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BOUNDEDNESS AND COMPACTNESS OF CERTAIN CLASSES OF GENERALIZED INTEGRAL OPERATORS

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Abstract. Integral operators in general and the Cesàro integral operator in particular have long been used in the study of the various branches of analysis such as geometric function theory. We consider certain classes of generalized integral operators of Cesàro type and investigate their topological properties such as boundedness and compactness. We conclude our paper with the extension of two results on the Cesàro integral operators.

1. Introduction

Integral operators in general and the Cesàro integral operator

$$C_\mu f(z) = \frac{1}{z}\int_0^z f(\xi)\xi^{\mu-1}(1-\xi)^{-1}d\xi,$$

in particular have long been used in the study of the various branches of analysis such as geometric function theory. Dahlner [5] provided a study of the Cesàro integral operator on Hardy and Bergman spaces. Persson [9] imposed a complete spectral feature in H^p, $p \geq 1$ as well as $L_a^{p,\rho}$, $p \geq 1$, $\rho \geq 1$. Ballamoole, Miller and Miller [3] introduced instructions regarding spectral properties of the Cesàro-like operators on weighted Bergman spaces. Alemann and Persson [2] extended the Cesàro integral operator in the form

$$C_g f(z) = \frac{1}{z}\int_0^z f(\xi)g'(\xi)d\xi,$$

where g' is defined in the Banach space of analytic functions.

The Cesáro operator $\mathcal{C}$ expansion of the power series $f(z) = \sum_{n=0}^{\infty} a_n(f)z^n$ may be written as

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\sum_{k=0}^{n} a_k(f)\Big)z^n.$$

The history of the Cesáro operator can be traced back to Hardy (e.g. see Rudin [10]), who was amidst the first to show that $\mathcal{C}$ is bounded on H^2. The fact that

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the Cesáro operator is bounded follows from the work of Siskakis [11],[12]. The boundedness of $\mathcal{C}$ on H^1 was proved by utilizing a result of Hardy and Littlewood by Miao [8] who showed that $\mathcal{C}$ is bounded on $H^p, p \in (0,1)$. Further studies can be found in [6]. We note that the Cesáro operator is unbounded on H^∞ (e.g. see [10]) so that it is reasonable to work in a larger space of analytic functions. Miao in [8] determined the coefficient inequalities for concave Cesáro operators on non-concave analytic functions in the unit disk and a generalization of Cesáro operators has recently appeared in the work of Albert and Miller [1] and Ballamoole, Miller and Miller [4].

In the present paper, we consider certain classes of generalized integral operators of Cesàro type and investigate their topological properties such as boundedness and compactness in the space $\mathcal{B}_{\log}$ and its extension $\mathcal{B}^{\beta}_{\log}$, $\beta \in (0,1)$. We conclude the paper with two results that extend the Cesàro integral operator $C_g f$ to the space $L^2(U;H)$.

2. Boundedness and Compactness

Let $\mathcal{A}$ be a class of functions f which are analytic and normalized by $f(0) - f'(0) - 1 = 0$ in the open unit disk $U = \{z : |z| < 1\}$. For the functions f and g in $\mathcal{A}$, we consider the boundedness and compactness of the operator $C_g f(z)$ and the space $\mathcal{B}_{\log}$ of all functions $f \in \mathcal{A}$ which satisfy the condition

$$\|f\|_{\mathcal{B}_{\log}} = \sup_{z\in U}(1-|z|^2)|\frac{f'(z)}{f(z)}| \ln \frac{1}{(1-|z|^2)} < \infty, \quad (z \in U).$$

A function $f \in \mathcal{A}$ is said to be in the class Σ if and only if it has the norm

$$\|f\| = \sup_{z\in U}(1-|z|^2)|\frac{f''(z)}{f'(z)}| < \infty, \ \ (z \in U).$$

Note that the fraction $T_f := \frac{f''(z)}{f'(z)}$ is called pre-Schwarzian derivative which is usually used to discuss the univalency of analytic functions (e.g. see [7, 13, 14, 15]). Now we are ready to state and prove our first theorem

Theorem 2.1. Let the functions f and g be in $\mathcal{A}$ so that $f \in \mathcal{B}_{\log}$. Then the integral operator $C_g f$ is bounded in Σ if and only if $g \in \Sigma$.

Proof. First, assume that $C_g f$ is bounded. If $f(z) = 1$ then $g \in \Sigma$ by the maximum principle theorem. Conversely, assume that. Then we have

$$\frac{(C_g f)''(z)}{(C_g f)'(z)} = \frac{f(z)g''(z)}{f(z)g'(z) - C_g f} + \frac{f'(z)g'(z)}{f(z)g'(z) - C_g f} - \frac{2}{z}.$$

Consequently, by the maximum principle, we obtain

$$|\frac{(C_g f)''(z)}{(C_g f)'(z)}| \le \frac{|f(z)g''(z)|}{|f(z)g'(z)| - |\int_0^z f(\xi)g'(\xi)d\xi|} + \frac{|f'(z)g'(z)|}{|f(z)g'(z)| - |\int_0^z f(\xi)g'(\xi)d\xi|} + 2.$$

Now, by using the inequality

$$\Big|\int_0^x h(x)\Big| \le \frac{2}{8} \max_{x\in[0,1]} |h''|, \quad h(1) = 0$$

and taking in account that f vanishes on ∂U, we obtain

$$|\frac{(C_gf)''(z)}{(C_gf)'(z)}| \le \frac{|f(z)g''(z)|}{|f(z)g'(z)| - \frac{2}{8}\max_{z\in U}|(fg')''(z)|} + \frac{|f'(z)g'(z)|}{|f(z)g'(z)| - \frac{2}{8}\max_{z\in U}|(fg')''(z)|} + 2.$$

Thus, by utilizing the Gagliardo-Nirenberg-Sobolev inequality, for the second derivative and taking the maximum value, we conclude that there is a constant $\epsilon > 0$ such that

$$\begin{aligned}(1-|z|^2)|\frac{(C_gf)''(z)}{(C_gf)'(z)}| &\le \epsilon(1-|z|^2)|\frac{f(z)g''(z)+f'(z)g'(z)}{f(z)g'(z)}| + 2\\ &= \epsilon(1-|z|^2)[|\frac{g''(z)}{g'(z)} + \frac{f'(z)}{f(z)}|] + 2\\ &\le \epsilon\Big(\|g\| + \frac{(1-|z|^2)|\frac{f'(z)}{f(z)}|\ln[\frac{1}{1-|z|^2}]}{\ln[\frac{1}{1-|z|^2}]} + \frac{2}{\epsilon}\Big)\\ &\le \epsilon\Big(\|g\| + \frac{\|f\|_{\mathcal{B}_{\log}}}{\ln[\frac{1}{1-|z|^2}]} + \frac{2}{\epsilon}\Big)\\ &\le \epsilon\Big(\|g\| + \frac{\|f\|_{\mathcal{B}_{\log}}}{\ln[\frac{1}{1-|z|^2}](1-|z|^2)} + \frac{2}{\epsilon}\Big).\end{aligned}$$

Now the boundedness of the operator C_gf follows upon taking the supremum for the last assertion over U and using the finiteness of the quantity

$$\sup_{a\in(0,1]} a(\ln\frac{1}{a}).$$

This completes the proof.

In our second theorem, we determine a sufficient condition for functions to be in the class $\mathcal{B}_{\log}$.

Theorem 2.2. If $g \in \mathcal{A}$ and $C_gf : \Sigma \to \Sigma$ is bounded, then $f \in \mathcal{B}_{\log}$.

Proof. Assume that $C_g : \Sigma \to \Sigma$ is bounded. Then there is a positive constant C such that

$$\|C_gf\| \le C\|f\|$$

for every $f \in \Sigma$. Set

$$h_\omega(z) = \frac{(\overline{\omega}z - 1)}{\overline{\omega}}[(1+\ln\frac{1}{1-\overline{\omega}z})^2 + 1][\ln\frac{1}{1-|\omega|^2}]^{-1},$$

for $\omega \in U$ such that $\sqrt{1-\frac{1}{e}} < |\omega| < 1$. Then we have

$$h'_\omega(z) = (\ln\frac{1}{1-\overline{\omega}z})^2[\ln\frac{1}{1-|\omega|^2}]^{-1}$$

and

$$h''_\omega(z) = \frac{2\overline{\omega}}{1-\overline{\omega}z}(\ln\frac{1}{1-\overline{\omega}z})[\ln\frac{1}{1-|\omega|^2}]^{-1}.$$

Thus

$$\frac{h''_\omega(z)}{h'_\omega(z)} = \frac{2\overline{\omega}}{1-\overline{\omega}z}[\ln\frac{1}{1-\overline{\omega}z}]^{-1} \tag{1}$$

and then

$$\frac{h''_\omega(\omega)}{h'_\omega(\omega)} = \frac{2\overline{\omega}}{1-|\omega|^2}[\ln\frac{1}{1-|\omega|^2}]^{-1}.$$

It is clear that the relation (1) is finite when $|z| < 1$ and hence $\|h_\omega(z)\| < \infty$. Setting

$$M := \sup_{\sqrt{1-\frac{1}{e}}<|\omega|<1} \|h_\omega(z)\| < \infty$$

we have

$$\begin{aligned} \infty &> \|C_{h_\omega} f\| \\ &\geq \sup_{z\in U}(1-|z|^2)|\frac{h''_\omega(z)}{h'_\omega(z)} + \frac{f'(z)}{f(z)}| \\ &\geq (1-|\omega|^2)|\frac{h''_\omega(\omega)}{h'_\omega(\omega)} + \frac{f'(\omega)}{f(\omega)}| \\ &\geq |\frac{2\overline{\omega}}{\ln\frac{1}{1-|\omega|^2}} + (1-|\omega|^2)\frac{f'(\omega)}{f(\omega)}| \\ &\geq \frac{-2|\omega| + (1-|\omega|^2)|\frac{f'(\omega)}{f(\omega)}|\ln\frac{1}{1-|\omega|^2}}{\ln\frac{1}{1-|\omega|^2}}. \end{aligned} \tag{2}$$

Now let

$$g_\omega(z) := 2\frac{(\overline{\omega}z-1)}{\overline{\omega}}[(1+\ln\frac{1}{1-\overline{\omega}z})^2+1][\ln\frac{1}{1-|\omega|^2}]^{-1} - \int_0^z \ln\frac{1}{1-\overline{\omega}x}dx$$

for $a \in U$ such that $\sqrt{1-\frac{1}{e}} < |\omega| < 1$. Then we obtain

$$g'_\omega(z) = 2(\ln\frac{1}{1-\overline{\omega}z})^2[\ln\frac{1}{1-|\omega|^2}]^{-1} - \ln\frac{1}{1-\overline{\omega}z}$$

and

$$g''_\omega(z) = \frac{4\overline{\omega}}{1-\overline{\omega}z}(\ln\frac{1}{1-\overline{\omega}z})[\ln\frac{1}{1-|\omega|^2}]^{-1} - \frac{\overline{\omega}}{1-\overline{\omega}z}.$$

Thus, we conclude that

$$\frac{g''_\omega(\omega)}{g'_\omega(\omega)} = \frac{\frac{3|\omega|}{1-|\omega|^2}}{\ln\frac{1}{1-|\omega|^2}}.$$

Similarly, we have

$$N := \sup_{\sqrt{1-\frac{1}{e}}<|\omega|<1} \|f_a\| < \infty.$$

Consequently

$$
\begin{aligned}
\infty &> \|C_{g_\omega} f\| \\
&\geq \sup_{z\in U}(1-|z|^2)|\frac{g''_\omega(z)}{g'_\omega(z)} + \frac{f'(z)}{f(z)}| \\
&\geq (1-|\omega|^2)|\frac{g''_\omega(\omega)}{g'_\omega(\omega)} + \frac{f'(\omega)}{f(\omega)}| \\
&\geq (1-|\omega|^2)|\frac{\frac{3|\omega|}{1-|\omega|^2}}{\ln\frac{1}{1-|\omega|^2}} + \frac{f'(\omega)}{f(\omega)}| \\
&\geq \frac{-3|\omega| + (1-|\omega|^2)|\frac{f'(\omega)}{f(\omega)}|\ln\frac{1}{1-|\omega|^2}}{\ln\frac{1}{1-|\omega|^2}}.
\end{aligned} \tag{3}
$$

From (2) and (3) it follows that

$$
(1-|\omega|^2)|\frac{f'(\omega)}{f(\omega)}|\ln\frac{1}{(1-|\omega|^2)} < \infty \tag{4}
$$

for all $\sqrt{1-\frac{1}{e}} < |\omega| < 1$. Also, we have

$$
\sup_{|\omega|\leq\sqrt{1-\frac{1}{e}}}(1-|\omega|^2)|\frac{f'(\omega)}{f(\omega)}|\ln\frac{1}{(1-|\omega|^2)} \leq \sup_{\sqrt{1-\frac{1}{e}}\leq|\omega|<1}(1-|\omega|^2)|\frac{f'(\omega)}{f(\omega)}|\ln\frac{1}{(1-|\omega|^2)}. \tag{5}
$$

Now from (4) and (5) we conclude that $f \in \mathcal{B}_{\log}$, as desired.

Our next two theorems are on the compactness of the integral operator C_g in U.

Theorem 2.3. Assume that g is an analytic function on U. Then for $f \in \mathcal{B}_{\log}$, the integral operator $C_g f$ is compact if and only if $g \in \Sigma$.

Proof. If $C_g f$ is compact, then it is bounded, and by Theorem 2.1 it follows that $g \in \Sigma$. Now assume that $g \in \Sigma$ and that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}_{\log}$ such that $\max_{z\in U}(f_n g')''(z) = 0$ and that $f_n \to 0$ uniformly on $\overline{U}$ as $n \to \infty$. For every $\varepsilon > 0$ there is $\delta \in (0,1)$ such that

$$
\frac{1}{1-|z|^2} < \varepsilon,
$$

where $\delta < |z| < 1$. Since δ is arbitrary, we chose $\ln\frac{1}{1-|z|^2} > 1$ for $\delta < |z| < 1$ and

$$
\begin{aligned}
\|C_g f_n\| &= \sup_{z\in U}(1-|z|^2)|\frac{(C_g f_n)''(z)}{(C_g f_n)'(z)}| \\
&\leq \sup_{z\in U}(1-|z|^2)|\frac{f_n(z)g''(z) + f'_n(z)g'(z)}{f_n(z)g'(z)}| \\
&\leq \sup_{z\in U}(1-|z|^2)|\frac{g''(z)}{g'(z)}| + \sup_{z\in U}(1-|z|^2)|\frac{f'_n(z)}{f_n(z)}|(\ln\frac{1}{1-|z|^2}) \\
&\leq \frac{\|g\|}{1-|z|^2} + \|f_n\|_{\mathcal{B}_{\log}} \\
&< \varepsilon\|g\| + \|f_n\|_{\mathcal{B}_{\log}}.
\end{aligned}
$$

Since for $f_n \to 0$ on $\overline{U}$ we have $\|f_n\|_{\mathcal{B}_{\log}} \to 0$, and that ε is an arbitrary positive number, by letting $n \to \infty$ in the last inequality, we obtain that $lim_{n\to\infty}\|C_g f_n\| = 0$. Therefore, the operator C_g is compact.

Theorem 2.4. Assume that g is an analytic function on U. Then the integral operator $C_g : \Sigma \to \Sigma$ is compact if and only if f is a non-zero constant.

Proof. Without loss of generality, we assume that $g(z) = z$. Then it is clear that $C_g f$ is compact. Conversely, assume that $C_g : \Sigma \to \Sigma$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in U such that $|z_n| \to 1$ as $n \to \infty$. We aim to show that $f'(z_n) \to 0$ as $n \to \infty$. Therefore, by the maximum modulus theorem, we have f is a constant. In fact, setting

$$g_n(z) = 2\frac{(\overline{z}_n z - 1)}{\overline{z}_n}[(1 + \ln\frac{1}{1-\overline{z}_n z})^2 + 1][\ln\frac{1}{1-|z|^2}]^{-1} - 4\int_0^z \ln\frac{1}{1-\overline{z}_n w}dw,$$

we obtain

$$g_n'(z) = 2(\ln\frac{1}{1-\overline{z}_n z})^2[\ln\frac{1}{1-|z|^2}]^{-1} - 4[\ln\frac{1}{1-\overline{z}_n z}]$$

and

$$g_n''(z) = \frac{4\overline{z}_n}{1-\overline{z}_n z}(\ln\frac{1}{1-\overline{z}_n z})[\ln\frac{1}{1-|z|^2}]^{-1} - \frac{4\overline{z}_n}{1-\overline{z}_n z}.$$

Consequently, we have

$$\frac{g_n''(z_n)}{g_n'(z_n)} = 0.$$

Similar to the proof of Theorem 2.2, we see that $g_n \to 0$ uniformly on $\overline{U}$. Since $C_g : \Sigma \to \Sigma$ is compact then we get

$$\|C_g f_n\| \to 0, \quad n \to \infty.$$

Thus

$$\begin{aligned}|\frac{f'(z_n)}{f(z_n)}| &\le \sup_{z\in U}|\frac{f'(z)}{f(z)}| + \sup_{z\in U}|\frac{g_n''(z)}{g_n'(z)}| \\ &\le \|C_g f_n\| \to 0\end{aligned}$$

implies that $f_n'(z) \to 0$ and so f is a constant as desired.

Let H be a complex Hilbert space and $B(H, H)$ be a bounded space on H. Recall that the operator P is called an *accretive* if $\Re(Pu, u)_H \ge 0$, $\forall u \in H$. Moreover, the space $L^2(U; H)$ is a Hilbert space with the inner product

$$(f, g)_{L^2(U;H)} = \int_0^1 (f(z), g(z))_H dz, \quad z \in U.$$

We proceed to extend the Cesàro integral operator $C_g f$ to the space $L^2(U; H)$. We are now ready to state and prove the following:

Theorem 2.5. Let $f \in L^2(U; H)$. Then $C_g f \in B(L^2(U; H))$. Moreover, if $\Re(g'(z)) > 0, z \in U$ then $C_g f$ is an accretive operator.

Proof. By making use the Young inequality, it follows that

$$\|C_g f\|_{L^2(U;H)} \le \|g'\|_{L^1(U)}\|u\|_{L^2(U;H)} \le C\|u\|_{L^2(U;H)}.$$

To prove that $C_g f$ is an accretive operator, it suffices to show that

$$\Re\Big(\int_0^z f(\xi)g'(\xi)d\xi,\, f\Big)_{L^2(U;H)} \ge 0,$$

where f is in the domain of C_g. By the assumption $\Re(g'(z)) > 0, z \in U$, we have

$$\begin{aligned}\Re\Big(\int_0^z f(\xi)g'(\xi)d\xi,\, f\Big)_{L^2(U;H)} &= \Re\Big(\int_0^1 \big(\int_0^z f(\xi)g'(\xi)d\xi,\, f\big)_H dz\Big)\\ &= \Re\Big(\int_0^1 \big(\int_0^z f^2(\xi)g'(\xi)d\xi\big)dz\Big) \ge 0\end{aligned}$$

where f is analytic and non-negative in the domain of C_g. Therefore $C_g f$ is an accretive operator.

For $\beta > 0$, we consider the space $\mathcal{B}^{\beta}_{\log}$ of all functions $f \in \mathcal{H}$, $f(0) = 0$ so that

$$\|f\|_{\mathcal{B}^{\beta}_{\log}} = \sup_{z\in U}(1-|z|^2)^{\beta}\Big|\frac{f'(z)}{f(z)}\Big| \ln\frac{1}{(1-|z|^2)^{\beta}} < \infty, \quad (z \in U).$$

Obviously, for $0 < \beta_1 < 1 < \beta_2 < \infty$, we obtain

$$\mathcal{B}^{\beta_1}_{\log} \subsetneqq \mathcal{B}_{\log} \subsetneqq \mathcal{B}^{\beta_2}_{\log}.$$

Theorem 2.6. Let $0 < \beta < 1$ and $f, g \in \mathcal{H}(U)$, $g \in \mathcal{B}^{\beta}_{\log}$. Then the integral operator C_g is bounded on $\mathcal{B}^{\beta}_{\log}$ if and only if $f \in H^{\infty}$. Moreover,

$$\|C_g\| = \|f\|_{H^{\infty}}.$$

Proof. Without loss of generality, we let

$$\|g\|_{\mathcal{B}^{\beta}_{\log}} = \sup_{z\in U}(1-|z|^2)^{\beta}\Big|\frac{g'(z)}{g(z)}\Big| \ln\frac{1}{(1-|z|^2)^{\beta}} = 1.$$

The maximum principle and the Cauchy-Schwarz inequality for integral imply that

$$\begin{aligned}\|C_g f\|_{\mathcal{B}^{\beta}_{\log}} &= \sup_{z\in U}(1-|z|^2)^{\beta}\Big|\frac{(C_g f)'(z)}{(C_g f)(z)}\Big| \ln\frac{1}{(1-|z|^2)^{\beta}}\\ &\le \sup_{z\in U}(1-|z|^2)^{\beta}\Big(\Big|\frac{(fg')(z)}{\int_0^z (fg')(z)d\zeta}\Big| + 1\Big) \ln\frac{1}{(1-|z|^2)^{\beta}}\\ &\le \sup_{z\in U}(1-|z|^2)^{\beta}\Big(\Big|\frac{(fg')(z)}{\int_0^z (fg')(z)d\zeta}\Big|\Big) \ln\frac{1}{(1-|z|^2)^{\beta}} + \sup_{z\in U}(1-|z|^2)^{\beta} \times \ln\frac{1}{(1-|z|^2)^{\beta}}\\ &\le \|f\|_{\mathcal{B}^{\beta}_{\log}} + \rho, \quad \rho \to 0.\end{aligned}$$

Define $\kappa := \sup_{z\in U}|f|$. Given $\epsilon >$, there exists $z_1 \in U$ such that $|f(z_1)| > \kappa - \epsilon$. Set

$$F(z) = \int_0^z \frac{(1-|z_1|^2)^{\beta}}{(1-\overline{z_1}\zeta)^{2\beta}}d\zeta.$$

By [15, Theorem 13.11], F is analytic in U with

$$F'(z) = \frac{(1-|z_1|^2)^\beta}{(1-\overline{z_1}z)^{2\beta}}.$$

Now, we have

$$\begin{aligned}\|F\|_{\mathcal{B}^\beta_{\log}} &= \sup_{z\in U}(1-|z|^2)^\beta \Big|\frac{F'(z)}{F(z)}\Big| \ln\frac{1}{(1-|z|^2)^\beta}\\ &= \sup_{z\in U}(1-|z|^2)^\beta \Big(\Big|\frac{\frac{(1-|z_1|^2)^\beta}{(1-\overline{z_1}z)^{2\beta}}}{\int_0^z \frac{(1-|z_1|^2)^\beta}{(1-\overline{z_1}\zeta)^{2\beta}}d\zeta}\Big|\Big) \ln\frac{1}{(1-|z|^2)^\beta} \leq 1.\end{aligned}$$

On the other hand

$$\begin{aligned}\|F\|_{\mathcal{B}^\beta_{\log}} &= \sup_{z\in U}(1-|z|^2)^\beta \Big|\frac{F'(z)}{F(z)}\Big| \ln\frac{1}{(1-|z|^2)^\beta}\\ &= \sup_{z\in U}(1-|z|^2)^\beta \Big(\Big|\frac{\frac{(1-|z_1|^2)^\beta}{(1-\overline{z_1}z)^{2\beta}}}{\int_0^z \frac{(1-|z_1|^2)^\beta}{(1-\overline{z_1}\zeta)^{2\beta}}d\zeta}\Big|\Big) \ln\frac{1}{(1-|z|^2)^\beta}\\ &\geq (1-|z|^2)^\beta \Big(\Big|\frac{\frac{(1-|z_1|^2)^\beta}{(1-|z_1|^2)^{2\beta}}}{\int_0^z \frac{(1-|z_1|^2)^\beta}{(1-|z_1|^2)^{2\beta}}d\zeta}\Big|\Big) \ln\frac{1}{(1-|z|^2)^\beta}\\ &\geq (1-|z|^2)^\beta \ln\frac{1}{(1-|z|^2)^\beta}\\ &\geq 1,\end{aligned}$$

for sufficient small $0 < \beta < 1$ and $|z| < 1$. Thus $\|F\|_{\mathcal{B}^\beta_{\log}} = 1$. Now the proof is complete since for $F(z) \equiv g(z)$ in $C_g f$ and for arbitrary small ϵ we obtain

$$\begin{aligned}\|C_g\| \geq \|C_g f\|_{\mathcal{B}^\beta_{\log}} &= \sup_{z\in U}(1-|z|^2)^\beta \Big|\frac{(C_g f)'(z)}{(C_g f)(z)}\Big| \ln\frac{1}{(1-|z|^2)^\beta}\\ &= \sup_{z\in U}(1-|z|^2)^\beta \Big(\Big|\frac{(fg')(z)}{\int_0^z (fg')(z)dz}\Big| + 1\Big) \ln\frac{1}{(1-|z|^2)^\beta}\\ &\geq (1-|z_1|^2)^\beta \Big(\Big|\frac{f(z_1)\frac{(1-|z_1|^2)^\alpha}{(1-\overline{z_1}z)^{2\beta}}}{\int_0^z f(z_1)\frac{(1-|z_1|^2)^\beta}{(1-\overline{z_1}z)^{2\beta}}dz}\Big|\Big) \ln\frac{1}{(1-|z_1|^2)^\beta} + (1-|z_1|^2)^\beta\\ &\times \ln\frac{1}{(1-|z_1|^2)^\beta}\\ &> |f(z_1)| > \kappa - \epsilon.\end{aligned}$$

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