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CERTAIN SUBCLASS OF $p-$VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH LIU-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce the class $\Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$ of meromorphic $p-$valent functions of order $\gamma(0 \leq \gamma < p)$ and types β in the punctured unit disc $\mathbb{U}^*$ which are defined by making use of Liu-Srivastava operator. We investigate various properties and characteristics of this class.

1. Introduction

The class of meromorphic functions which are analytic and p-valent in $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$ and has the form:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \ (p,\ n \in \mathbb{N} = \{1, 2, ...\},\ p < n), \tag{1}$$

is denoted by $\sum_p$. For the function $f(z)$ in this form and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p} \ (p \in \mathbb{N}), \tag{2}$$

the Hadamard products (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} = (g * f)(z). \tag{3}$$

For complex numbers $\alpha_1, ..., \alpha_q, \beta_1, ..., \beta s$ $(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ..., \}; j = 1, 2, ..., s)$, Liu and Srivastava [8] considered the linear operator

$$M_{p,q,s}(\alpha_1)f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_q)_n} \frac{a_{n-p}}{n!} z^{n-p}, \tag{4}$$

where $(\theta)_\mu$ is the Pochhammer symbol defined by

$$(\theta)_\mu = \frac{\Gamma(\theta + \mu)}{\Gamma(\theta)} = \begin{cases} 1 & (\mu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ \theta(\theta + 1)..(\theta + \mu - 1), & (\mu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{5}$$

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For fixed parameters ($0 \leq \gamma < p, 0 < \beta \leq 1, \frac{1}{2} \leq \varrho \leq 1$ and $p \in \mathbb{N}$), we say that a function $f(z) \in \sum_p$ is in the class $\Lambda_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$ if it also satisfies:

$$\left|\frac{z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + p}{(2\varrho-1)z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + (2\varrho\gamma - p)}\right| < \beta \ (z \in \mathbb{U}^*). \tag{6}$$

Let $\sum_p^*$ be the class of missing functions of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n|\, z^n \,(p \in \mathbb{N}; z \in \mathbb{U}^*)\,. \tag{7}$$

Furthermore we say that a function $f(z) \in \Lambda^+_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$ when $f(z)$ is of the form (7) and satisfies (6).

We note that:
$\Lambda_{p,q,s}(\alpha_1;\gamma,1,1) = \sum^+_{p,q,s}(\alpha_1;1,-1,\lambda)$ (see Aouf [3]).
For more details of meromorphic multivalent functions see ([1],[2],[4],[5],[6],[7],[9],[10]).

In this paper we investigate various important properties and characteristics of the class $\Lambda^+_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$.

2. Properties of the class $\Lambda^+_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$

In the reminder of this paper we assume that: $\alpha_j > 0 (j = 1,...,q), \beta_j > 0 (j = 1,...,s), 0 \leq \gamma < p, 0 < \beta \leq 1, \frac{1}{2} \leq \varrho \leq 1, p \in \mathbb{N}, z \in \mathbb{U}^*$.

Theorem 1. *The function $f(z) \in \Lambda^+_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$ if and only if*

$$\sum_{n=p}^{\infty} n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)\,|a_n| \leq 2\beta\varrho(p-\gamma), \tag{8}$$

where for convenience

$$\Gamma_c(\alpha_1) = \frac{(\alpha_1)_c ... (\alpha_q)_c}{(\beta_1)_c ... (\beta_s)_c c!}. \tag{9}$$

Proof. Let $f(z) \in \Lambda^+_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$. Then from (6) and (7), we have

$$\left|\frac{z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + p}{(2\varrho-1)z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + (2\varrho\gamma - p)}\right|$$

$$= \left|\frac{\sum\limits_{n=p}^{\infty} n\Gamma_{n+p}(\alpha_1)\,|a_n|\, z^{n+p}}{2\varrho(p-\gamma) - \sum\limits_{n=p}^{\infty} n(2\varrho-1)\Gamma_{n+p}(\alpha_1)\,|a_n|\, z^{n+p}}\right| < \beta. \tag{10}$$

Since $Re(z) \leq |z|\,(z \in \mathbb{C})$, choosing z to be real and letting $z \to 1^-$, then (10) yields

$$\sum_{n=p}^{\infty} n\Gamma_{n+p}(\alpha_1)\,|a_n| \leq 2\beta\varrho(p-\gamma) - \sum_{n=p}^{\infty} n\beta(2\varrho-1)\Gamma_{n+p}(\alpha_1)\,|a_n|\,, \tag{11}$$

which leads to (8).

In order to prove the converse, we assume that the inequality (8) holds. Then , if we let $z \in \partial\mathbb{U}$, we find from (7) and (8) that

$$\left|\frac{z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + p}{(2\varrho-1)z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + (2\varrho\gamma - p)}\right|$$

$$\leq \frac{\sum\limits_{n=p}^{\infty} n\Gamma_{n+p}(\alpha_1)\left|a_n\right|\left|z\right|^{n+p}}{2\varrho(p-\gamma) - \sum\limits_{n=p}^{\infty} n(2\varrho-1)\Gamma_{n+p}(\alpha_1)\left|a_n\right|\left|z\right|^{n+p}}$$

$$< \beta(z \in \partial\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}). \tag{12}$$

Hence , by the maximum modulus theorem , we have $f(z) \in \Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$. This completes the proof of Theorem 1. □

Corollary 2. *If $f(z) \in \Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$, then*

$$|a_n| \leq \frac{2\beta\varrho(p-\gamma)}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)} \ (n \geq p). \tag{13}$$

The result is sharp for f given by

$$f(z) = z^{-p} + \frac{2\beta\varrho(p-\gamma)}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)} z^n \ (n \geq p). \tag{14}$$

Next we prove the growth and distortion properties for the class $\Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$.

Theorem 3. *If a function $f(z) \in \Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$ and the sequence $\{D_n\}$ is nondecreasing, then*

$$\left(\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!}\cdot\frac{(p-\gamma)}{D_p}r^{2p}\right)r^{-(p+m)}$$

$$\leq \left|f^{(m)}(z)\right| \leq \left(\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!}\cdot\frac{p-\gamma}{D_p}r^{2p}\right)r^{-(p+m)}$$

$$(0 < |z| = r < 1; m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}; p \in \mathbb{N}; p > m), \tag{15}$$

where

$$D_n = n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1). \tag{16}$$

Equality holds for

$$f(z) = z^{-p} + \frac{2\beta\varrho(p-\gamma)}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)} z^p. \tag{17}$$

Proof. In view of Theorem 1, we have

$$\frac{D_p}{p!}\sum_{n=p}^{\infty} n!\left|a_n\right| \leq \sum_{n=p}^{\infty} D_n\left|a_n\right| \leq p-\gamma,$$

which yields

$$\sum_{n=p}^{\infty} n!\left|a_n\right| \leq \frac{2p!\beta\varrho(p-\gamma)}{D_p}. \tag{18}$$

Now, differentiating both sides of (7) $m-$times with respect to z, we have

$$f^{(m)}(z) = (-1)^{(m)}\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)} + \sum_{n=p}^{\infty}\frac{n!}{(n-m)!}\left|a_n\right| z^{n-m}, \tag{19}$$

$$(m \in \mathbb{N}_0 : p \in \mathbb{N}; p > m).$$

From (18) and (19), we have

$$\left|f^{(m)}(z)\right| \leq \frac{(p+m-1)!}{(p-1)!} r^{-(p+m)} + r^{p-m} \sum_{n=p}^{\infty} \frac{n!}{(n-m)!} |a_n|$$

$$\leq \left(\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!} \cdot \frac{p-\gamma}{D_p} r^{2p} \right) r^{-(p+m)}$$

and

$$\left|f^{(m)}(z)\right| \geq \frac{(p+m-1)!}{(p-1)!} r^{-(p+m)} - r^{p-m} \sum_{n=p}^{\infty} \frac{n!}{(n-m)!} |a_n|$$

$$\geq \left(\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!} \cdot \frac{p-\gamma}{D_p} r^{2p} \right) r^{-(p+m)}$$

This completes the proof of Theorem 3. □

Theorem 4. *Let $f(z) \in \Lambda^{+}_{p,q,s}(\alpha_1; \gamma, \varrho, \beta)$. Then*

(i) $f(z)$ is meromorphically $p-$valent starlike of order $\delta(0 \leq \delta < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_{n \geq p} \left\{ \frac{n(p-\delta)[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)[(p+n)+(p-\delta)]} \right\}^{\frac{1}{n+p}}, \tag{20}$$

(ii) $f(z)$ is meromorphically $p-$valent convex of order $\delta(0 \leq \delta < p)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_{n \geq p} \left\{ \frac{(p-\delta)[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)[(p+n)+(p-\delta)]} \right\}^{\frac{1}{n+p}}. \tag{21}$$

Each of these results is sharp for the function $f(z)$ given by (14).

Proof. (i) From (7), we easily get

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq \frac{\sum\limits_{n=p}^{\infty} (p+n) |a_n| z^{n+p}}{1 - \sum\limits_{n=p}^{\infty} |a_n| z^{n+p}}. \tag{22}$$

Thus,

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \delta(0 \leq \delta < p) \tag{23}$$

if

$$\sum_{n=p}^{\infty} \frac{(p+n)+(p-\delta)}{(p-\delta)} |a_n| |z|^{n+p} \leq 1. \tag{24}$$

Hence, by Theorem 1, (14) will be true if

$$\frac{(p+n)+(p-\delta)}{(p-\delta)} |z|^{n+p} \leq \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}, \tag{25}$$

that is if

$$|z|^{n+p} \leq \frac{n(p-\delta)[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)[(p+n)+(p-\delta)]}.$$

(ii) Also from (7), we have

$$\left|1+\frac{zf''(z)}{f'(z)}+p\right| \leq \frac{\sum\limits_{n=p}^{\infty} n(p+n)\left|a_n\right| z^{n+p}}{1-\sum\limits_{n=p}^{\infty} n\left|a_n\right| z^{n+p}} \tag{26}$$

Thus,

$$\left|1+\frac{zf''(z)}{f'(z)}+p\right| \leq p-\delta(0\leq\delta<p) \tag{27}$$

if

$$\sum_{n=p}^{\infty}\frac{n[(p+n)+(p-\delta)]}{(p-\delta)}\left|a_n\right|\left|z\right|^{n+p} \leq 1. \tag{28}$$

Hence, by Theorem 1, (28) will be true if

$$\frac{n[(p+n)+(p-\delta)]}{(p-\delta)}\left|z\right|^{n+p} \leq \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}, \tag{29}$$

the proof of Theorem 4 is completed by merely verifying that each assertion is sharp for the function given by (14). □

For functions

$$f_i(z)=z^{-p}+\sum_{n=p}^{\infty}\left|a_{n,i}\right| z^n \ (i=1,2;p\in\mathbb{N}), \tag{30}$$

the Hadamard products (or convolution) of functions $f_1(z)$ and $f_2(z)$, is given by

$$(f_1*f_2)(z)=z^{-p}+\sum_{n=p}^{\infty}\left|a_{n,1}\right|\left|a_{n,2}\right| z^n. \tag{31}$$

Theorem 5. *Let $f_i(z)$ $(i=1,2)\in\Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$, where $f_i(z)$ $(i=1,2)$ are in the form (30). Then $(f_1*f_2)(z)\in\Lambda^{+}_{p,q,s}(\alpha_1;\delta,\varrho,\beta)$, where*

$$\delta=p-\frac{2\beta\varrho(p-\gamma)^2}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)}. \tag{32}$$

Sharppness holds for functions

$$f_i(z)=z^{-p}+\frac{2\beta\varrho(p-\gamma)}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)}z^p \ (i=1,2;\ p\in\mathbb{N}). \tag{33}$$

Proof. Using the technique for analytic functions, we need to find the largest real parameter δ such that

$$\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\delta)}\left|a_{n,1}\right|\left|a_{n,2}\right| \leq 1. \tag{34}$$

Since $f_i(z)\in\Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\kappa,\beta)$ $(i=1,2)$, we have

$$\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}\left|a_{n,1}\right| \leq 1 \tag{35}$$

and

$$\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}\left|a_{n,2}\right| \leq 1. \tag{36}$$

By Cauchy-Schwarz inequality we have

$$\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}\sqrt{|a_{n,1}|\,|a_{n,2}|}\leq 1. \tag{37}$$

thus, it is sufficient to show that

$$\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\delta)}\,|a_{n,1}|\,|a_{n,2}|\leq\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}\sqrt{|a_{n,1}|\,|a_{n,2}|} \tag{38}$$

or, equivalently, that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|}\leq\frac{(p-\delta)}{(p-\gamma)}. \tag{39}$$

Hence, in the light of the inequality (37), it is sufficient to prove that

$$\frac{2\beta\varrho(p-\gamma)}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}\leq\frac{(p-\delta)}{(p-\gamma)}. \tag{40}$$

It follows from (40) that

$$\delta\leq p-\frac{2\beta\varrho(p-\gamma)^2}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}. \tag{41}$$

Let

$$M(n)=p-\frac{2\beta\varrho(p-\gamma)^2}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}, \tag{42}$$

then $M(n)$ is increasing function of $n(n\geq p)$. Therefore, we conclude that

$$\delta\ \leq M\ (p)\ =p-\frac{2\beta\varrho(p-\gamma)^2}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)} \tag{43}$$

and hence the proof of Theorem 5 is completed. □

Theorem 6. *Let* $f_1(z)\in\Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$ *and* $f_2(z)\in\Lambda^{+}_{p,q,s}(\alpha_1;\lambda,\varrho,\beta)$*, where* $f_i(z)(i=1,2)$ *are in the form (31). Then* $(f_1*f_2)(z)\in\Lambda^{+}_{p,q,s}(\alpha_1;\zeta,\varrho,\beta)$*,where*

$$\xi=p-\frac{2\beta\varrho(p-\gamma)(p-\lambda)}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)}. \tag{44}$$

Sharpness holds for

$$f_1(z)=z^{-p}+\frac{2\beta\varrho(p-\gamma)}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)}z^p \tag{45}$$

and

$$f_2(z)=z^{-p}+\frac{2\beta\varrho(p-\lambda)}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)}z^p. \tag{46}$$

Proof. We need to find the largest real parameter ξ such that

$$\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\xi)}\,|a_{n,1}|\,|a_{n,2}|\leq 1. \tag{47}$$

Since $f_1(z)\in\Lambda^{+}_{p,q,s}(\alpha_1;\gamma,\varrho,\beta)$, we have

$$\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}\,|a_{n,1}|\leq 1 \tag{48}$$

and $f_2(z) \in \Lambda^+_{p,q,s}(\alpha_1;\lambda,\varrho,\beta)$, we have

$$\sum_{n=p}^{\infty} \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\lambda)} |a_{n,2}| \leq 1. \tag{49}$$

By Cauchy-Schwarz inequality we have

$$\sum_{n=p}^{\infty} \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho\sqrt{p-\gamma}\sqrt{p-\lambda}} \sqrt{|a_{n,1}|\,|a_{n,2}|} \leq 1, \tag{50}$$

thus it is sufficient to show that

$$\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\xi)} |a_{n,1}|\,|a_{n,2}| \leq \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho\sqrt{p-\gamma}\sqrt{p-\lambda}} \sqrt{|a_{n,1}|\,|a_{n,2}|} \tag{51}$$

or, equivalently, that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{(p-\xi)}{\sqrt{p-\gamma}\sqrt{p-\lambda}}. \tag{52}$$

Hence, in the light of the inequality (50), it is sufficient to prove that

$$\frac{2\beta\varrho\sqrt{p-\gamma}\sqrt{p-\lambda}}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)} \leq \frac{(p-\xi)}{\sqrt{p-\gamma}\sqrt{p-\lambda}}. \tag{53}$$

It follows from (53) that

$$\xi \leq p - \frac{2\beta\varrho(p-\gamma)(p-\lambda)}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}. \tag{54}$$

Let

$$A(n) = p - \frac{2\beta\varrho(p-\gamma)(p-\lambda)}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}, \tag{55}$$

then $A(n)$ is increasing function of $n (n \geq p)$. Therefore, we conclude that

$$\xi \ \leq A(p) = p - \frac{2\beta\varrho(p-\gamma)(p-\lambda)}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)} \tag{56}$$

and hence the proof of Theorem 6 is completed. □

Theorem 7. *Let $f_i(z)(i=1,2) \in \Lambda^+_{p,q,s}(\alpha_1;\alpha,\varrho,\beta)$, where $f_i(z)(i=1,2)$ are in the form (31). Then*

$$h\ (z) = z^{-p} + \sum_{n=p}^{\infty} (|a_{n,1}|^2 + |a_{n,2}|^2) z^n \tag{57}$$

belongs to the class $\Lambda^+_{p,q,s}(\alpha_1;\varphi,\varrho,\beta)$, where

$$\varphi = p - \frac{4\beta\varrho(p-\gamma)^2}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)}. \tag{58}$$

Sharpness holds for $f_i(z)(i=1,2)$ defined by (33).

Proof. By using Theorem 5, we obtain

$$\sum_{n=p}^{\infty} \left\{ \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)} \right\}^2 |a_{n,1}|^2 \leq \left\{ \sum_{n=p}^{\infty} \frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)} |a_{n,1}| \right\}^2 \leq 1, \tag{59}$$

and

$$\sum_{n=p}^{\infty}\left\{\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}\right\}|a_{n,2}|^2 \leq \left\{\sum_{n=p}^{\infty}\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)}|a_{n,2}|\right\}^2 \leq 1. \tag{60}$$

It follows form (59) and (60) that

$$\sum_{n=p}^{\infty}\frac{1}{8}\left\{\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{\beta\varrho(p-\gamma)}\right\}^2(|a_{n,1}|^2+|a_{n,2}|^2) \leq 1. \tag{61}$$

Therefore, we need to find the largest φ such that

$$\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\varphi)} \leq \frac{1}{8}\left\{\frac{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}{\beta\varrho(p-\gamma)}\right\}^2 \tag{62}$$

that is

$$\varphi \leq p-\frac{4\beta\varrho(p-\gamma)^2}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}. \tag{63}$$

Let

$$H(n)=p-\frac{4\beta\varrho(p-\gamma)^2}{n[1+\beta(2\varrho-1)]\Gamma_{n+p}(\alpha_1)}, \tag{64}$$

then $H(n)$ is increasing function of $n(n \geq p)$. Therefore, we conclude that

$$\varphi \leq H(p)=p-\frac{4\beta\varrho(p-\gamma)^2}{p[1+\beta(2\varrho-1)]\Gamma_{2p}(\alpha_1)} \tag{65}$$

and hence the proof of Theorem 7 is completed. □

Remark 8. . *Putting $\varrho=1$ in our results we obtain the results obtained by Aouf* [3, with $A=1$and $B=-1$].

References

[1] M. K. Aouf, A generalization of meromorphic multivalent functions with positive coefficients, Math.Japon., 35 (1990), 609-614.

[2] M. K. Aouf, On a class of meromorphic multivalent functions with positive coefficients, Math.Japon., 35 (1990), 603-608.

[3] M. K. Aouf, Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, Comput., Math. Appl., 55 (2008), 494-509.

[4] M. K. Aouf, The generalized hypergeometric function and associated families of meromorphically multivalent functions, J. Math. Ineq., 3 (2009), no. 1, 43-62.

[5] M. K. Aouf and H. M. Hossen, New criteria for meromorphic $p-$valent starlike functions, Tsukuba J. Math., 17 (1993), 481-486.

[6] M. K. Aouf, H. M. Hossen and H. E. Elattar, A certain class of meromorphic multivalent functions with positive coefficients with fixed second coefficients, Punjab Univ. J. Math., 33 (2000), 115-124..

[7] S. B. Joshi and M. K. Aouf, Meromorphic multivalent functions with positive coefficients and fixed second coefficients, Kyungpook Math. J., 35 (1995), 163-169.A

[8] J. L. Liu and H. M. Srivastava, Class of meromorphically multivalent functions associated with generalized hypergeometric function, Math. Comput. Modelling, 39 (2004), 21-34.

[9] S. Owa, H. E. Darwish and M. K. Aouf, Meromorphic multivalent functions with positive coefficients and fixed second coefficients, Math. Japon., 46 (1997), no. 2, 231-236.

[10] H. M. Srivastava, H. M. Hossen and M. K. Aouf, A unified presentation of some classes of meromorphically multivalent functions, Comput. Math. Appl., 38 (1999), 63-70.

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