introduced new spaces by using ideal convergence, Orlicz function and an infinite matrix.

For example, Hazarika et al [19] introduced paranorm ideal convergent sequence spaces by using Zweir transform and Orlicz function as follows:

$$\mathcal{Z}^{\mathcal{I}}(M,p) = \left\{x \in \omega : \left\{n \in \mathbb{N} : \left[M\left(\frac{|(Z^p x)_n - L|}{\rho}\right)\right]^{p_n} \geq \varepsilon\right\} \in \mathcal{I} \text{ for some } L \in \mathbb{C}\right\}$$

and

$$\mathcal{Z}_0^{\mathcal{I}}(M,p) = \left\{x \in \omega : \left\{n \in \mathbb{N} : \left[M\left(\frac{|(Z^p x)_n|}{\rho}\right)\right]^{p_n} \geq \varepsilon\right\} \in \mathcal{I}\right\},$$

where the Zweir matrix $Z^p = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} p, & (n = k) \\ 1-p, & (n-1 = k); (n,k \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Ideal convergent sequence spaces combined by a sequence of Orlicz functions (M_i) and the matrix Λ

$$c^{\mathcal{I}}(\mathcal{M}, \Lambda, p) = \left\{ x \in \omega : \mathcal{I} - \lim_i M_i \left(\frac{|\Lambda_i(x) - L|}{\rho} \right)^{p_i} = 0, \text{ for some } L \text{ and } \rho > 0 \right\},$$

$$c_0^{\mathcal{I}}(\mathcal{M}, \Lambda, p) = \left\{ x \in \omega : \mathcal{I} - \lim_i M_i \left(\frac{|\Lambda_i(x)|}{\rho} \right)^{p_i} = 0, \text{ for some } \rho > 0 \right\}$$

and

$$\ell_\infty(\mathcal{M}, \Lambda, p) = \left\{ x \in \omega : \sup_i M_i \left(\frac{|\Lambda_i(x)|}{\rho} \right)^{p_i} < \infty, \text{ for some } \rho > 0 \right\}$$

were defined by Mursaleen and Sharma [33], where $\Lambda_i(x) = \frac{1}{\lambda_i} \sum_{m=1}^{i} (\lambda_m - \lambda_{m-1}) x_m$ and (λ_m) is a strictly increasing sequence of positive real numbers tending to infinity.

Kara and İlkhan [21] defined the following spaces:

$$c^{\mathcal{I}}(M, A, p) = \left\{ x \in \omega : \mathcal{I} - \lim_n \left[M \left(\frac{|A_n(x) - L|}{\rho} \right) \right]^{p_n} = 0, \text{ for some } L \text{ and } \rho > 0 \right\},$$

$$c_0^{\mathcal{I}}(M, A, p) = \left\{ x \in \omega : \mathcal{I} - \lim_n \left[M \left(\frac{|A_n(x)|}{\rho} \right) \right]^{p_n} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$\ell_\infty(M, A, p) = \left\{ x \in \omega : \sup_n \left[M \left(\frac{|A_n(x)|}{\rho} \right) \right]^{p_n} < \infty, \text{ for some } \rho > 0 \right\}.$$

In the literature, there are more papers related to sequence spaces defined by using ideal convergence, an Orlicz function and an infinite matrix. For some of these papers, one can see [18, 20, 35, 37, 43].

An increasing integer sequence $\theta = (k_r)$ is called a *lacunary sequence* if $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} is written as $q_r = k_r/k_{r-1}$.

Freedman et al [13] defined the space of lacunary strongly convergent sequences N_θ in the following way:

$$N_\theta = \left\{ x \in \omega : \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - l| = 0, \text{ for some } l \right\}$$

for any lacunary sequence $\theta = (k_r)$.

The sequence spaces defined by combining a lacunary sequence, an Orlicz function and an infinite matrix were investigated by many authors. We refer to Bhardwaj and Singh [3], Bilgin [4, 5], Savaş and Rhoades [38], Karakaya [22], Güngör et al [14] for some of related papers.

In [44], the authors defined the notions of *lacunary $\mathcal{I}$-convergence* and *lacunary $\mathcal{I}$-null sequences*. If for every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} |x_i - l| \geq \varepsilon \right\} \in \mathcal{I} \text{ and } \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} |x_i| \geq \varepsilon \right\} \in \mathcal{I},$$

then the sequence $x = (x_i)$ is said to be lacunary $\mathcal{I}$- convergent to l and lacunary $\mathcal{I}$-null, respectively.

It can be seen in [9, 11, 23, 24, 39, 45], the authors used the concept of ideal convergence and lacunary sequence to define different types of sequence spaces.

In this paper, we define some new sequence spaces using the concept of ideal convergence, lacunary sequence, Orlicz function and A-transform as follows:

$$\mathcal{I}{-}N_\theta^0(A, M, p) = \left\{ x \in \omega : \mathcal{I} - \lim_r \frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{p_i} = 0, \text{ for some } \rho > 0 \right\},$$

$$\mathcal{I}{-}N_\theta(A, M, p) = \left\{ x \in \omega : \mathcal{I} - \lim_r \frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(x) - L|}{\rho} \right) \right]^{p_i} = 0, \text{ for some } L \text{ and } \rho > 0 \right\},$$

$$\mathcal{I}{-}N_\theta^\infty(A, M, p) = \left\{ x \in \omega : \left(\frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{p_i} \right) \text{ is } \mathcal{I} - \text{bounded for some } \rho > 0 \right\}.$$

In the case, $M(x) = x$ for all $x \in [0, \infty)$, we write $\mathcal{I} - N_\theta^0(A, p)$, $\mathcal{I} - N_\theta(A, p)$ and $\mathcal{I} - N_\theta^\infty(A, p)$ instead of the spaces $\mathcal{I} - N_\theta^0(A, M, p)$, $\mathcal{I} - N_\theta(A, M, p)$ and $\mathcal{I} - N_\theta^\infty(A, M, p)$, respectively.

In the case, $p_i = 1$ for all $i \in \mathbb{N}$, we write $\mathcal{I} - N_\theta^0(A, M)$, $\mathcal{I} - N_\theta(A, M)$ and $\mathcal{I} - N_\theta^\infty(A, M)$ instead of the spaces $\mathcal{I} - N_\theta^0(A, M, p)$, $\mathcal{I} - N_\theta(A, M, p)$ and $\mathcal{I} - N_\theta^\infty(A, M, p)$, respectively.

If $A = I$, $p_i = 1$ for all $i \in \mathbb{N}$ and $M(x) = x$ for all $x \in [0, \infty)$, the spaces $\mathcal{I} - N_\theta^0(A, M, p)$ and $\mathcal{I} - N_\theta(A, M, p)$ reduce to the spaces of lacunary $\mathcal{I}$-null and lacunary $\mathcal{I}$-convergent sequences defined by Tripathy et al [44].

The main purpose of this paper is to introduce the spaces $\mathcal{I} - N_\theta^0(A, M, p)$, $\mathcal{I} - N_\theta(A, M, p)$ and $\mathcal{I} - N_\theta^\infty(A, M, p)$ and give some inclusion theorems.

2. **Main results**

In this section, we investigate some inclusion relations related to the spaces $\mathcal{I} - N_\theta^0(A, M, p)$, $\mathcal{I} - N_\theta(A, M, p)$ and $\mathcal{I} - N_\theta^\infty(A, M, p)$. Firstly, we prove that these spaces are linear over the set of complex or real numbers.

Theorem 1 $\mathcal{I} - N_\theta^0(A, M, p)$, $\mathcal{I} - N_\theta(A, M, p)$ and $\mathcal{I} - N_\theta^\infty(A, M, p)$ are linear spaces.

Proof. Let $x, y \in \mathcal{I} - N_\theta^0(A, M, p)$ and α, β be scalars. Then there exist $\rho_1, \rho_2 > 0$ such that for every $\varepsilon > 0$

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(x)|}{\rho_1} \right) \right]^{p_i} \geq \frac{\varepsilon}{2D} \right\} \in \mathcal{I},$$

$$A_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(y)|}{\rho_2} \right) \right]^{p_i} \geq \frac{\varepsilon}{2D} \right\} \in \mathcal{I}.$$

To prove that $\alpha x+\beta y \in \mathcal{I}-N_\theta^0(A,M,p)$, let define $\rho=\max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Suppose that $r \notin A_1 \cup A_2$. By using inequalitty (1), we have

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(\alpha x+\beta y)|}{\rho}\right)\right]^{p_i} &\le D\left\{\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho_1}\right)\right]^{p_i}+\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(y)|}{\rho_2}\right)\right]^{p_i}\right\}\\ &< D\left\{\frac{\varepsilon}{2D}+\frac{\varepsilon}{2D}\right\}=\varepsilon\end{aligned}$$

since M is an Orlicz function and A is a linear transformation. Hence $r \notin A_0 = \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(\alpha x+\beta y)|}{\rho}\right)\right]^{p_i}\ge\varepsilon\right\}$. This implies that $A_0 \subset A_1 \cup A_2$. By additivity and heritability of $\mathcal{I}$, we have $A_0 \in \mathcal{I}$. Consequently, $\mathcal{I}-N_\theta^0(A,M,p)$ is a linear space.

In a similar way, one can prove that $\mathcal{I}-N_\theta(A,M,p)$ and $\mathcal{I}-N_\theta^\infty(A,M,p)$ are linear spaces.

Now, we give some inclusion relations.

Theorem 2 The following inclusion relations hold:

$$\mathcal{I}-N_\theta^0(A,M,p)\subset\mathcal{I}-N_\theta(A,M,p)\subset\mathcal{I}-N_\theta^\infty(A,M,p).$$

Proof. Clearly, the first inclusion is true. To prove that the inclusion $\mathcal{I}-N_\theta(A,M,p)\subset\mathcal{I}-N_\theta^\infty(A,M,p)$ holds, let $x\in\mathcal{I}-N_\theta(A,M,p)$. Then there exists $\rho_1>0$ such that for every $\varepsilon>0$

$$A_0=\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)-L|}{\rho_1}\right)\right]^{p_i}\ge\varepsilon\right\}\in\mathcal{I}.$$

Let define $\rho=2\rho_1$. Since M is non-decreasing and convex, we have

$$M\left(\frac{|A_i(x)|}{\rho}\right)\le M\left(\frac{|A_i(x)-L|}{\rho_1}\right)+M\left(\frac{|L|}{\rho_1}\right).$$

Suppose that $r\notin A_0$. Hence by the last inequality and (1), the following inequality is obtained:

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i} &\le D\left\{\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)-L|}{\rho_1}\right)\right]^{p_i}+\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|L|}{\rho_1}\right)\right]^{p_i}\right\}\\ &< D\left\{\varepsilon+\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|L|}{\rho_1}\right)\right]^{p_i}\right\}.\end{aligned}$$

Because of the fact that $\left[M\left(\frac{|L|}{\rho_1}\right)\right]^{p_i}\le\max\left\{1,\left[M\left(\frac{|L|}{\rho_1}\right)\right]^H\right\}$, we have

$$\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|L|}{\rho_1}\right)\right]^{p_i}<\infty.$$

Put $K=D\left\{\varepsilon+\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|L|}{\rho_1}\right)\right]^{p_i}\right\}$. It follows that $\left\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}>K\right\}\in\mathcal{I}$ which means $x\in\mathcal{I}-N_\theta^\infty(A,M,p)$. This completes the proof.

Theorem 3 Let M and M' be Orlicz functions which satisfies Δ_2-condition. Then the following inclusion relations hold:

(1) $\mathcal{I}-N_\theta^0(A,M,p)\subseteq\mathcal{I}-N_\theta^0(A,M'\circ M,p)$, $\mathcal{I}-N_\theta(A,M,p)\subseteq\mathcal{I}-N_\theta(A,M'\circ M,p)$ and $\mathcal{I}-N_\theta^\infty(A,M,p)\subseteq\mathcal{I}-N_\theta^\infty(A,M'\circ M,p)$.

(2) $\mathcal{I}-N_\theta^0(A,M,p)\cap\mathcal{I}-N_\theta^0(A,M',p)\subseteq\mathcal{I}-N_\theta^0(A,M+M',p)$, $\mathcal{I}-N_\theta(A,M,p)\cap\mathcal{I}-N_\theta(A,M',p)\subseteq\mathcal{I}-N_\theta(A,M+M',p)$ and $\mathcal{I}-N_\theta^\infty(A,M,p)\cap\mathcal{I}-N_\theta^\infty(A,M',p)\subseteq\mathcal{I}-N_\theta^\infty(A,M+M',p)$.

Proof. We prove only the third inclusions since the others can be proved similarly.

(1) Let $x\in\mathcal{I}-N_\theta^\infty(A,M,p)$. Then there exists $K_1>0$ such that

$$A_0=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}>K_1\right\}\in\mathcal{I}$$

for a $\rho>0$. Since M' is nondecreasing and convex, and satisfies Δ_2-condition, we obtain the inequality

$$\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)>\delta}\left[M'\left(M\left(\frac{|A_i(x)|}{\rho}\right)\right)\right]^{p_i}\leq\max\left\{1,\left(K\frac{1}{\delta}M'(2)\right)^H\right\}\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)>\delta}\left[M\left(\frac{A_i(x)}{\rho}\right)\right]^{p_i}\tag{2}$$

for $K\geq1$. By continuity of M', we have

$$\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)\leq\delta}\left[M'\left(M\left(\frac{|A_i(x)|}{\rho}\right)\right)\right]^{p_i}\leq\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)\leq\delta}\varepsilon^{p_i}\leq\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)\leq\delta}\max\{\varepsilon^h,\varepsilon^H\}.\tag{3}$$

Suppose that $r\notin A_0$. Then by using the inequalities (2) and (3), we have

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\left[M'\left(M\left(\frac{|A_i(x)|}{\rho}\right)\right)\right]^{p_i}&=\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)>\delta}\left[M'\left(M\left(\frac{|A_i(x)|}{\rho}\right)\right)\right]^{p_i}+\frac{1}{h_r}\sum_{i\in I_r,\ M\left(\frac{A_i(x)}{\rho}\right)\leq\delta}\left[M'\left(M\left(\frac{|A_i(x)|}{\rho}\right)\right)\right]^{p_i}\\&\leq\max\left\{1,\left(K\frac{1}{\delta}M'(2)\right)^H\right\}K_1+\max\{\varepsilon^h,\varepsilon^H\}=K_2.\end{aligned}$$

Hence $r\notin B=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{i\in I_r}\left[M'\left(M\left(\frac{|A_i(x)|}{\rho}\right)\right)\right]^{p_i}>K_2\right\}$ and so $B\subset A_0$ which implies $B\in\mathcal{I}$. We conclude that $x\in\mathcal{I}-N_\theta^\infty(A,M'\circ M,p)$.

(2) Let $x\in\mathcal{I}-N_\theta^\infty(A,M,p)\cap\mathcal{I}-N_\theta^\infty(A,M',p)$. There exist $K_1>0$ and $K_2>0$ such that

$$A_1=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}>K_1\right\}\in\mathcal{I}$$

and

$$A_2=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}>K_2\right\}\in\mathcal{I}$$

for some $\rho>0$. Let $r\notin A_1\cup A_2$. Then we have

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\left[(M+M')\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}&\leq D\left\{\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}+\frac{1}{h_r}\sum_{i\in I_r}\left[M'\left(\frac{|A_i(x)|}{\rho}\right)\right]^{p_i}\right\}\\&<D\left\{K_1+K_2\right\}=K.\end{aligned}$$

Thus r is not contained in $B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \left[(M + M') \left(\frac{|A_i(x)|}{\rho} \right) \right]^{p_i} > K \right\}$. We have $A_1 \cup A_2 \in \mathcal{I}$ and $B \subset A_1 \cup A_2$ which imply $B \in \mathcal{I}$. This means $x \in \mathcal{I} - N_\theta^\infty(A, M + M', p)$ and completes the proof.

Corollary 1 Let M be an Orlicz function which satisfies Δ_2-condition. Then the inclusions $\mathcal{I} - N_\theta^0(A, p) \subset \mathcal{I} - N_\theta^0(A, M, p)$, $\mathcal{I} - N_\theta(A, p) \subset \mathcal{I} - N_\theta(A, M, p)$ and $\mathcal{I} - N_\theta^\infty(A, p) \subset \mathcal{I} - N_\theta^\infty(A, M, p)$ hold.

Proof. The proof follows from the first part of Theorem 3 by using $M(x) = x$ and $M'(x) = M(x)$ for all $x \in [0, \infty)$.

Theorem 4 Let $0 < p_i \leq q_i$ and $\left(\frac{q_i}{p_i} \right)$ be bounded. Then the following inclusions hold:

$$\mathcal{I} - N_\theta^0(A, M, q) \subset \mathcal{I} - N_\theta^0(A, M, p) \text{ and } \mathcal{I} - N_\theta(A, M, q) \subset \mathcal{I} - N_\theta(A, M, p).$$

Proof. We prove only the first inclusion and the other inclusion can be carried out by using a similar technique. Let $x \in \mathcal{I} - N_\theta^0(A, M, q)$. Write $t_i = \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{q_i}$ and $\lambda_i = \frac{p_i}{q_i}$, so that $0 < \lambda \leq \lambda_i \leq 1$. By using Hölder inequality, we obtain

$$\begin{aligned}
\frac{1}{h_r} \sum_{i \in I_r} (t_i)^{\lambda_i} &= \frac{1}{h_r} \sum_{i \in I_r,\ t_i \geq 1} (t_i)^{\lambda_i} + \frac{1}{h_r} \sum_{i \in I_r,\ t_i < 1} (t_i)^{\lambda_i} \\
&\leq \frac{1}{h_r} \sum_{i \in I_r,\ t_i \geq 1} t_i + \frac{1}{h_r} \sum_{i \in I_r,\ t_i < 1} (t_i)^{\lambda} \\
&= \frac{1}{h_r} \sum_{i \in I_r,\ t_i \geq 1} t_i + \sum_{i \in I_r,\ t_i < 1} \left(\frac{1}{h_r} t_i \right)^{\lambda} \left(\frac{1}{h_r} \right)^{1-\lambda} \\
&\leq \frac{1}{h_r} \sum_{i \in I_r,\ t_i \geq 1} t_i + \left(\sum_{i \in I_r,\ t_i < 1} \left[\left(\frac{1}{h_r} t_i \right)^{\lambda} \right]^{\frac{1}{\lambda}} \right)^{\lambda} \left(\sum_{i \in I_r,\ t_i < 1} \left[\left(\frac{1}{h_r} \right)^{1-\lambda} \right]^{\frac{1}{1-\lambda}} \right)^{1-\lambda} \\
&\leq \frac{1}{h_r} \sum_{i \in I_r,\ t_i \geq 1} t_i + \left(\frac{1}{h_r} \sum_{i \in I_r,\ t_i < 1} t_i \right)^{\lambda}.
\end{aligned}$$

Hence for every $\varepsilon > 0$ we have

$$\begin{aligned}
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{p_i} \geq \varepsilon \right\} \subset & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r,\ t_i \geq 1} \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{q_i} \geq \frac{\varepsilon}{2} \right\} \\
& \cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r,\ t_i < 1} \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{q_i} \geq \left(\frac{\varepsilon}{2} \right)^{\frac{1}{\lambda}} \right\}.
\end{aligned}$$

This last inclusion implies that $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \left[M \left(\frac{|A_i(x)|}{\rho} \right) \right]^{p_i} \geq \varepsilon \right\} \in \mathcal{I}$ and so $x \in \mathcal{I} - N_\theta^0(A, M, p)$.

Corollary 2

(1) If $0 < \inf p_i \leq 1$, then the inclusions $\mathcal{I} - N_\theta^0(A, M) \subset \mathcal{I} - N_\theta^0(A, M, p)$ and $\mathcal{I} - N_\theta(A, M) \subset \mathcal{I} - N_\theta(A, M, p)$ hold.

(2) If $1 \leq p_i \leq \sup p_i < \infty$, then the inclusions $\mathcal{I} - N_\theta^0(A, M, p) \subset \mathcal{I} - N_\theta^0(A, M)$ and $\mathcal{I} - N_\theta(A, M, p) \subset \mathcal{I} - N_\theta(A, M)$ hold.

Proof. The proof follows from Theorem 4 taking $t_k = 1$ for all k and changing the roles of p_k and t_k only for the second part of the corollary.

Theorem 5 If $\lim_{i\to\infty} p_i > 0$ and $x \to L(\mathcal{I} - N_\theta(A, M, p))$, then L is unique.

Proof. Let $\lim_{i\to\infty} p_i = p' > 0$ and assume that $x \to L(\mathcal{I} - N_\theta(A, M, p))$ and $x \to L'(\mathcal{I} - N_\theta(A, M, p))$ for $L \neq L'$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i\in I_r} \left[M\left(\frac{|A_i(x) - L|}{\rho_1} \right) \right]^{p_i} \geq \frac{\varepsilon}{2D} \right\} \in \mathcal{I}$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i\in I_r} \left[M\left(\frac{|A_i(x) - L'|}{\rho_2} \right) \right]^{p_i} \geq \frac{\varepsilon}{2D} \right\} \in \mathcal{I}.$$

Then we have

$$\frac{1}{h_r} \sum_{i\in I_r} \left[M\left(\frac{|L - L'|}{\rho} \right) \right]^{p_i} \leq D \left\{ \frac{1}{h_r} \sum_{i\in I_r} \left[M\left(\frac{|A_i(x) - L|}{\rho_1} \right) \right]^{p_i} + \frac{1}{h_r} \sum_{i\in I_r} \left[M\left(\frac{|A_i(x) - L'|}{\rho_2} \right) \right]^{p_i} \right\},$$

where $\rho = \max\{2\rho_1, 2\rho_2\}$.

Hence it is obtained that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i\in I_r} \left[M\left(\frac{|L - L'|}{\rho} \right) \right]^{p_i} \geq \varepsilon \right\} \in \mathcal{I}$$

Also we have

$$\left[M\left(\frac{|L - L'|}{\rho} \right) \right]^{p_i} \to \left[M\left(\frac{|L - L'|}{\rho} \right) \right]^{p'}$$

as $i \to \infty$ and so $\left[M\left(\frac{|L-L'|}{\rho} \right) \right]^{p'} = 0$. This implies that $L = L'$.

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