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BEHAVIOR OF SOLUTIONS OF A CLASS OF NONLINEAR RATIONAL DIFFERENCE EQUATION $x_{n+1} = \alpha x_{n-k} + \frac{\beta x^{\delta}_{n-\ell}}{\gamma x^{\delta}_{n-s}}$

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Abstract. The main objective of this paper is to study the qualitative behavior for a class of nonlinear rational difference equation. We study the local stability, periodicity, Oscillation, boundedness, and the global stability for the positive solutions of equation. Examples illustrate the importance of the results

1. Introduction

In this paper, we aim to achieve a qualitative study of some behavior and solutions in a non-linear differential equations

$$x_{n+1} = \alpha x_{n-k} + \frac{\beta x^{\delta}_{n-\ell}}{\gamma x^{\delta}_{n-s}},\ n = 0, 1, 2, ..., \tag{1}$$

where the coefficients α, β and $\gamma \in (0, \infty)$ while k, ℓ and s are positive integers. The initial conditions $x_{-j}, x_{-j+1}, ..., x_0$ are arbitrary positive real numbers such that $j = -\max\{k, \ell, s\}$. Consider $\delta \in [1, \infty)$. Qualitative analysis of difference equation is not only interesting in its own right, but it can provide insights into their continuous counterpararts, namely, differential equations.

There is a set of nonlinear difference equations, known as the rational difference equations, all of which consists of the ratio of two polynomials in the sequence terms in the same from .there has been many work about the global asympototic of solutions of rational difference equations [3], [6], [7], [8], [11], [12]

In the following we present some basic definitions and known results which will be useful in our study.

Definition 1. [2] *Consider a difference equation in the form*

$$x_{n+1} = F(x_{n-k}, x_{n-\ell}, x_{n-s}) \tag{2}$$

where F is a continuous function, while k,ℓ, $s \in (0, \infty)$ are positive integers. An equilibrium point $\overline{x}$ of this equation is a point that satisfies the condition $\overline{x}$ =

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$F(\overline{x},\overline{x},\overline{x})$. That is, the constant sequence $\{x_n\}$ with $x_n=\overline{x}$ for all $n\geqslant -k\geqslant -\ell$ is a solution of that equation.

Definition 2. [9]*Let $\overline{x}\in(0,\infty)$ be an equilibrium point of Eq.(2).Then we have*

(i) An equilibrium point $\overline{x}$ of Eq. is said to be locally stable if for every $\varepsilon>0$ there exists $\sigma>0$ such that, if $x_{-j},...,x_{-1},x_0\in(0,\infty)$ with $|x_{-j}-\overline{x}|+...+|x_{-1}-\overline{x}|+|x_0-\overline{x}|<\sigma$, then $|x_n-\overline{x}|<\varepsilon$ for all $n\geqslant -j$.

(ii) An equilibrium point $\overline{x}$ of Eq.(2) is said to be locally asymptotically stable if it is locally stable and there exists $y>0$ such that, $x_{-j},...,x_{-1},x_0\in(0,\infty)$ with $|x_{-j}-\overline{x}|+...+|x_{-1}-\overline{x}|+|x_0-\overline{x}|<y$, then $\lim_{n\to\infty} x_n=\overline{x}$.

(iii) An equilibrium point $\overline{x}$ of Eq.(2) is said to be a global attractor if for every $x_{-j},...,x_{-1},x_0\in(0,\infty)$ we have $\lim_{n\to\infty} x_n=\overline{x}$.

(iv) An equilibrium point $\overline{x}$ of Eq.(2) is said to be globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\overline{x}$ of Eq.(2) is said to be unstable if it is not locally stable.

Definition 3. [1]*The sequence $\{x_n\}$ is said to be periodic with period p if $x_{n+p}=x_n$ for $n=0,1,...,$*

Definition 4. [5]*Eq.(2) is said to be permanent and bounded if there exists numbers m and M with $0<m<M<\infty$ such that for any initial conditions $x_{-j},...,x_{-1},x_0\in(0,\infty)$ there exists a positive integer N which depends on these initial conditions such that $m\leq x_n\leq M$ for all $n\geq N$.*

Definition 5. [4]*A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be nonoscillatory about the point $\overline{x}$ if there is exists $N\geq -k$ such that either $x_n>\overline{x}$ for all $n\geq N$ or $x_n<\overline{x}$ for all $n\geq N$. Otherwise $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about $\overline{x}$.*

Definition 6. [3]*The linearized equation of Eq.(2) about the equilibrium point $\overline{x}$ is defined by the equation.*

$$y_{n+1}=p_0y_{n-k}+p_1y_{n-\ell}+p_2y_{n-s} \tag{3}$$

$$p_0=\frac{\partial f}{\partial x_{n-k}}(\overline{x},\overline{x},\overline{x}),p_1=\frac{\partial f}{\partial x_{n-\ell}}(\overline{x},\overline{x},\overline{x}),p_2=\frac{\partial f}{\partial x_{n-s}}(\overline{x},\overline{x},\overline{x})$$

The characteristic equation associated with Eq. (3) is

$$p(\lambda)=\lambda^{\ell+1}-p_o\lambda^{\ell}-p_1\lambda^{\ell-k}-p_2=0 \tag{4}$$

Theorem 1. [8]*Assume that p_0,p_1 and $p_2\in R$. Then*

$$|p_0|+|p_1|+|p_2|<1 \tag{5}$$

is a sufficient condition for the locally stability of Eq.(2).

2. Local stable of the equilibrium point

The equilibrium point of Eq.(1) is the positive solution of the equation

$$\overline{x}=\alpha\overline{x}+\frac{a\overline{x}^{\delta}}{b\overline{x}^{\delta}}$$

which gives

$$\overline{x}=\frac{\beta}{\gamma(1-\alpha)},\alpha<1 \tag{6}$$

Now let $f : (0,\infty)^3 \to (0,\infty)$ be a function defined by

$$f(u,v,w) = \alpha u + \frac{\beta v^{\delta}}{\gamma w^{\delta}}.$$

Then, we have

$$\frac{\partial f}{\partial u} = \alpha, \tag{7}$$

$$\frac{\partial f}{\partial v} = \frac{\beta \delta v^{\delta-1}}{\gamma w^{\delta}}, \tag{8}$$

and

$$\frac{\partial f}{\partial w} = \frac{-\beta \gamma \delta v^{\delta} w^{\delta-1}}{(\gamma w^{\delta})^2}. \tag{9}$$

Theorem 2. *If*

$$\alpha + 2\delta < 1 + 2\alpha\delta$$

then the equilibrium point $\overline{x}$ *of eq (1) is local stable.*

Proof. From (7) to (9), we get

$$\frac{\partial f}{\partial u}(\overline{x},\overline{x},\overline{x}) = \alpha = p_0,$$

$$\frac{\partial f}{\partial v}(\overline{x},\overline{x},\overline{x}) = \delta(1-\alpha) = P_1,$$

and

$$\frac{\partial f}{\partial w}(\overline{x},\overline{x},\overline{x}) = -\delta(1-\alpha) = P_2.$$

Thus, the linearized equation associated with Eq. (2) about $\overline{x}$, is

$$y_{n+1} = p_0 y_{n-k} + p_1 y_{n-\ell} + p_2 y_{n-s}.$$

It follows by Theorem 1 that Eq.(1) is localy stable if

$$|\alpha| + |\delta(1-\alpha)| + |-\delta(1-\alpha)| < 1,$$

after simplification and calculations, we get

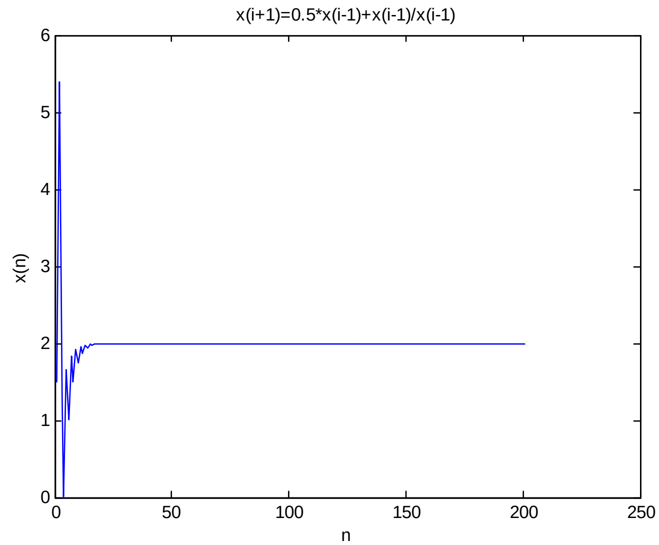
$$\alpha + 2\delta(1-\alpha) < 1,$$

which is true if

$$\alpha + 2\delta < 1 + 2\alpha\delta.$$

The proof is completed. □

Example 1. *Fig. 1, shows that Eq. (1) has Local stable solutions if* $\alpha = 0.5$, $\beta = \gamma = 1$, $k = \ell = s = \delta = 1$, $x_0 = 1.5$, $x_{-1} = 5.4$, $x_{-2} = 1.3$, $\overline{x} = 2$.



$fig.1$

3. Periodic solutions of Eq. (1)

In this part of the research we are studying the possibility of the existence of periodic solutions to the eq. (1).

Theorem 3. *If* $\delta = 1$. *In the all following cases, Equation (1) has no positive prime period-two solutions:*

(1) If k, ℓ and s are all even positive number.
(2) If k, ℓ and s are all odd positive number.
(3) If k is even and ℓ, s are both odd positive number.
(4) If k, ℓ are both even and s odd positive number.
(5) If k is odd and ℓ, s are both even positive number.
(6) If k, ℓ are both odd and s is even positive number.
(7) If k, s are both odd and ℓ is even positive number.

Proof. Case(1) Suppose that there exists a prime period-two solution

$$..., p, q, p, q, p, q, , ...$$

If k, ℓ even then $x_n = x_{n-k} = x_{n-\ell} = x_{n-s} = q$, $x_{n+1} = p$

$$p = \alpha q + \frac{\beta}{\gamma}, \tag{10}$$

also,

$$q = \alpha p + \frac{\beta}{\gamma}. \tag{11}$$

By (10) and (11), we have

$$(p - q)(\alpha + 1) = 0 \qquad \Longrightarrow \qquad p = q$$

Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed. □

The following theorem states the sufficient conditions that the Eq (1) has periodic solutions of prime period two.

Theorem 4. *Assume that k, s are both even and ℓ is odd positive integers and $\delta = 1$. If*

$$3\alpha < 1, \tag{12}$$

then Eq. (1) has prime period two solution.

Proof. Suppose that there exists a prime period-two solution

$$..., p, q, p, q, p, q, , ...$$

of (1). We will prove that condition (12) holds.

We see from (1) that if k, s are both even and ℓ is odd, then $x_n = x_{n-k} = x_{n-s} = q$, $x_{n+1} = x_{n-\ell} = p$

$$p = \alpha q + \frac{\beta p}{\gamma q},$$

and

$$q = \alpha p + \frac{\beta q}{\gamma p},$$

we have

$$\gamma pq = \alpha\gamma q^2 + \beta p, \tag{13}$$

and

$$\gamma pq = \alpha\gamma p^2 + \beta q. \tag{14}$$

By subtracting (13) and (14), we have

$$\alpha\gamma\left(q^2 - p^2\right) + \beta\left(p - q\right) = 0,$$

then,

$$(p + q) = \frac{\beta}{\alpha\gamma}. \tag{15}$$

By Combining (13) and (14), we have

$$2\gamma pq = \alpha\gamma(p^2 + q^2) + \beta(p + q), \tag{16}$$

then,

$$p^2 + q^2 = (p + q)^2 - 2pq. \tag{17}$$

Form (15), (16) and (17), we get

$$\begin{aligned} 2\gamma pq\left(1 + \alpha\right) &= \alpha\gamma\left[\frac{\beta}{\alpha\gamma}\right]^2 + \beta\left[\frac{\beta}{\alpha\gamma}\right], \\ pq &= \frac{\beta^2}{\alpha\gamma^2(\alpha + 1)}. \end{aligned}$$

We have,

$$u^2 + \left(p + q\right)u + pq = 0 \quad \text{and} \quad \left(p + q\right)^2 - 4pq > 0,$$

then,

$$\left[\frac{\beta}{\alpha\gamma}\right]^2 - \frac{4\beta^2}{\alpha\gamma^2(1 + \alpha)} > 0,$$

which is true if

$$3\alpha < 1.$$

Hence, the proof is completed. □

Example 2. *Fig. 2, shows that Eq. (1) has prime period two solutions if* $k = s = 0, \ell = 1,\ \alpha = (1/16)\,,\ \beta = 2,\ \gamma = \delta = 1,$ *(see Table 1)*

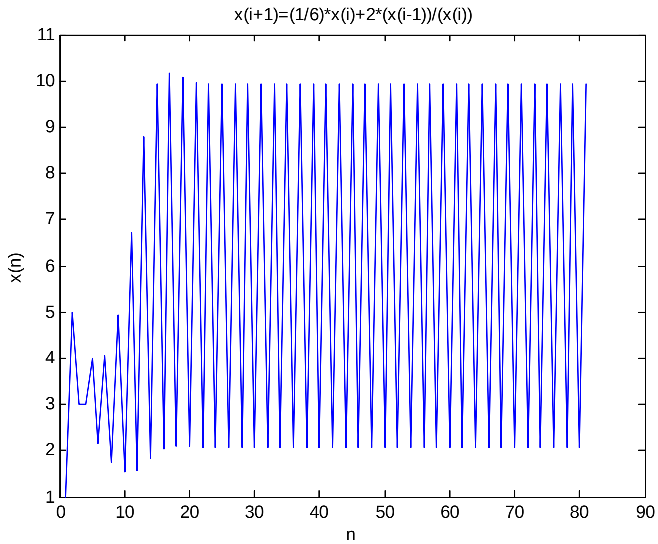


fig 2.

n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$
1	1.0000	17	10.1640	33	9.9283	49	9.9279
2	5.0000	18	2.0923	34	2.0721	50	2.0721
3	3.0000	19	10.0644	35	9.9281	51	9.9279
4	3.0000	20	2.0932	36	2.0721	52	2.0721
5	4.0000	21	9.9652	37	9.9280	53	9.9279
6	2.1667	22	2.0810	38	2.0721	54	2.0721
7	4.0534	23	9.9243	39	9.9279	55	9.9279
8	1.7446	24	2.0734	40	2.0721	56	2.0721
9	4.9375	25	9.9185	41	9.9279	57	9.9279
10	1.5296	26	2.0712	42	2.0721	58	2.0721
11	6.7109	27	9.9228	43	9.9279	59	9.9279
12	1.5743	28	2.0713	44	2.0721	60	2.0721
13	8.7877	29	9.9267	45	9.9279	61	9.9279
14	1.8229	30	2.0718	46	2.0721	62	2.0721
15	9.9452	31	9.9281	47	9.9279	63	9.9279
16	2.0241	32	2.0720	48	2.0721	64	2.0721

Tabe 1

4. Global stability

Theorem 5. *If* $\alpha < 1$, *then the equilibrium point* $\overline{x}$ *of Eq. 1 is global attractor.*

Proof. We consider the following function

$$f(u,v,w) = \alpha u + \frac{\beta v^{\delta}}{\gamma w^{\delta}},$$

f are increasing for u, v and decreasing for w.

Let $m = f(m,m,M)$ and $M = f(M,M,m)$

$$m = \alpha m + \frac{\beta m^{\delta}}{\gamma M^{\delta}}, \tag{18}$$

$$M = \alpha M + \frac{\beta M^{\delta}}{\gamma m^{\delta}}, \tag{19}$$

from (18)

$$\gamma m M^{\delta}(1-\alpha) = \beta m^{\delta}, \tag{20}$$

from (19)

$$\gamma M m^{\delta}(1-\alpha) = \beta M^{\delta}. \tag{21}$$

Subtracting Equation (20) of (21) produces

$$\gamma(1-\alpha)\left(mM^{\delta} - Mm^{\delta}\right) - \beta\left(m^{\delta} - M^{\delta}\right) = 0,$$

then

$$M = m$$

Hence, the proof is completed. □

5. Oscillatory solution

Theorem 6. *Eq.(1) has an oscilatory solution If $k = \max\{k,\ell,s\}$ and k,ℓ is odd and s is even.*

Proof. First, assume that,

$$x_{-k}, x_{-k+2}, x_{-k+4}, ..., x_{-1} > \overline{x} \quad \textit{and} \quad x_{-k+1}, x_{-k+3}, ..., x_0 < \overline{x}$$

so

$$x_1 = \alpha x_{-k} + \frac{\beta x_{-\ell}^{\delta}}{\gamma x_{-s}^{\delta}},$$

then

$$x_1 > \alpha\overline{x} + \frac{\beta\overline{x}^{\gamma}}{\gamma\overline{x}^{\delta}},$$

and

$$x_1 > \frac{\beta}{\gamma(1-\alpha)} = \overline{x}.$$

So, we have

$$x_2 = \alpha x_{-k+1} + \frac{\beta x_{-\ell+1}^{\delta}}{\gamma x_{-s+1}^{\delta}},$$

so,

$$x_2 < \alpha + \frac{a\overline{x}^{\gamma}}{b\overline{x}^{\gamma} + c\overline{x}^{\gamma}},$$

then,

$$x_2 < \frac{\beta}{\gamma(1-\alpha)} = \overline{x}.$$

Secandiy assume that,

$$x_{-k}, x_{-k+2}, x_{-k+4}, ..., x_{-1} < \overline{x} \quad \text{and} \quad x_{-k+1}, x_{-k+3}, ..., x_0 > \overline{x},$$

$$x_1 = \alpha x_{-k} + \frac{\beta x_{-\ell}^{\delta}}{\gamma x_{-s}^{\delta}},$$

then,

$$x_1 < \alpha \overline{x} + \frac{\beta \overline{x}^{\gamma}}{\gamma \overline{x}^{\delta}},$$

and

$$x_1 < \frac{\beta}{\gamma(1-\alpha)} = \overline{x}.$$

So, we have

$$x_2 = \alpha x_{-k+1} + \frac{\beta x_{-\ell+1}^{\delta}}{\gamma x_{-s+1}^{\delta}},$$

so,

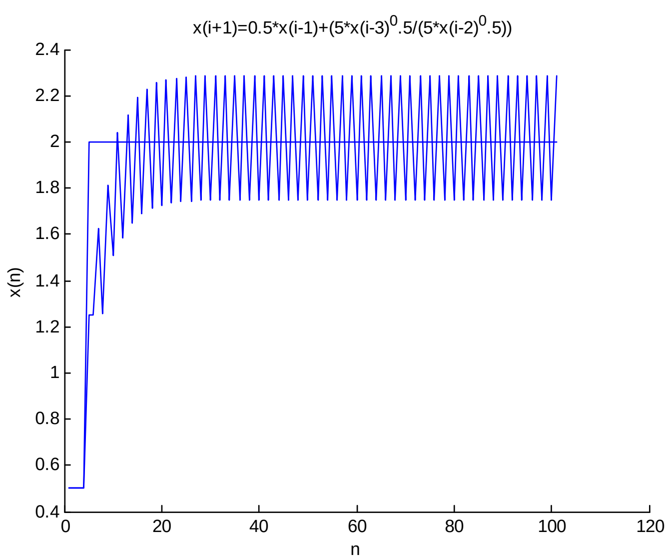
$$x_2 > \alpha \overline{x} + \frac{\beta \overline{x}^{\gamma}}{\gamma \overline{x}^{\delta}},$$

then,

$$x_2 > \frac{\beta}{\gamma(1-\alpha)} = \overline{x}.$$

One camproceed in prove manwer to show that $x_3 < \overline{x}$ and $x_4 > \overline{x}$ and soon. Hence, the proof is completed. □

Example 3. *Fig. 3, shows that Eq.(1) has 0scilatory solution if* $\alpha = 0.5$, $\beta = 5$, $\gamma = 5$, $\delta = 0.5$, $\overline{x} = 2$.(*see Table* 2)



$fig.3$

1	0.5000	17	2.2298	33	2.2878	49	2.2888
2	0.5000	18	1.7129	34	1.7471	50	1.7476
3	0.5000	19	2.2541	35	2.2882	51	2.2888
4	0.5000	20	1.7273	36	1.7473	52	1.7477
5	1.2500	21	2.2680	37	2.2884	53	2.2888
6	1.2500	22	1.7354	38	1.7475	54	1.7477
7	1.6250	23	2.2764	39	2.2886	55	2.2888
8	1.2575	24	1.7404	40	1.7475	56	1.7477
9	1.8125	25	2.2814	41	2.2887	57	2.2888
10	1.5058	26	1.7433	42	1.7476	58	1.7477
11	2.0430	27	2.2844	43	2.2887	59	2.2888
12	1.5858	28	1.7451	44	1.7476	60	1.7477
13	2.1186	29	2.2862	45	2.2887	61	2.2888
14	1.6514	30	1.7461	46	1.7476	62	1.7477
15	2.1944	31	2.2872	47	2.2888	63	2.2888
16	1.6909	32	1.7467	48	1.7476	64	1.7477

Table 2

6. Boundedness of the solutions

Theorem 7. *Let* $\{x_n\}_{n=-\max\{k,\ell,s\}}^{\infty}$ *be asolution of Eq (1), then the following statements are true :-*

(1) Assume that $\beta < \gamma$ and let for some $N \geq 0, x_{N-\ell+1}, ..., x_{N-1}, x_N \in \left[\frac{\beta}{\gamma}, 1\right]$ are valid, then we have

$$\frac{\alpha\beta^{\delta}}{\gamma^{\delta}} + \frac{\beta^{\delta}}{\gamma^{\delta-1}} \leq x_n \leq \alpha + \frac{\gamma^{\delta-1}}{\beta^{\delta-1}}$$

(2) Assume that $\beta > \gamma$ and for some $N \geq 0, x_{N-\ell+1}, ..., x_N \in \left[1, \frac{\beta}{\gamma}\right]$ are valid , Then we have

$$\alpha + \frac{\gamma^{\delta-1}}{\beta^{\delta-1}} \leq x_n \leq \frac{\alpha\beta^{\delta}}{\gamma^{\delta}} + \frac{\beta^{\delta}}{\gamma^{\delta-1}}$$

Proof. (1) If $\beta < \gamma$ then $x_{N-\ell+1}, ..., x_{N-1}, x_N \in \left[\frac{\beta}{\gamma}, 1\right]$

$$x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}^{\delta}}{\gamma x_{n-s}^{\delta}},$$

then,

$$\begin{aligned} &\leq \quad \alpha + \frac{\beta}{\gamma\left(\frac{\beta}{\gamma}\right)^{\delta}}, \\ &\leq \quad \alpha + \frac{\gamma^{\delta-1}}{\beta^{\delta-1}}, \end{aligned}$$

and

$$x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}^{\delta}}{\gamma x_{n-s}^{\delta}},$$

then,

$$\geq \frac{\alpha\beta}{\gamma} + \frac{\beta^{\delta+1}}{\gamma^{\delta-1}}.$$

Then

$$\frac{\alpha\beta}{\gamma} + \frac{\beta^{\delta+1}}{\gamma^{\delta-1}} \leq x_n \leq \alpha + \frac{\gamma^{\delta-1}}{\beta^{\delta-1}}.$$

Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed. □

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