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SOME FAMILIES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. By using Sălăgean operator we introduce the subclass $S_{m,n,j}^{\lambda}(\alpha,\beta)$ of normalized analytic functions with negative coefficients. In this paper, we obtain coefficients estimates, distortion theorems, several results for the modified Hadamard products of functions belonging to this class. Applications involving an integral operator and some fractional calculus operators are also considered.

1. Introduction

Denote by $\mathcal{A}_j$ the class of functions of form

$$f(z)=z+\sum_{k=j+1}^{\infty}a_k z^k \,(j\in\mathbb{N}=\{1,2,3,...\}), \tag{1}$$

which are analytic in the open unit disc $\mathbb{U}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}$. We note that $\mathcal{A}_1=\mathcal{A}$. For a function $f(z)$ in $\mathcal{A}_j$, let

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= zf'(z) \\ D^n f(z) &= D\left(D^{n-1}f(z)\right) \quad (n\in\mathbb{N}) \end{aligned}$$

$$= z+\sum_{k=j+1}^{\infty}k^n a_k z^k \quad (\mathbb{N}_0=\mathbb{N}\cup\{0\}) \tag{2}$$

The differential operator D^n was introduced by Sălăgean [16]. With the help of the differential operator D^n, for $0\le\alpha<1$, $0\le\lambda\le1$, $0<\beta\le1$, $n\in\mathbb{N}_0$ and m $\in\mathbb{N}$, let $S_j(m,n,\lambda,\alpha,\beta)$ denote the subclass of $\mathcal{A}_j$ consisting of functions $f(z)$ of the form (1) and satisfying the condition

$$\left|\frac{F_{m,n,\lambda}(z)-1}{F_{m,n,\lambda}(z)+1-2\alpha}\right|<\beta \quad (z\in\mathbb{U}), \tag{3}$$

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where, for convenience,

$$F_{n,m,\lambda}(z)=\frac{(1-\lambda)\,z\,(D^nf(z))'+\lambda z\,(D^{n+m}f(z))'}{(1-\lambda)\,(D^nf(z))+\lambda D^{n+m}f(z)}=\frac{\phi_{n,m,\lambda}(z)}{\Psi_{n,m,\lambda}(z)}.$$

The operator D^{n+m} was studied by Sekine [18], Hossen et al. [10] and Aouf and Sălăgean [5]. Denote by T_j the subclass of $\mathcal{A}_j$ consisting of the functions of the form

$$f(z)=z-\sum_{k=j+1}^{\infty}a_kz^k \quad (a_k\geq 0,k\geq j+1;j\in\mathbb{N}). \tag{4}$$

We note that $T_1=T$. Further, define the class $S_{m,n,j}^{\lambda}(\alpha,\beta)$ by

$$S_{m,n,j}^{\lambda}(\alpha,\beta)=S_j(m,n,\lambda,\alpha,\beta)\cap T_j. \tag{5}$$

Remark 1. Putting $\lambda=0$ and $j=1$, in (3), we get the following class (see Sălăgean [14])

$$T_n(\alpha,\beta)=\left\{f(z)\in T:\left|\frac{\frac{z(D^nf(z))'}{D^nf(z)}-1}{\frac{z(D^nf(z))'}{D^nf(z)}+1-2\alpha}\right|<\beta,\ z\in\mathbb{U}\right\}, \tag{6}$$

which reduces to the classes $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ by taking $n=0$ and $n=1$, respectively (see Gupta and Jain [9]). Also $T_n(\alpha,1)=T(n,\alpha)$ (see Hur and Oh [11]).

Also by replacing $\frac{z(D^nf(z))'}{D^nf(z)}$ by $(D^nf(z))'$ in (6), we obtain the class $P_n(\alpha,\beta,\gamma)$ defined by Aouf et al. [6, with $\gamma=\alpha=1$].

We note that, specializing the parameters n,λ,α,β and m, we obtain the following subclasses studied by various authors.

(i) $S_{m,n,1}^{0}(\alpha,1)=T(n,\alpha)$ (see see Hur and Oh [11]);
(ii) $S_{m,0,1}^{0}(\alpha,\beta)=S^*(\alpha,\beta)$ (see Sălăgean [15]);
(iii) $S_{m,n,j}^{\lambda}(\alpha,1)=T_j(m,n,\lambda,\alpha)$ (see Aouf et al. [3]);
(iv) $S_{1,n,j}^{\lambda}(\alpha,1)=\mathbb{P}(j,\lambda,\alpha,n)$ (see Aouf and Srivastava [7]);
(v) $S_{1,0,j}^{\lambda}(\alpha,1)=P(n,\lambda,\alpha)$ (see Altintas [1]);
(vi) $S_{m,0,j}^{0}(\alpha,1)=T^*(\alpha)$ and $S_{1,0,1}^{1}(\alpha,1)=C(\alpha)$ (see Silverman [19]);
(vii) $S_{0,m,j}^{0}(\alpha,1)=T_\alpha(j)$ and $S_{1,0,1}^{0}(\alpha,1)=C_\alpha(j)$ (Chatterjea [8] and Srivastava et al. [21]);
(viii) $S_{1,\Omega,1}^{\lambda}(\alpha,1)=T_\Omega(1,1,\lambda,\alpha,0)$(see Kairnar and More [12]);
(ix) $S_{m,n,1}^{0}(\gamma,\beta)=T_1^m(1,0,\gamma,\beta,1)\,(0\leq\gamma<\frac{1}{2})$ (Aouf et al. [4]).

2. Coefficient estimates and other properties of the class $S_{m,n,j}^{\lambda}(\alpha,\beta)$

Theorem 1. Let the function $f(z)$ be defined by (4). Then $f(z)\in S_{m,n,j}^{\lambda}(\alpha,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty}k^n[1+\lambda(k^m-1)]\left\{(k-1)(1+\beta)+2\beta(1-\alpha)\right\}a_k\leq 2\beta(1-\alpha). \tag{7}$$

Proof. Assume that the inequality (7) holds. Then, for $|z|=r<1$, we must show that

$$|\phi_{m,n,\lambda}(z)-\Psi_{m,n,\lambda}(z)|-\beta\,|\phi_{m,n,\lambda}(z)+(1-2\alpha)\,\Psi_{m,n,\lambda}(z)|<0.$$

We have

$$\left| -\sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k-1)\,a_k z^{k-1} \right| - \beta \left| 2\,(1-\alpha) - \sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k+1-2\alpha)\,a_k z^{k-1} \right|$$

$$\leq \sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k-1)\,a_k - \beta \left[2\,(1-\alpha) - \sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k+1-2\alpha)\,a_k \right]$$

$$\leq \sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,\{(k-1)\,(1+\beta)+2\beta\,(1-\alpha)\} - 2\beta\,(1-\alpha) \leq 0.$$

Hence, $f(z) \in S^{\lambda}_{m,n,j}(\alpha,\beta)$.

Conversely, let $f(z) \in S^{\lambda}_{m,n,j}(\alpha,\beta)$. Then we have

$$\left| \frac{F_{m,n,\lambda}(z)-1}{F_{m,n,\lambda}(z)+1-2\alpha} \right|$$

$$\left| \frac{-\sum\limits_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k-1)\,a_k z^{k-1}}{2\,(1-\alpha) - \sum\limits_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k+1-2\alpha)\,a_k z^{k-1}} \right| < \beta.$$

Since $Re\,(z) < |z|$ for all z, we obtain the inequality:

$$Re\left\{ \frac{\sum\limits_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k-1)\,a_k z^{k-1}}{2\,(1-\alpha) - \sum\limits_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k+1-2\alpha)\,a_k z^{k-1}} \right\} < \beta \;\; (z \in \mathbb{U})\,. \tag{8}$$

Now choose values of z on the real axis so that $F_{m,n,\lambda}(z)$ is real. Upon clearing the denominator in (8) and letting $z \to 1^{-}$ through real values, we find that

$$\sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k-1)\,a_k \leq 2\beta\,(1-\alpha) - \beta \sum_{k=j+1}^{\infty} k^n[1+\lambda(k^m-1)]\,(k+1-2\alpha)\,a_k,$$

which leads us readily to the inequality (7). Thus we complete our proof of Theorem 1. □

Corollary 1. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$. Then

$$a_k \le \frac{2\beta(1-\alpha)}{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}\ (k\ge j+1; j\in\mathbb{N}). \tag{9}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z-\frac{2\beta(1-\alpha)}{k^n[1+\lambda(k^m-1)][(k-1)(1+\beta)+2\beta(1-\alpha)]}z^k(k\ge j+1; j\in\mathbb{N}). \tag{10}$$

Remark 2. Since

$$[1+\lambda(k^m-1)]\le 1-\mu+\mu k^m\ \ (k\ge j+1;\ j\in\mathbb{N};\ 0\le\lambda\le\mu\le 1),$$

we have the inclusion property

$$S^{\mu}_{m,n,j}(\alpha,\beta)\subseteq S^{\lambda}_{m,n,j}(\alpha,\beta)\qquad(0\le\lambda\le\mu\le 1).$$

Furthermore, for $0\le\alpha_1\le\alpha_2<1$, it is easily verified that

$$\frac{\{(k-1)(1+\beta)+2\beta(1-\alpha_1)\}}{(1-\alpha_1)}\le\frac{\{(k-1)(1+\beta)+2\beta(1-\alpha_2)\}}{(1-\alpha_2)},$$

so that, with the aid of Theorem 1, we obtain the inclusion property

$$S^{\lambda}_{m,n,j}(\alpha_2,\beta)\subseteq S^{\lambda}_{m,n,j}(\alpha_1,\beta)\quad(0\le\alpha_1\le\alpha_2<1).$$

Theorem 2. For each $n\in\mathbb{N}_0$,

$$S^{\lambda}_{m,n+1,j}(\alpha,\beta)\subseteq S^{\lambda}_{m,n,j}(\delta,\beta),$$

where

$$\delta=\frac{(1+\beta)(j+\alpha)+2\beta(1-\alpha)}{(j+1)(1+\beta)+2\beta(1-\alpha)}. \tag{11}$$

Proof. Suppose that the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n+1,j}(\alpha,\beta)$. Then, by Theorem 1, we have

$$\sum_{k=j+1}^{\infty}k^{n+1}[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}a_k\le 2\beta(1-\alpha). \tag{12}$$

Then we must prove that,

$$\sum_{k=j+1}^{\infty}k^{n}[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\delta)\}a_k\le 2\beta(1-\delta). \tag{13}$$

In view of (12), (13) will hold true if

$$\frac{\{(k-1)(1+\beta)+2\beta(1-\delta)\}}{(1-\delta)}\le\frac{k\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{(1-\alpha)}$$

$$(k\ge j+1; j\in\mathbb{N}),$$

which leads to

$$\delta\le\frac{(1+\beta)(k-1+\alpha)+2\beta(1-\alpha)}{k(1+\beta)+2\beta(1-\alpha)}\ \ (k\ge j+1; j\in\mathbb{N}). \tag{14}$$

Since the right-hand side of (14) is an increasing function of k, letting $k=j+1$ in (14), we obtain

$$\delta\le\frac{(1+\beta)(j+\alpha)+2\beta(1-\alpha)}{(j+1)(1+\beta)+2\beta(1-\alpha)}. \tag{15}$$

This completes the proof of Theorem 2. □

Remark 3. Since $\delta > \alpha$, it follows from Remark 2 that

$$S^{\lambda}_{m,n,j}(\delta,\beta) \subset S^{\lambda}_{m,n,j}(\alpha,\beta) \ \ (n \in \mathbb{N}_0)$$

and hence

$$S^{\lambda}_{m,n+1,j}(\alpha,\beta) \subset S^{\lambda}_{m,n,j}(\delta,\beta) \subset S^{\lambda}_{m,n,j}(\alpha,\beta) \quad (n \in \mathbb{N}_0)\,,$$

where δ is defined by (11).

Theorem 3. Let $0 \leq \alpha_k < 1 \ \ (k = 1, 2)$ and $0 < \beta_k \leq 1 \ (k = 1, 2)$. Then

$$S^{\lambda}_{m,n,j}(\alpha_1,\beta_1) = S^{\lambda}_{m,n,j}(\alpha_2,\beta_2) \tag{16}$$

if and only if

$$\frac{\beta_1(1-\alpha_1)}{1+\beta_1} = \frac{\beta_2(1-\alpha_2)}{1+\beta_2}. \tag{17}$$

In particular, if $0 \leq \alpha < 1$ and $0 < \beta \leq 1$,then

$$S^{\lambda}_{m,n,j}(\alpha,\beta) = S^{\lambda}_{m,n,j}(\frac{1-\beta+2\alpha\beta}{1+\beta},1) = \mathbb{P}(j,\lambda,\frac{1-\beta+2\alpha\beta}{1+\beta},n). \tag{18}$$

Proof. Let us first assume that the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha_1,\beta_1)$ and let the condition (16) hold true. Then, by (7), we have

$$\sum_{k=j+1}^{\infty} \frac{k^n[1+\lambda(k^m-1)]\left\{(k-1)(1+\beta_2)+2\beta_2(1-\alpha_2)\right\}a_k}{2\beta_2(1-\alpha_2)}$$

$$= \sum_{k=j+1}^{\infty} \frac{k^n[1+\lambda(k^m-1)]\left\{(k-1)(1+\beta_1)+2\beta(1-\alpha_1)\right\}a_k}{2\beta_1(1-\alpha_1)} \leq 1$$

which shows that $f(z) \in S^{\lambda}_{m,n,j}(\alpha_2,\beta_2)$. Again with the aid of Theorem 1, reversing the above steps, we can similarly prove that, under the condition (17),

$$f(z) \in S^{\lambda}_{m,n,j}(\alpha_2,\beta_2) \implies f(z) \in S^{\lambda}_{m,n,j}(\alpha_1,\beta_1).$$

Conversely, the assertion (16) can easily be shown to imply the condition (17). The proof of Theorem 3 is thus completed by observing that (18) is a special case of (16) when $\alpha_1 = \alpha, \beta_1 = \beta$, and $\beta_2 = 1$. □

3. Distortion theorems

Theorem 4. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$. Then for $|z| = r < 1$, we have

$$\left|D^i f(z)\right| \geq r - \frac{2\beta(1-\alpha)}{(j+1)^{n-i}[1+\lambda((j+1)^m-1)][j(1+\beta)+2\beta(1-\alpha)]}r^{j+1}, \tag{19}$$

$$\left|D^i f(z)\right| \leq r + \frac{2\beta(1-\alpha)}{(j+1)^{n-i}[1+\lambda((j+1)^m-1)][j(1+\beta)+2\beta(1-\alpha)]}r^{j+1}, \tag{20}$$

for $z \in \mathbb{U}$ and $0 \leq i \leq n$. The equalities in (19) and (20) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{2\beta(1-\alpha)}{(j+1)^n\,[1+\lambda((j+1)^m-1)]\,[j\,(1+\beta)+2\beta\,(1-\alpha)]}z^{j+1} \ \ (z = \pm r)\,. \tag{21}$$

Proof. Note that $f(z) \in S^{\lambda}_{m,n,j}(\alpha,\beta)$ if and only if $D^i f(z) \in S^{\lambda}_{m,n-i,j}(\alpha,\beta)$ and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k \ . \tag{22}$$

By Theorem 1,we have

$$(j+1)^{n-i}\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]\sum_{k=j+1}^{\infty} k^i a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n\left[1+\lambda\left(k^m-1\right)\right]\left[\left(k-1\right)\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]a_k \leq 2\beta(1-\alpha),$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \ \leq \frac{2\beta(1-\alpha)}{(j+1)^{n-i}\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]}. \tag{23}$$

The assertions (19) and (20) of Theorem 4 would now follow readily form (22) and (23). Finally, we note that equalities in (19) and (20) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{2\beta(1-\alpha)}{(j+1)^{n-i}\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]} z^{j+1}. \tag{24}$$

This completes the proof of Theorem 4. □

Corollary 2. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$. Then for $|z| = r < 1$

$$|f(z)| \geq r - \frac{2\beta(1-\alpha)}{(j+1)^n \ \ \left[1+\lambda((1+j)^m-1)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]} r^{j+1} \tag{25}$$

and

$$|f(z)| \leq r + \frac{2\beta(1-\alpha)}{(j+1)^n \ \ \left[1+\lambda((1+j)^m-1)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]} r^{j+1}. \tag{26}$$

The equalities in (25) and (26) are attained for the function $f(z)$ given by (21).

Proof. Taking $i = 0$ in Theorem 4, we immediately obtain (25) and (26), respectively. □

Corollary 3. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$.Then for $|z| = r < 1$

$$|f'(z)| \geq 1 - \frac{2\beta(1-\alpha)}{(j+1)^{n-1} \ \ \left[1+\lambda((1+j)^m-1)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]} r^j \tag{27}$$

and

$$|f'(z)| \leq 1 + \frac{2\beta(1-\alpha)}{(j+1)^{n-1} \ \ \left[1+\lambda((1+j)^m-1)\right]\left[j\left(1+\beta\right)+2\beta\left(1-\alpha\right)\right]} r^j \tag{28}$$

The equalities in (27) and (28) are attained for the function $f(z)$ given by (21).

Proof. Setting $i = 1$ in Theorem 4, and making use of the definition of D^n, we arrive at Corollary 3. □

4. Modified Hadamard products

Let $f_v(z)\,(v=1,2)$ be defined by

$$f_v(z)=z-\sum_{k=j+1}^{\infty}a_{k,v}\,z^k \quad (a_{\nu,k}\geq 0; v=1,2)\,. \tag{29}$$

Then the modified Hadamard product of the functions $f_v(z)\,(v=1,2)$ is defined by

$$(f_1 * f_2)(z)=z-\sum_{k=j+1}^{\infty}a_{k,1}a_{k,2}z^k. \tag{30}$$

.

Theorem 5. Let the functions $f_v(z)\,(v=1,2)$ defined by (29) belong to the class $S_{m,n,j}^{\lambda}(\alpha,\beta)$. Then $(f_1*f_2)(z)\in S_{m,n,j}^{\lambda}(\eta,\beta)$, where

$$\eta=1-\frac{2\beta j\,(1-\alpha)^2\,(1+\beta)}{(j+1)^n\,[1+\lambda((j+1)^m-1)]\,\{j\,(1+\beta)+2\beta\,(1-\alpha)\}^2-4\beta^2\,(1-\alpha)^2}. \tag{31}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [17], we need to find the largest η such that

$$\sum_{k=j+1}^{\infty}k^n[1+\lambda(k^m-1)]\,\{(k-1)\,(1+\beta)+2\beta\,(1-\eta)\}\,a_{k,1}a_{k,2}\leq 2\beta\,(1-\eta)\,. \tag{32}$$

Since

$$\sum_{k=j+1}^{\infty}k^n[1+\lambda(k^m-1)]\,\{(k-1)\,(1+\beta)+2\beta\,(1-\alpha)\}\,a_{k,1}\leq 2\beta\,(1-\alpha) \tag{33}$$

and

$$\sum_{k=j+1}^{\infty}k^n\;[1+\lambda(k^m-1)]\,\{(k-1)\,(1+\beta)+2\beta\,(1-\alpha)\}\,a_{k,2}\leq 2\beta\,(1-\alpha)\,, \tag{34}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty}\frac{k^n[1+\lambda(k^m-1)]\,\{(k-1)\,(1+\beta)+2\beta\,(1-\alpha)\}}{2\beta\,(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}\leq 1. \tag{35}$$

Thus it is sufficient to show that

$$\begin{aligned}&\frac{k^n[1+\lambda(k^m-1)]\{(k-1)\,(1+\beta)+2\beta\,(1-\eta)\}}{2\beta\,(1-\eta)}a_{k,1}a_{k,2}\\ \leq\quad &\frac{k^n[1+\lambda(k^m-1)]\,\{(k-1)\,(1+\beta)+2\beta\,(1-\alpha)\}}{2\beta\,(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}\\ &(k\geq j+1; j\in\mathbb{N})\,,\end{aligned}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}}=\frac{\{(k-1)\,(1+\beta)+2\beta\,(1-\alpha)\}\,(1-\eta)}{\{(k-1)\,(1+\beta)+2\beta\,(1-\eta)\}\,(1-\alpha)}. \tag{36}$$

Since (35) implies that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{2\beta(1-\alpha)}{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}, \tag{37}$$

we need only to prove that

$$\frac{2\beta(1-\alpha)}{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}} \leq \frac{\{(k-1)(1+\beta)+2\beta(1-\alpha)\}(1-\eta)}{\{(k-1)(1+\beta)+2\beta(1-\eta)\}(1-\alpha)}$$

or, equivalently, that

$$\eta \leq 1-\{2\beta(1-\alpha)^2(k-1)(1+\beta)\}.$$

$$.\{k^n[1+\lambda(k^m-1)][(k-1)(1+\beta)+2\beta(1-\alpha)]^2-4\beta^2(1-\alpha)^2\}^{-1}. \tag{38}$$

Since the right-hand side of (38) is an increasing function of k, by letting $k=j+1$ in (38), we obtain

$$\eta \leq 1-\frac{2\beta j(1-\alpha)^2(1+\beta)}{(j+1)^n[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}^2-4\beta^2(1-\alpha)^2},$$

which proves the main assertion of Theorem 5.The sharpness of the result of Theorem 5 follows if we take

$$f_\nu(z) = z-\frac{2\beta(1-\alpha)}{(j+1)^n[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}}z^{j+1}\,(v=1,2). \tag{39}$$

□

Theorem 6. For each function $f_1(z)$ and $f_2(z)$ belongs to the same class $S^\lambda_{m,n,j}(\alpha,\beta)$, then $(f_1*f_2)(z)\in S^\lambda_{m,n,j}(\sigma,1)$, where

$$\sigma = \frac{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+2\beta(1-\alpha)]^2-(j+1)[2\beta(1-\alpha)]^2}{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+2\beta(1-\alpha)]^2-[2\beta(1-\alpha)]^2}. \tag{40}$$

The result is sharp for the functions $f_v(z)\,(v=1,2)$ defined by (39).

Proof. Proceeding as in the proof of Theorem 5, we need to find the largest σ such that

$$\sum_{k=j+1}^{\infty}\frac{k^n[1+\lambda(k^m-1)](k-\sigma)}{(1-\sigma)}a_{k,1}a_{k,2}\leq 1. \tag{41}$$

From (35) and (41). Thus it is sufficient to show that

$$\frac{k^n[1+\lambda(k^m-1)](k-\sigma)}{(1-\sigma)}a_{k,1}a_{k,2}\leq$$

$$\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{2\beta(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}(k\geq j+1;j\in\mathbb{N}),$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{\{(k-1)(1+\beta)+2\beta(1-\alpha)\}(1-\sigma)}{2\beta(1-\alpha)(k-\sigma)}. \tag{42}$$

Since (37) and (42), we need only to prove that

$$\frac{2\beta(1-\alpha)}{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}} \leq$$

$$\frac{\{(k-1)(1+\beta)+2\beta(1-\alpha)\}(1-\sigma)}{2\beta(1-\alpha)(k-\sigma)},$$

or, equivalently, that

$$\sigma \leq \frac{k^n[1+\lambda(k^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}^2-k\{2\beta(1-\alpha)\}^2}{k^n[1+\lambda(k^m-1]\{j(1+\beta)+2\beta(1-\alpha)\}^2-\{2\beta(1-\alpha)\}^2}. \tag{43}$$

Since the right-hand side of (43) is an increasing function of k, by letting $k=j+1$ in (43), we obtain

$$\sigma = \frac{(j+1)^n[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}^2-(j+1)\{2\beta(1-\alpha)\}^2}{(j+1)^n[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}^2-\{2\beta(1-\alpha)\}^2},$$

which proves the main assertion of Theorem 6. □

Theorem 7. Let $f_1(z) \in S^{\lambda}_{m,n,j}(\alpha,\beta)$ and $f_2(z) \in S^{\lambda}_{m,n,j}(\gamma,\beta)$. Then $(f_1 * f_2)(z) \in S^{\lambda}_{m,n,j}(\sigma,\beta)$, where

$$\sigma \leq 1-\{2\beta j(1-\alpha)(1-\gamma)\}.\{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+2\beta(1-\alpha)]$$
$$.[j(1+\beta)+2\beta(1-\gamma)]-4\beta^2(1-\alpha)(1-\gamma)\}^{-1}. \tag{44}$$

The result is best possible for the functions

$$f_1(z) = z - \frac{(1-\alpha)}{(j+1)^n(1+\lambda((j+1)^m-1))\{j(1+\beta)+2\beta(1-\alpha)\}}z^{j+1}, \tag{45}$$

and

$$f_2(z) = z - \frac{(1-\gamma)}{(j+1)^n(1+\lambda((j+1)^m-1))\{j(1+\beta)+2\beta(1-\gamma)\}}z^{j+1}. \tag{46}$$

Proof. Proceeding as in the proof of Theorem 5, we get

$$\sigma \leq 1-$$

$$\frac{2\beta(k-1)(1-\alpha)(1-\gamma)}{k^n[1+\lambda(k^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\{(k-1)(1+\beta)+2\beta(1-\gamma)\}-4\beta^2(1-\alpha)(1-\gamma)}. \tag{47}$$

Since the right-hand side of (47) is an increasing function of k, setting $k=j+1$ in (47), we obtain (44). This completes the proof of Theorem 7. □

Corollary 4. Let the functions $f_v(z)\,(v=1,2,3)$ defined by (29) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$. Then $(f_1 * f_2 * f_3)(z) \in S^{\lambda}_{m,n,j}(\xi,\beta)$, where

$$\xi \leq 1-\frac{4\beta j(1-\alpha)^3(1+\beta)}{(j+1)^{2n}[1+\lambda((j+1)^m-1)]\{(j+1)(1+\beta)+2\beta(1-\alpha)\}^3-8\beta^3(1-\alpha)^3}. \tag{48}$$

The result is the best possible for the functions $f_v(z)(v=1,2,3)$ given by (39).

Proof. From Theorem 5, we have $(f_1 * f_2)(z) \in S^{\lambda}_{m,n,j}(\eta, \beta)$, where η is given by (31). Now, using Theorem 7, we get $(f_1 * f_2 * f_3)(z) \in S^{\lambda}_{m,n,j}(\xi, \beta)$, where

$$\xi = 1 - \frac{4\beta j(1-\alpha)^3(1+\beta)}{(j+1)^{2n}[1+\lambda((j+1)^m-1)]\{(j+1)(1+\beta)+2\beta(1-\alpha)\}^3 - 8\beta^3(1-\alpha)^3}.$$

This completes the proof of Corollary 4. □

Theorem 8. Let the functions $f_v(z)\,(v=1,2)$ defined by (29) be in the same class $S^{\lambda}_{m,n,j}(\alpha, \beta)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=j+1}^{\infty}(a^2_{k,1} + a^2_{k,2})z^k \tag{49}$$

belongs to the class $S^{\lambda}_{m,n,j}(\phi, \beta)$, where

$$\phi = 1 - \{4\beta j(1+\beta)(1-\alpha)^2\}.$$
$$.\{(j+1)^n[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}^2 - 8\beta^2(1-\alpha)^2\}^{-1}.$$

The result is sharp for the functions $f_v(z)\,(v=1,2)$ defined by (39).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=j+1}^{\infty}\left(\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{2\beta(1-\alpha)}\right)^2 a^2_{k,1}$$
$$\leq \left(\sum_{k=j+1}^{\infty}\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{2\beta(1-\alpha)}a_{k,1}\right)^2 \leq 1. \tag{50}$$

Similarly, we have

$$\sum_{k=j+1}^{\infty}\left(\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{2\beta(1-\alpha)}\right)^2 a^2_{k,2} \leq 1. \tag{51}$$

It follows from (50) and (51) that

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left(\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{2\beta(1-\alpha)}\right)^2 (a^2_{k,1}+a^2_{k,2}) \leq 1.$$

Therefore, we need to find the largest ϕ such that

$$\sum_{k=j+1}^{\infty}\left(\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\phi)\}}{2\beta(1-\phi)}\right)(a^2_{k,1}+a^2_{k,2}) \leq 1$$

that is,

$$\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\phi)\}}{2\beta(1-\phi)}$$
$$\leq \frac{1}{2}\left(\frac{k^n[1+\lambda(k^m-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{2\beta(1-\alpha)}\right)^2$$
$$(k \geq j+1; j \in \mathbb{N}),$$

or, equivalently, that

$$\phi \leq 1 - \{4\beta(1+\beta)(k-1)(1-\alpha)^2\}.$$

$$.\{k^n[1+\lambda(k^m-1)][(k-1)(1+\beta)+2\beta(1-\alpha)]^2-8\beta^2(1-\alpha)^2\}^{-1}. \quad (52)$$

Since the right-hand side of (52) is an increasing function of k, we readily have

$$\phi \leq 1-[4\beta j(1+\beta)(1-\alpha)^2].$$

$$.\{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+2\beta(1-\alpha)^2]-8\beta^2(1-\alpha)^2\}^{-1} \quad (53)$$

and Theorem 8 follows at once. □

5. A family of integral operator

Theorem 9. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$, and let c be a real number such that $c>-1$. Then the function $F(z)$ defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad (c>-1) \quad (54)$$

also belongs to the class $S^{\lambda}_{m,n,j}(\rho,\beta)$, where

$$\rho \leq \frac{(1+\beta)[\alpha(c+1)+j]+2\beta(1-\alpha)}{(c+j+1)(1+\beta)+2\beta(1-\alpha)}.$$

The result is sharp for the function $f(z)$ defined by (21).

Proof. From the representation (54) of $F(z)$, it follows that

$$F(z)=z-\sum_{k=j+1}^{\infty}\left(\frac{c+1}{c+k}\right)a_k z^k.$$

Therefore, we have

$$\frac{\{(k-1)(1+\beta)+2\beta(1-\rho)\}(c+1)}{(1-\rho)(c+k)} \leq \frac{\{(k-1)(1+\beta)+2\beta(1-\alpha)\}}{(1-\alpha)}$$

or, equivalently,

$$\rho \leq \frac{(1+\beta)[\alpha(c+1)+(k-1)]+2\beta(1-\alpha)}{(c+k)(1+\beta)+2\beta(1-\alpha)}. \quad (55)$$

The right-hand side of (55) being an increasing function of k, setting $k=j+1$ in (55), we obtain

$$\rho \leq \frac{(1+\beta)[\alpha(c+1)+j]+2\beta(1-\alpha)}{(c+j+1)(1+\beta)+2\beta(1-\alpha)},$$

which completes the proof of Theorem 9. □

Proceeding as in the proof of Theorem 9, we can obtain the following theorem.

Theorem 10. If $f(z)\in S^{\lambda}_{m,n,j}(\alpha,\beta)$, then the function $F(z)$ defined by (54) belongs to the class $S^{\lambda}_{m,n,j}(\nu,1)$, where

$$\nu \leq \frac{(1+\beta)[c+j+1]+2c\beta(1-\alpha)}{(c+j+1)(1+\beta)+2\beta(1-\alpha)}.$$

The result is sharp, the extremal function $f(z)$ being given by (21).

Theorem 11. Let the function $F(z)$ given by

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \qquad (a_k \geq 0;\ k = j+1;\ j \in \mathbb{N}),$$

be in the class $S^{\lambda}_{m,n,j}(\alpha, \beta)$, and let c be a real number such that $c > -1$. Then the function $f(z)$defined by (54) is univalent in $|z| < R$, where

$$R = \inf_{k\ \geq j+1} \left(\frac{k^{n-1}[1 + \lambda(k^m - 1)]\{(1+\beta)(k-1) + 2\beta(1-\alpha)\}(c+1)}{2\beta(1-\alpha)(c+k)} \right)^{\frac{1}{k-1}}. \quad (56)$$

The result is sharp for the function $f(z)$ defined by (21).

Proof. We find to from (54) that

$$f(z) = \frac{z^{1-z}(z^c F(z))'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1} \right) d_k z^k.$$

In order to obtain the desired result, it suffices to show that

$$|f'(z) - 1| < 1 \qquad \text{whenever} \quad |z| < R,$$

where R is given by (56). Now

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} d_k |z|^{k-1}.$$

Thus we have $|f'(z) - 1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} d_k |z|^{k-1} < 1. \quad (57)$$

But, by Theorem 1, we know that

$$\sum_{k=j+1}^{\infty} \frac{k^n[1 + \lambda(k^m - 1)]\{(1+\beta)(k-1) + 2\beta(1-\alpha)\}}{2\beta(1-\alpha)} d_k \leq 1.$$

Hence (57) will be satisfied if

$$\frac{k(c+k)}{c+1} |z|^{k-1} < \frac{k^n[1 + \lambda(k^m - 1)]\{(1+\beta)(k-1) + 2\beta(1-\alpha)\}}{2\beta(1-\alpha)},$$

that is, if

$$|z| < \left(\frac{k^n[1 + \lambda(k^m - 1)]\{(1+\beta)(k-1) + 2\beta(1-\alpha)\}(c+1)}{2\beta(1-\alpha)(c+k)} \right)^{\frac{1}{k-1}} (k \geq j+1;\ j \in \mathbb{N}). \quad (58)$$

Therefore, the function $f(z)$ given by (54) is univalent in $|z| < R$, where R is defined by (56). The sharpness of the result follows if we take

$$f(z) = z - \frac{2\beta(1-\alpha)(c+k)}{k^n[1 + \lambda(k^m - 1)][(k-1)(1+\beta) + 2\beta(1-\alpha)](c+1)} z^k. \quad (59)$$

□

6. Applications of fractional calculus

We recall here the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals), which were used earlier by Owa [13] (and, more recently, by Srivastava and Aouf [20]; see also Aouf [2]).

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\mu \quad (\mu > 0), \tag{60}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\zeta)^{1-\mu}$ is removed, by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{\mu}} \qquad (0 \le \mu < 1), \tag{61}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\mu$, is defined, for a function $f(z)$, by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} \{D_z^{\mu} f(z)\} \qquad (0 \le \mu < 1; n \in \mathbb{N}_0). \tag{62}$$

Theorem 12. Let the function $f(z)$ defined by (4) be in the class $S_{m,n,j}^{\lambda}(\alpha, \beta)$.Then

$$\begin{aligned} &\left|D_z^{-\mu}(D^i f(z))\right| \\ \ge\ & \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}\,[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2+\mu)} r^j\right) \end{aligned}$$

$$(|z| = r < 1; \quad \mu > 0; \quad i \in \{0,1,2,......,n\}) \tag{63}$$

and

$$\begin{aligned} &\left|D_z^{-\mu}(D^i f(z))\right| \\ \le\ & \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}\,[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2+\mu)} r^j\right) \end{aligned}$$

$$(|z| = r < 1; \quad \mu > 0; \quad i \in \{0,1,2,......,n\}). \tag{64}$$

Each of the assertions (63) and (64) is sharp.

Proof. We observe that

$$f(z) \in\ S_{m,n,j}^{\lambda}(\alpha, \beta) \Longleftrightarrow D^i f(z) \in S_{m,n-i,j}^{\lambda}(\alpha, \beta)$$

and that (cf. Equation (2))

$$D^{i}f(z)=z-\sum_{k=j+1}^{\infty}k^{i}a_{k}z^{k}\qquad(i\in\mathbb{N}_{0}).$$

In view of Theorem 1, we have

$$(j+1)^{n-i}[1+\lambda((j+1)^{m}-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\sum_{k=j+1}^{\infty}k^{i}a_{k}$$

$$\begin{aligned}&\leq \sum_{k=j+1}^{\infty}k^{n}[1+\lambda(k^{m}-1)]\{(k-1)(1+\beta)+2\beta(1-\alpha)\}a_{k}\\&\leq 2\beta(1-\alpha),\end{aligned}$$

so that, from (23).Consider the function $G(z)$ defined by

$$\begin{aligned}G(z)&=\Gamma(2+\mu)\,z^{-\mu}D_{z}^{-\mu}(D^{i}f(z))\\&=z-\sum_{k=j+1}^{\infty}\frac{\Gamma(k+1)\,\Gamma(2+\mu)}{\Gamma(k+1+\mu)}k^{i}a_{k}z^{k}\\&=z-\sum_{k=j+1}^{\infty}\phi(k)\,k^{i}a_{k}z^{k},\end{aligned}$$

where

$$\phi(k)=\frac{\Gamma(k+1)\,\Gamma(2+\mu)}{\Gamma(k+1+\mu)}\,(k\geq j+1;\ j\in\mathbb{N};\quad \mu>0).$$

Since $\phi(k)$ is a decreasing function of k, we get

$$0<\phi(k)\leq\phi(j+1)=\frac{\Gamma(j+2)\,\Gamma(2+\mu)}{\Gamma(j+2+\mu)}\,(k\geq j+1;\ j\in\mathbb{N};\ \mu>0\ ).\tag{65}$$

Thus, by using (23) and (65), we see that

$$\begin{aligned}|G(z)|&\geq r-\phi(j+1)r^{j+1}\sum_{k=j+1}^{\infty}k^{i}a_{k}\\&\geq r-\frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}[1+\lambda((j+1)^{m}-1)\lambda]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2+\mu)}r^{j+1}\end{aligned}$$

and

$$\begin{aligned}|G(z)|&\leq r+\phi(j+1)r^{j+1}\sum_{k=j+1}^{\infty}k^{i}a_{k}\\&\leq r+\frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}[1+\lambda((j+1)^{m}-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2+\mu)}r^{j+1},\end{aligned}$$

which prove the inequalities (63) and (64) of Theorem 11. The equalities in (63) and (64) are attained for the function $f(z)$ given by (24). This completes the proof of Theorem 12. □

Putting, $i=0$ in Theorem 12, we obtain, the following corollary.

Corollary 3. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$. Then

$$\begin{aligned}&\left|D_z^{-\mu}f(z)\right|\\ \geq\ &\frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1-\frac{2\beta(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n\left[1+\lambda((j+1)^m-1)\right]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2+\mu)}r^j\right)\\ &(|z|=r<1;\quad \mu>0) \qquad (66)\end{aligned}$$

and

$$\begin{aligned}&\left|D_z^{-\mu}f(z)\right|\\ \leq\ &\frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1+\frac{2\beta(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n\left[1+\lambda((j+1)^m-1)\right]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2+\mu)}r^j\right)\\ &(|z|=r<1;\quad \mu>0). \qquad (67)\end{aligned}$$

The estimates in (66) and (67) are sharp for the function $f(z)$ given by (24) with $i=0$.

Theorem 13. Let the function $f(z)$ defined by (4) be in the class $S^{\lambda}_{m,n,j}(\alpha,\beta)$.Then

$$\begin{aligned}&\left|D_z^{\mu}(D^i f(z))\right|\\ \geq\ &\frac{r^{1+\mu}}{\Gamma(2-\mu)}\left(1-\frac{2\beta(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2-\mu)}r^j\right)\\ &(|z|=r<1;\quad 0\leq\mu<1; i\in\{0,1,2,3,...,n-1\}) \qquad (68)\end{aligned}$$

and

$$\begin{aligned}&\left|D_z^{\mu}(D^i f(z))\right|\\ \leq\ &\frac{r^{1+\mu}}{\Gamma(2-\mu)}\left(1+\frac{2\beta(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2-\mu)}r^j\right)\\ &(|z|=r<1;\quad 0\leq\mu<1; i\in\{0,1,2,3,...,n-1\}). \qquad (69)\end{aligned}$$

Each of the assertions (68) and (69) is sharp.

Proof. Consider the function $H(z)$ defined by

$$\begin{aligned}H(z)&=\Gamma(2-\mu)\,z^{\mu}D_z^{\mu}(D^i f(z))\\ &=z-\sum_{k=j+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)}k^i a_k z^k\\ &=z-\sum_{k=j+1}^{\infty}\Psi(k)\,k^{i+1}a_k z^k,\end{aligned}$$

where

$$\Psi(k)=\sum_{k=j+1}^{\infty}\frac{\Gamma(k)\Gamma(2-\mu)}{\Gamma(k+1-\mu)}\left(k\geq j+1;\quad j\in\mathbb{N};\quad 0\leq\quad \mu<1\right). \qquad (70)$$

It is easily seen from (70) that

$$0<\Psi(k)\leq\Psi(j+1)=\sum_{k=j+1}^{\infty}\frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+2-\mu)}. \qquad (71)$$

Consequently, with the aid of (23) and (71), we have

$$\begin{aligned} |H(z)| &\geq r-\Psi(j+1)\, r^{j+1}\sum_{k=j+1}^{\infty}k^{i+1}a_k \\ &\geq r-\frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2-\mu)}{(j+1)^{n-i}[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2-\mu)}r^{j+1} \end{aligned}$$

$$(|z|=r<1;\quad 0\leq\mu<1; i\in\{0,1,2,3,...,n-1\}) \tag{72}$$

and

$$\begin{aligned} |H(z)| &\leq r+\Psi(j+1)\, r^{j+1}\sum_{k=j+1}^{\infty}k^{i+1}a_k \\ &\leq r-\frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2-\mu)}{(j+1)^{n-i}[1+((j+1)^m-1)\lambda]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2-\mu)}r^{j+1} \end{aligned}$$

$$(|z|=r<1;\quad 0\leq\mu<1; i\in\{0,1,2,3,...,n-1\}). \tag{73}$$

The estimates in (68) and (69) follow from (72) and (73), respectively. Each of these estimates is sharp for the function $f(z)$ given by (24). □

Letting $i=0$ in Theorem 13, we have the following corollary

Corollary 6. Let the function $f(z)$ defined by (4) be in the class $S_{m,n,j}^{\lambda}(\alpha,\beta)$. Then

$$\begin{aligned} &|D_z^{\mu}f(z)| \\ \geq\ &\frac{r^{1-\mu}}{\Gamma(2-\mu)}\left(1-\frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2-\mu)}{(j+1)^n\,(1+\lambda((j+1)^m-1))\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2-\mu)}r^j\right) \end{aligned}$$

$$(|z|=r<1;\quad 0\leq\mu<1) \tag{74}$$

and

$$\begin{aligned} &|D_z^{\mu}f(z)| \\ \leq\ &\frac{r^{1-\mu}}{\Gamma(2-\mu)}\left(1+\frac{2\beta(1-\alpha)\Gamma(j+2)\,\Gamma(2-\mu)}{(j+1)^n\,[1+\lambda((j+1)^m-1)]\{j(1+\beta)+2\beta(1-\alpha)\}\Gamma(j+2-\mu)}r^j\right) \end{aligned}$$

$$(|z|=r<1;\quad 0\leq\mu<1). \tag{75}$$

The estimates in (74) and (75) are sharp for the function $f(z)$ given by (24).

Remark 4. Putting $\mu=0$ in Corollary 6 , we obtain the result of Corollary 2.

Remark 5. We note that our results in Corollaries 5 and 6, respectively, are also obtained by Srivastava et al. [22, Corollaries 2 and 3, respectively, with $\gamma=1$].

7. **Acknowledgment**

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