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GROWTH PROPERTIES OF THE JACOBI-DUNKL TRANSFORM IN THE SPACE $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$

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Abstract. In this paper, we prove the estimate for the Jacobi-Dunkl transform associated to $\Lambda_{\alpha,\beta}$ in $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$,$1 < p \leq 2$, as applied to some classes of functions characterized by a generalized modulus of continuity.

1. INTRODUCTION

Integral transform and their inverses (e.g. Jacobi-Dunkl transform) are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see, [11] and [12]).
In [7], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator .
The main aim of this paper is to generalis the Theorem 1 in [7] on certain classes of functions characterized by a generalized continuity modulus and connected with the Jacobi-Dunkl transform in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$,$1 < p \leq 2$. For this purpose, we use a generalized Jacobi-Dunkl translation .
In section 2, we give some definitions and preliminaries concerning the Jacobi-Dunkl transform. The estimate are proved in section 3.

2. NOTATION AND PRELIMINARIES

In this section, we recapitulate from ([1],[2],[3],[5],[6]) some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$.
The Jacobi-Dunkl function with parameters α and β whith $\alpha \geq \beta \geq \frac{-1}{2}$ and $\alpha \neq \frac{-1}{2}$, defined by the formula

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) & \text{if } \lambda \in \mathbb{C}\backslash\{0\}, \\ 1 & \text{if } \lambda = 0, \end{cases}$$

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with $\lambda^2=\mu^2+\rho^2$, $\rho=\alpha+\beta+1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by:

$$\varphi_\mu^{\alpha,\beta}(x)=F\left(\frac{\rho+i\mu}{2},\frac{\rho-i\mu}{2},\alpha+1,-(\sinh(x))^2\right),$$

F is the Gausse hypergeometric function (see [1]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases}\Lambda_{\alpha,\beta}\mathcal{U}=i\lambda\mathcal{U} & ,\lambda\in\mathbb{C}\\ \\ \mathcal{U}(0)=1\end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}\mathcal{U}(x)=\frac{d\mathcal{U}(x)}{dx}+[(2\alpha+1)\coth x+(2\beta+1)\tanh x]\times\frac{\mathcal{U}(x)-\mathcal{U}(-x)}{2}]$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x)=\frac{d\mathcal{U}(x)}{dx}+\frac{A'(x)}{A(x)}\times\left(\frac{\mathcal{U}(x)-\mathcal{U}(-x)}{2}\right),$$

where $A(x)=|x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x)=A_{\alpha,\beta}(x)=2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x)=-\frac{\mu^2+\rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [1] and [2])

$$\forall x\in\mathbb{R},\psi_\lambda^{\alpha,\beta}(x)=\varphi_\mu^{\alpha,\beta}(x)+i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x).$$

For $\alpha>\frac{-1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(x)=\frac{2^\alpha\Gamma(\alpha+1)J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$is the Bessel function of the first Kind and $\Gamma(x)$ is the gamma-function (see [8]) .

Lemma 2.1. Let $\alpha\geq\beta\geq\frac{-1}{2},\alpha\neq\frac{-1}{2}$. Then for $|\nu|\leq\rho$, there exists a positive constant C_1 such that

$$|1-\varphi_{\mu+i\nu}^{\alpha,\beta}(x)|\geq C_1|1-j_\alpha(\mu x)|.$$

Proof. (See[4],Lemma 9).

Denote $L^p_{\alpha,\beta}(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$ such that

$$\begin{aligned}\|f\|_{p,\alpha,\beta} &= \left(\int_{\mathbb{R}}|f(x)|^pA_{\alpha,\beta}(x)dx\right)^{1/p}<+\infty, \quad \text{if} \quad 1\leq p<+\infty,\\ \|f\|_{\infty,\alpha,\beta} &= ess\sup_{x\in\mathbb{R}}|f(x)|<+\infty,\end{aligned}$$

and $L^p_\sigma(\mathbb{R}), p \geq 1$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\sigma} = \left(\int_{\mathbb{R}} |f(x)|^p d\sigma(x)\right)^{1/p} < +\infty$$

where $d\sigma$ is the measure given by :

$$d\sigma(\lambda) = C(\lambda)d\lambda = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|c_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|} 1_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$c_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C}\backslash(i\mathbb{N})$$

and $1_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
Using the eigenfunctions $\psi^{\alpha,\beta}_\lambda$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels , we define the Jacobi-Dunkl transform of a function $f \in L^p_{\alpha,\beta}(\mathbb{R})$ by:

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(x)\psi^{\alpha,\beta}_\lambda(x)A_{\alpha,\beta}(x)dx, \quad \lambda \in \mathbb{R}$$

and the inversion formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda)\psi^{\alpha,\beta}_{-\lambda}(x)d\sigma(\lambda).$$

The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$, i.e.

$$\|f\|_{2,\alpha,\beta} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{2,\sigma}. \tag{1}$$

By Plancherel's theorem (1) and the Marcinkiewics interpolation theorem (see [13]) we get for $f \in L^p_{\alpha,\beta}(\mathbb{R})$ with $1 < p \leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$,

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_{q,\sigma} \leq K\|f\|_{p,\alpha,\beta}, \tag{2}$$

where K is a positive constant (see [6]).
The operator of Jacobi-Dunkl translation is defined by :

$$T_xf(y) = \int_{\mathbb{R}} f(z)d\nu^{\alpha,\beta}_{x,y}(z), \quad \forall x,y \in \mathbb{R}$$

where $\nu^{\alpha,\beta}_{x,y}(z), \quad x,y \in \mathbb{R}$ are the signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } x,y \in \mathbb{R}^* \\ \\ \delta_x & \text{if } y=0 \\ \delta_y & \text{if } x=0 \end{cases}$$

Here, δ_x is the Dirac measure at x. And,

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}1_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \\ &\times \quad (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta \\ I_{x,y} &= [-|x|-|y|,-||x|-|y||] \cup [||x|-|y||,|x|+|y|] \\ \rho_\theta(x,y,z) &= 1-\sigma^\theta_{x,y,z}+\sigma^\theta_{z,x,y}+\sigma^\theta_{z,y,x} \end{aligned}$$

$$\forall z\in\mathbb{R},\theta\in[0,\pi],\sigma^{\theta}_{x,y,z}=\begin{cases}\frac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)} & ,\text{if } xy\neq 0\\ 0 & ,\text{if } xy=0\end{cases}$$

$$g_\theta(x,y,z)=1-\cosh^2(x)-\cosh^2(y)-\cosh^2(z)+2\cosh(x)\cosh(y)\cosh(z)\cos\theta$$

$$t_+=\begin{cases} t & ,\text{if } t>0\\ 0 & ,\text{if } t\leq 0\end{cases}$$

and,

$$M_{\alpha,\beta}=\begin{cases}\frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & ,\text{if } \alpha>\beta\\ 0 & ,\text{if } \alpha=\beta\end{cases}$$

For $f\in L^p_{\alpha,\beta}(\mathbb{R})$, we have (see [2])

$$\mathcal{F}_{\alpha,\beta}(T_hf)(\lambda)=\psi^{\alpha,\beta}_\lambda(h)\mathcal{F}_{\alpha,\beta}(f), \tag{3}$$

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}f)(\lambda)=i\lambda\mathcal{F}_{\alpha,\beta}(f). \tag{4}$$

The generalized modulus of continuity of a function $f\in L^p_{\alpha,\beta}(\mathbb{R})$ is defined by

$$\omega(f,\delta)_{p,\alpha,\beta}=\sup_{0<h\leq\delta}\|N_hf\|_{p,\alpha,\beta},\quad \delta>0,$$

where $N_h:=T_h+T_{-h}-2I$ and I is the identity operator in $L^p_{\alpha,\beta}(\mathbb{R})$.
Let $W^r_{p,\phi}(\Lambda_{\alpha,\beta})$ denote the class of functions $f\in L^p_{\alpha,\beta}(\mathbb{R})\cap C^r(\mathbb{R})$ that have generalized derivatives in the sense of Levi (see[9]) satisfying the estimate

$$\omega(\Lambda^r_{\alpha,\beta}f,\delta)_{p,\alpha,\beta}=O(\phi(\delta)),\quad \delta\to 0,$$

where $\phi(x)$ is any nonnegative function given on $[0,\infty)$, and $\Lambda^0_{\alpha,\beta}f=f$, $\Lambda^r_{\alpha,\beta}f=\Lambda_{\alpha,\beta}(\Lambda^{r-1}_{\alpha,\beta}f)$; $r=1,2,...$
i.e

$$W^r_{p,\phi}(\Lambda_{\alpha,\beta})=\{f\in L^p_{\alpha,\beta}(\mathbb{R})\cap C^r(\mathbb{R}),\Lambda^r_{\alpha,\beta}f\in L^p_{\alpha,\beta}(\mathbb{R})\text{and}\quad \omega(\Lambda^r_{\alpha,\beta}f,\delta)_{p,\alpha,\beta}=O(\phi(\delta)),\delta\to 0\}.$$

3. MAIN RESULT

In this section, we estimate the integral

$$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)$$

in certain classes of functions in $L^p_{\alpha,\beta}(\mathbb{R})$.

Lemma 3.1. For $f\in L^p_{\alpha,\beta}(\mathbb{R})$, we have

$$\left(\int_{\mathbb{R}}|\lambda|^{qr}|1-\varphi^{\alpha,\beta}_\mu(h)|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}}\leq\frac{K}{2}\|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}.$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From formula (4), we obtain

$$\mathcal{F}_{\alpha,\beta}(\Lambda^r_{\alpha,\beta}f)(\lambda)=i^r\lambda^r\mathcal{F}_{\alpha,\beta}(f)(\lambda),\quad r=0,1,2,.... \tag{5}$$

We us formulas (3) and (5), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda^r_{\alpha,\beta}f)(\lambda)=i^r(\psi^{(\alpha,\beta)}_\lambda(h)+\psi^{(\alpha,\beta)}_\lambda(-h)-2)\lambda^r\mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_\lambda^{(\alpha,\beta)}(h) = \varphi_\mu^{\alpha,\beta}(h) + i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(h),$$

and $\varphi_\mu^{\alpha,\beta}$ is even (see [2]), then

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda^r_{\alpha,\beta}f)(\lambda) = 2i^r(\varphi_\mu^{\alpha,\beta}(h)-1)\lambda^r\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By the inequality (2), we have the result.

Theorem 3.2. For functions $f \in L^p_{\alpha,\beta}(\mathbb{R})$ in the class $W^r_{p,\phi}(\Lambda_{\alpha,\beta})$,we have

$$\sup_{W^r_{p,\phi}(\Lambda_{\alpha,\beta})}\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) = O\left(N^{-2rq}\phi^q(\frac{c}{N})\right),$$

where $r = 0,1,2,....; c > 0$ is a fixed constant, and $\phi(t)$ is any nonnegative function defined on the interval $[0,\infty)$.

Proof. In the terms of $j_\alpha(x)$, the normalized Bessel function of the first kind, we have (see[10])

$$\begin{aligned} 1 - j_\alpha(x) &= O(1), \quad x \geq 1 && (6)\\ 1 - j_\alpha(x) &= O(x^2), \quad 0 \leq x \leq 1 && (7)\\ \sqrt{hx}J_\alpha(hx) &= O(1), \quad hx \geq 0. && (8) \end{aligned}$$

Let $f \in W^r_{p,\phi}(\Lambda_{\alpha,\beta})$. By the Hölder inequality and lemma 1.1, we have

$$\begin{aligned} &\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) - \int_{|\lambda|\geq N} j_\alpha(\lambda h)|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\\ = &\int_{|\lambda|\geq N}(1-j_\alpha(\lambda h))|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q C(\lambda)d\lambda\\ = &\int_{|\lambda|\geq N}(1-j_\alpha(\lambda h))\left(|\mathcal{F}_{\alpha,\beta}f(\lambda)|C(\lambda)^{\frac{1}{q}}\right)^q d\lambda\\ = &\int_{|\lambda|\geq N}(1-j_\alpha(\lambda h))\left(|\mathcal{F}_{\alpha,\beta}f(\lambda)|C(\lambda)^{\frac{1}{q}}\right)^{q-1}\left(|\mathcal{F}_{\alpha,\beta}f(\lambda)|C(\lambda)^{\frac{1}{q}}\right)d\lambda\\ \leq &\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q C(\lambda)d\lambda\right)^{\frac{q-1}{q}}\left(\int_{|\lambda|\geq N}|1-j_\alpha(\lambda h)|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q C(\lambda)d\lambda\right)^{\frac{1}{q}}\\ \leq &\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\left(\int_{|\lambda|\geq N}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}}\\ \leq &\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\left(\int_{|\lambda|\geq N}|\lambda|^{-rq+rq}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}}\\ \leq &N^{-r}\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\left(\int_{|\lambda|\geq N}|\lambda|^{qr}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}}. \end{aligned}$$

In view of lemma 3.1, we conclude that

$$\left(\int_{|\lambda|\geq N}|\lambda|^{qr}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}} \leq \frac{K}{2}\|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}.$$

Therefore

$$\begin{aligned}\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) &\leq \int_{|\lambda|\geq N} j_\alpha(\lambda h)|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\\ &+ N^{-r}\frac{K}{2}\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda))\right)^{\frac{q-1}{q}}\|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}.\end{aligned}$$

From formula (8) and definition of $j_\alpha(x)$, we have

$$j_\alpha(\lambda h) = O((\lambda h)^{-\alpha-\frac{1}{2}}).$$

Then

$$\begin{aligned}&\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\\ =\ & O\left(\int_{|\lambda|\geq N}|\lambda h|^{-\alpha-\frac{1}{2}}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)\\ +\ & O\left(N^{-r}\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}\right)\\ =\ & O\left((Nh)^{-\alpha-\frac{1}{2}}\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)\\ +\ & O\left(N^{-r}\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}\right).\end{aligned}$$

Or

$$\begin{aligned}(1-(Nh)^{-\alpha-\frac{1}{2}})\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) &= O(N^{-r})\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\\ &\times \|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}.\end{aligned}$$

For $f\in W^r_{p,\phi}(\Lambda_{\alpha,\beta})$ there exist a constant $C_2>0$ such that

$$\|N_h\Lambda^r_{\alpha,\beta}f\|_{p,\alpha,\beta}\leq C_2\phi(h).$$

Setting $h=\frac{c}{N}$ in the last equality and choosing $c>0$ such that $1-O(c^{-\alpha-\frac{1}{2}})\geq\frac{1}{2}$, we obtain

$$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) = O(N^{-r})\left(\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{q-1}{q}}\phi(\frac{c}{N}).$$

Then

$$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) = O\left(N^{-rq}\phi^q(\frac{c}{N})\right),$$

which completes the proof.

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