Electronic Journal of Mathematical Analysis and Applications,
Vol. 4(1) Jan. 2016, pp. 42-62.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

SOLVABILITY OF DEGENERATED $p(x)$-PARABOLIC EQUATIONS WITH THREE UNBOUNDED NONLINEARITIES.

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Abstract. In this paper, we study the existence of renormalized solutions for the nonlinear $p(x)$-parabolic problem with $f \in L^1(Q)$ and $b(x,u_0) \in L^1(\Omega)$. The main contribution of our work is to prove the existence of renormalized solutions of the weighted variable exponent Sobolev spaces and we suppose that $H(x,t,u,\nabla u)$ is the nonlinear term satisfying some growth condition but no sign condition or the coercivity condition.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1), T$ is a positive real number, and $Q = \Omega \times (0,T)$. We are interested in existence of renormalized solutions to the following nonlinear parabolic problem

$$(\mathcal{P}) \begin{cases} \frac{\partial b(x,u)}{\partial t} - div(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f \text{ in } Q = \Omega \times (0,T) \\ b(x,u) \mid_{t=0} = b(x,u_0) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $f \in L^1(Q)$, $b(x,u_0) \in L^1(\Omega)$. The operator $-div(a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on $L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega))$ (see assumption (3.3)-(3.5) of section 3) which is coercive $b(x,u)$ is an unbounded function of u, H is a nonlinear lower order term. The notion of renormalized solutions was introduced by R. J. Diperna and P. L. Lions [10] for the study of the Boltzmann equation. It was then used by L. Boccardo and al [6] when the right hand side is in $W^{-1,p'}(\Omega)$ and by J. M Rakoston [11] when the right hand side is in $L^1(\Omega)$.
It is our purpose to prove the existence of renormalized solution of weighted variable exponent Sobolev spaces for the problem $(\mathcal{P})$ setting without the sign condition and without the coercivity condition, the critical growth condition on H is only with respect to ∇u and not with respect to u (see assumption H2). Where the right hand side is assumed to satisfy: f belongs to $L^1(Q)$. Other work in this direction can be found in [[1],[4],[19],[20]].
For the convenience of the readers, we recall some definitions and basic properties of

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2010 *Mathematics Subject Classification.* A7A15, A6A32, 47D20.
Key words and phrases. Weighted variable exponent Sobolev space,Young's Inequality, Unbounded Nonlinearities, Renomalized Solution, Parabolic problems.
Submitted April 2, 2015.

the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$. Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}.$$

For any $p \in C_+(\overline{\Omega})$, we define $p^+ = \max_{x\in\overline{\Omega}} p(x)$, $p^- = \min_{x\in\overline{\Omega}} p(x)$.
For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega) = \{u : \Omega \to \mathbb{R}, measurable, \int_\Omega |u(x)|^{p(x)}\omega(x)dx < \infty\}.$$

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\omega(x)dx \le 1\}$$

becomes a normed space. When $\omega(x) \equiv 1$, we have $L^{p(x)}(\Omega,\omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,w)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega) = \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega,\omega)\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega,\omega)} \tag{1.1}$$

or, equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} + \omega(x)|\frac{\nabla u(x)}{\lambda}|^{p(x)}dx \le 1\}$$

for all $u \in W^{1,p(x)}(\Omega,\omega)$.
It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [21] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|} \tag{1.2}$$

for every x, y with $|x - y| \le \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [12]).
$W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega,\omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$.
Throughout the paper, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.
This paper is organized as follows. In Section 2, we state some basic results for the weighted variable exponent Lebesgue-Sobolev spaces which is given in [16]. In Section 3, we make precise all the assumption on b, a, H, f and $b(x, u_0)$ and give the definition of a renormalized solution of the problem $(\mathcal{P})$ and main results, which is proved in Section 4.

2. Preliminaries.

In this Section, we state some elementary properties for the (weighted) variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is when $\omega(x) \equiv 1$ can be found from [13, 15].

Lemma 2.1. *(See [13, 15].)(Generalised Hölder inequality).*

- **i)** *For any functions* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*, we have* $|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})||u||_{p(\cdot)}||v||_{p'(.)} \leq 2||u||_{p(\cdot)}||v||_{p'(.)}$.
- **ii)** *For all* $p,\ q \in C_+(\overline{\Omega})$ *such that* $p(x) \leq q(x)$ *a.e. in* Ω*, we have* $L^{q(.)} \hookrightarrow L^{p(\cdot)}$ *and the embedding is continuous.*

Lemma 2.2. *(See [16].) Denote* $\rho(u) = \int_\Omega \omega(x)|u(x)|^{p(x)}dx$ *for all* $u \in L^{p(x)}(\Omega, \omega)$*. Then,*

$$|u|_{L^{p(x)}(\Omega,\omega)} < 1 (= 1; > 1) \text{ if and only if } \rho(u) < 1 (= 1; > 1), \tag{2.1}$$

$$\text{if } |u|_{L^{p(x)}(\Omega,\omega)} > 1 \text{ then } |u|^{p^-}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega,\omega)}, \tag{2.2}$$

$$\text{if } |u|_{L^{p(x)}(\Omega,\omega)} < 1 \text{ then } |u|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{2.3}$$

Remark 2.3. *([17].) If we set*

$$I(u) = \int_\Omega |u(x)|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)}dx.$$

Then, following the same argumen, we have
$\min\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\} \leq I(u) \leq \max\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\}.$

Throughout the paper, we assume that ω is a measurable positive and a.e.finite function in Ω satisfying that
$(\mathbf{W}_1)$ $\omega \in L^1_{loc(\Omega)}$ and $\omega^{-\frac{1}{(p(x)-1)}} \in L^1_{loc}(\Omega)$;
$(\mathbf{W}_2)$ $\omega^{-s(x)} \in L^1(\Omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.
The reasons that we assume $(\mathbf{W}_1)$ and $(\mathbf{W}_2)$ can be found in [16].

Remark 2.4. *([16].)*
(i) If ω *is a positive measurable and finite function, then* $L^{p(x)}(\Omega, \omega)$ *is a reflexive Banach space.*
(ii) Moreover, if $(\boldsymbol{W}_1)$ *holds, then* $W^{1,p(x)}(\Omega, \omega)$ *is a reflexive Banach space.*

For $p, s \in C_+(\overline{\Omega})$, denote
$p_s(x) = \frac{p(x)s(x)}{s(x)+1} < p(x)$, where $s(x)$ is given in $(\mathbf{W}_2)$.
Assume that we fix the variable exponent restrictions

$$\begin{cases} p_s^*(x) = \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ p_s^*(x) \text{ arbitrary} & \text{if } N \leq p_s(x) \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our paper. We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

Lemma 2.5. *([16].) Let* $p, s \in C_+(\overline{\Omega})$ *satisfy the log-Hölder continuity condition (1.2), and let* $(\boldsymbol{W}_1)$ *and* $(\boldsymbol{W}_2)$ *be satisfied. If* $r \in C_+(\overline{\Omega}))$ *and* $1 < r(x) \leq p_s^*$*. Then, we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p^*_{s(x)}$ *for all* $x \in \overline{\Omega}$*.*

From Lemma 2.5, we have Poincaré-type inequality immediately.

Corollary 2.6. *([16].) Let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.2). If ($\boldsymbol{W}_1$) and ($\boldsymbol{W}_2$) hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds, for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.2) and $X := W_0^{1,p(x)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega,\omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} \omega(x)dx \leq 1\},$$

which is equivalent to the norm (1.1) due to Corollary 2.6. The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega,\omega))^*$, which is analogous to [[15], Theorem 3.16]. We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(x)}(\Omega,\omega)$ is equivalent to $W^{-1,p'(x)}(\Omega,\omega)$, where $\omega^* = \omega^{1-p'(x)}$.

Lemma 2.7. *([5].) Let $g \in L^{p(\cdot)}(Q,\omega)$ and let $g_n \in L^{p(\cdot)}(Q,\omega)$, with $\|g_n\|_{L^{p(\cdot)}(Q,\omega)} \leq c$, $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Q, then $g_n \rightharpoonup g$ in $L^{p(\cdot)}(Q,\omega)$, where $\rightharpoonup$ denotes weak convergence and ω is a weight function on Q.*

We will also use the standard notation for Bochner spaces, i.e., if $q \geq 1$ and X is a Banach space then $L^q(0,T;X)$ denotes the space of strongly measurable function $u : (0,T) \to X$ for which $t \to \|u(t)\|_X \in L^q(0,T)$ Morever, $C([0;T];X)$ denotes the space of continuous function $u : [0;T] \to X$ endowed with the norm $\|u\|_{C([0;T];X)} = max_{t\in[0;T]}\|u\|_X$,

$$L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) = \{u : (0,T) \to W_0^{1,p(\cdot)}(\Omega,\omega) \text{ measurable};$$
$$\Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega,\omega)}\Big)^{1/p^-} < \infty\}$$

and we define the space

$$L^\infty(0,T;X) = \{u : (0,T) \to X \text{ measurable}; \exists C > 0/\|u(t)\|_X \leq C \text{ a.e.}\}$$

where the norm is defined by:

$$\|u\|_{L^\infty(0,T;X)} = \inf\{C > 0; \|u(t)\|_X \leq C \text{ a.e.}\}.$$

We introduce the functional space see [5]

$$V = \{f \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)); |\nabla f| \in L^{p(\cdot)}(Q,\omega)\}, \tag{2.4}$$

which endowed with the norm:

$$\|f\|_V = \|\nabla f\|_{L^{p(\cdot)}(Q,\omega)}$$

or, the equivalent norm :

$$|||f|||_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} + \|\nabla f\|_{L^{p(\cdot)}(Q,\omega)},$$

is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(\cdot)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.8. *Let V be defined as in (2.4) and its dual space be denote by V^*. Then,*
i) We have the following continuous dense embeddings:

$$L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(\cdot)}(\Omega,\omega))^*)\hookrightarrow V^*\hookrightarrow L^{(p^+)'}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))^*).$$

Note that, we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(\cdot)}(\Omega,\omega))\hookrightarrow L^{p(\cdot)}(Q,\omega)\hookrightarrow L^{p^-}(0,T;L^{p(\cdot)}(\Omega,\omega)).$$

ii) One can represent the elements of V^* as follows: if $T\in V^*$, then there exists $F=(f_1,....,f_N)\in(L^{p'(\cdot)}(Q))^N$ such that $T=div_X F$ and

$$\langle T,\xi\rangle_{V^*,V}=\int_0^T\int_\Omega F\cdot\nabla\xi dxdt$$

for any $\xi\in V$. Moreover, we have

$$\|T\|_{V^*}=\max\{\|f_i\|_{L^{p(\cdot)}(Q,\omega)},\ i=1,....,n\}.$$

Remark 2.9. *The space $V\cap L^\infty(Q)$, is endowed with the norm definie by the formula:*

$$\|v\|_{V\cap L^\infty(Q)}=\max\{\|v\|_V,\|v\|_{L^\infty(Q)}\},\ v\in V\cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V+L^1(Q)$ endowed with the norm:

$$\|v\|_{V^*+L^1(Q)}:=\inf\{\|v_1\|_{V^*}+\|v_2\|_{L^1(Q)}\};\ v=v_1+v_2,\ v_1\in V^*,v_2\in L^1(Q).$$

2.1. **Some Technical Results.**

Lemma 2.10. *Assume (3.3) -(3.5) and let $(u_n)_n$ be a sequence in $L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega,\omega))$ such that $u_n\rightharpoonup u$ weakly in $L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega,\omega))$ and*

$$\int_Q\Big(a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla u)\Big)\cdot\nabla(u_n-u)dxdt\to 0. \tag{2.5}$$

Then, $u_n\to u$ strongly in $L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega,\omega))$.

Proof.
Let $D_n=[a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla u)]\nabla(u_n-u)$, thanks to (3.4), we have D_n is a positive function, and by (2.5) , $D_n\to 0$ in $L^1(Q)$ as $n\to\infty$.
Extracting a subsequence, still denoted by u_n, we can write $u_n\rightharpoonup u$ a.e. in Q and since $D_n\to 0$ a.e. in Q. There exists a subset B in Q with measure zero such that for all $(t,x)\in Q\backslash B$,

$$|u(x,t)|<\infty,\quad |\nabla u(x,t)|<\infty,\quad K(x,t)<\infty,\quad u_n\to u,\quad D_n\to 0.$$

Taking $\xi_n=\nabla u_n$ and $\xi=\nabla u$, we have

$$\begin{aligned}D_n(x,t)&=[a(x,t,u_n,\xi_n)-a(x,t,u_n,\xi)]\cdot(\xi_n-\xi)\\&=a(x,t,u_n,\xi_n)\xi_n+a(x,t,u_n,\xi)\xi-a(x,t,u_n,\xi_n)\xi-a(x,t,u_n,\xi)\xi_n\\&\geq\alpha\omega(x)|\xi_n|^{p(x)}+\alpha\omega(x)|\xi|^{p(x)}\\&-\beta\omega^{1/p(x)}(x)\Big(k(x,t)+\omega^{1/p'(x)}(x)|u_n|^{p(x)-1}+\omega^{1/p'(x)}(x)|\xi_n|^{p(x)-1}\Big)|\xi|\\&-\beta\omega^{1/p(x)}(x)\Big(k(x,t)+\omega^{1/p'(x)}(x)|u_n|^{p(x)-1}+\omega^{1/p'(x)}(x)|\xi|^{p(x)-1}\Big)|\xi_n|\\&\geq\alpha\omega(x)|\xi_n|^{p(x)}-C_{x,t}[1+\omega^{1/p'(x)}(x)|\xi_n|^{p(x)-1}+\omega^{1/p(x)}(x)|\xi_n|],\end{aligned}$$

where $C_{x,t}$ depending on x, but does not depend on n. (Since $u_n(x,t)\to u(x,t)$ then, $(u_n)_n$ is bounded), we obtain

$$D_n(x,t)\geq|\xi_n|^{p(x)}\Big(\alpha\omega(x)-\frac{C_{x,t}}{|\xi_n|^{p(x)}}-\frac{C_{x,t}\omega^{\frac{1}{p'(x)}}}{|\xi_n|}-\frac{C_{x,t}\omega^{\frac{1}{p(x)}}}{|\xi_n|^{p(x)-1}}\Big),$$

by the standard argument $(\xi_n)_n$ is bounded almost everywhere in Q. Indeed, if $|\xi_n| \to \infty$ in a measurable subset $E \in Q$ then,

$$\lim_{n\to\infty} \int_Q D_n(x,t)dx \geq \lim_{n\to\infty} \int_E |\xi_n|^{p(x)} \Big(\alpha\omega(x) - \frac{C_{x,t}}{|\xi_n|^{p(x)}} - \frac{C_{x,t}\omega^{\frac{1}{p'(x)}}}{|\xi_n|} - \frac{C_{x,t}\omega^{\frac{1}{p(x)}}}{|\xi_n|^{p(x)-1}} \Big) = \infty,$$

which is absurd since $D_n(x,t) \to 0$ in $L^1(Q))$. Let ξ^* an accumulation point of $(\xi_n)_n$, we have $|\xi^*| < \infty$ and by continuity of $a(.,.,.,.)$, we obtain

$$a(x,t,u(x,t),\xi^*) - a(x,t,u(x,t),\xi)] \cdot (\xi_n - \xi) = 0,$$

thanks to (3.4), we have $\xi^* = \xi$, the uniqueness of the accumulation point implies that $\nabla u_n(x,t) \to \nabla u(x,t)$ a.e. in Q. Since the sequence $a(x,t,u,\nabla u_n)$ is bounded in $(L^{p'(x)}(Q,\omega^*))^N$ and $a(x,t,u,\nabla u_n) \to a(x,t,u,\nabla u)$ a.e. in Q, Lemma 2.7 implies

$$a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \quad \text{in } (L^{p'(x)}(Q,\omega^*))^N.$$

Let us taking $\bar{y}_n = a(x,t,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,t,u,\nabla u)\nabla u$, then $\bar{y}_n \to \bar{y}$ in $L^1(Q)$, according to the condition (3.5), we have

$$\alpha\omega(x)|\nabla u_n|^{p(x)} \leq a(x,t,u_n,\nabla u_n)\nabla u_n.$$

Let $z_n = |\nabla u_n|^{p(x)}\omega,\ z = |\nabla u|^{p(x)}\omega$ and $y_n = \frac{\bar{y}_n}{\alpha},\ y = \frac{\bar{y}}{\alpha}$. Then, by Fatou's Lemma, we obtain

$$\int_Q 2y dxdt \leq \liminf_{n\to\infty} \int_Q (y_n + y - |z_n - z|)dxdt,$$

i.e., $0 \leq \limsup_{n\to\infty} \int_Q |z_n - z| dxdt$, hence

$$0 \leq \liminf_{n\to\infty} \int_Q |z_n - z| dx \leq \leq \limsup_{n\to\infty} \int_Q |z_n - z| dx \leq 0,$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(Q,\omega))^N,$$

we deduce that

$$u_n \to u \quad \text{in } L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega,\omega)),$$

which completes our proof.

Let $X = L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, the dual space of X is $X^* = L^{p^-}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$.

Lemma 2.11. *(See[17].)*
$W := \Big\{ u \in V; u_t \in V^* + L^1(Q) \Big\} \hookrightarrow C([0,T];L^1(\Omega))$
and

$$W \cap L^\infty(Q) \hookrightarrow C([0,T];L^2(\Omega)).$$

Definition 2.12. *A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph*

$$G(T) = \Big\{ (u, T(u)) \in X \times X^* \ \textit{for all}\ u \in D(T) \Big\},$$

is not a proper subset of any monotone set in $X \times X^*$.
Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset

$$D(L) = \Big\{ v \in X : v' \in X^*, v(0) = 0 \Big\} \ \textit{of } X \textit{ in to } X^* \textit{ by}$$

$$\Big\langle Lu, v \Big\rangle_X = \int_0^T \langle u'(t), v(t) \rangle_V dt \qquad u \in D(L),\ v \in X.$$

Definition 2.13. *A mapping S is called pseudo-monotone with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \leq 0$, that we have*
$\lim_{n\to\infty} \sup \Big\langle S(u_n), u_n - u \Big\rangle = 0$ *and $S(u_n) \rightharpoonup S(u)$ as $n \to \infty$.*

3. Assumption and Main Results

Throughout the paper, we assume that the following assumption hold true.

Assumption (H1)

Let Ω be a bounded open subset of $\mathbb{R}^N(N \geq 1)$, $p \in C_+(\overline{\Omega})$ and
$b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1 function with

$$b(x, 0) = 0. \tag{3.1}$$

Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^{p(\cdot)}(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \text{ and } \Big| D_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \leq B_k(x), \tag{3.2}$$

for almost every $x \in \Omega$ and every s such that $|s| \leq k$, we denote by
$D_x(\partial b(x,s) \setminus \partial s)$ the gradient of $\partial b(x,s) \setminus \partial s$ defined in the sense of distributions.

Assumption (H2)

We consider a Leray -Lions operator defined by the formula:

$$Au = -div\ a(x,t,u,\nabla u),$$

where $a : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function i.e., (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, for almost every x in Ω) which satisfies the following conditions there exist $k \in L^{p'(\cdot)}(Q)$ and $\alpha > 0$, $\beta > 0$ such that for almost every $(x,t) \in Q$ all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

$$|a(x,t,s,\xi)| \leq \beta\omega^{1/p(x)}(x)[k(x,t) + \omega^{1/p'(x)}|s|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}], \tag{3.3}$$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)] \cdot (\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \in \mathbb{R}^N, \tag{3.4}$$

$$a(x,t,s,\xi) \cdot \xi \geq \alpha\omega|\xi|^{p(x)}. \tag{3.5}$$

Assumption (H3)

Let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)\omega|\xi|^{p(x)} \tag{3.6}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma \in L^1(Q)$.

We recall that, for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ defined by

$$T_k(s) = \begin{cases} s & if \quad |s| \leq k \\ k\frac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Definition 3.1. *Let $f \in L^1(Q)$ and $b(., u_0) \in L^1(\Omega)$. A real-valued function u defined on Q is renormalized solutions of problem $(\mathcal{P})$ if:*

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega)) \textit{ for all } k \geq 0,\ b(x,u) \in L^\infty(0,T; L^1(\Omega)), \tag{3.7}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \to 0 \textit{ as } m \to \infty, \tag{3.8}$$

$$\begin{aligned}\frac{\partial B_S(x,u)}{\partial t} &- div\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &+H(x,t,u,\nabla u)S'(u) = fS'(u) \textit{ in } D'(Q),\end{aligned} \tag{3.9}$$

for all $S \in W^{2,\infty}(\mathbb{R})$, which are piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x,u)\mid_{t=0} = B_S(x,u_0) \quad in \quad \Omega. \tag{3.10}$$

Remark 3.2. *Equation (3.9) is formally obtained through pointwise multiplication of problem* $(\mathcal{P})$ *by* $S'(u)$. *However, while* $a(x,t,u,\nabla u)$ *and* $H(x,t,u,\nabla u)$ *do not in general make sense in* $(\mathcal{P})$, *all the terms in (3.9) have a meaning in* $D'(Q)$. *Indeed, if M is such that supp* $S \subset [-M,M]$, *the following identifications are made in (3.9):*

- $S(u)$ *belongs to* $V \cap L^\infty(Q)$. *Since S is a bounded function.*
- $S'(u)\ a(x,t,u,\nabla u)$ *identifies with* $S'(u)\ a(x,t,T_M(u),\nabla T_M(u))$ *a.e. in Q,*

for any $\varphi \in D(Q)$, *using Hölder inequality*

$$\int_Q S'(u)a(x,t,u,\nabla u)\nabla\varphi dxdt = \int_Q S'(u)a(x,t,T_M(u),\nabla T_M(u))\nabla\varphi dxdt$$

$$\leq C_M\|S'\|_{L^\infty(Q)} \max\Big\{\Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega\Big)^{\frac{1}{p'^-}}, \Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega\Big)^{\frac{1}{p'^+}}\Big\}\|\nabla\varphi\|_{L^{p(\cdot)}(Q)},$$

where $M>0$ *is that supp* $S' \subset [-M,M]$. *As* $D(Q)$ *is dense in V, we deduce that*

$$div(S'(u)a(x,t,u,\nabla u)) \in V^*.$$

- $S''(u)\ a(x,t,u,\nabla u)\nabla u$ *identifies with* $S''(u)\ a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u)$ *and*
$$S''(u)a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u) \in L^1(Q).$$
- $S'(u)H(x,t,u,\nabla u)$ *identifies with* $S'(u)H(x,t,T_M(u),\nabla T_M(u))$ *a.e. in Q. Since* $|T_M(u)| \leq M$ *a.e. in Q and* $S'(u) \in L^\infty(Q)$, *we see from (3.6) and (3.7) that* $S'(u)H(x,t,T_M(u),\nabla T_M(u)) \in L^1(Q)$.
- $S'(u)\ f$ *belongs to* $L^1(Q)$.

The above considerations show that equation (3.9) hold in $D'(Q)$ *and that*

$$\frac{\partial B_S(x,u)}{\partial t} \in V^* + L^1(Q).$$

Due to the properties of S and (3.9), $\frac{\partial S(u)}{\partial t} \in V^* + L^1(Q)$, *using Lemma 2.11 which implies that* $S(u) \in C^0([0,T);L^1(\Omega))$. *So that the initial condition (3.10) makes sense since, due to the properties of S (increasing) and (3.2), we have*

$$\Big|(B_S(x,r) - B_S(x,r')\Big| \leq A_k(x)\Big|S(r)-S(r')\Big| \text{ for all } r,r' \in \mathbb{R}. \tag{3.11}$$

Theorem 3.3. *Let* $f \in L^1(Q)$, $p(\cdot) \in C_+(\bar\Omega)$ *and assume that* u_0 *is a measurable function such that* $b(.,u_0) \in L^1(\Omega)$. *Assume that* $(H1)-(H3)$ *hold true. Then there, exists a renormalized solution u of problem* $(\mathcal{P})$ *in the sense of Definition 3.1.*

4. Proof of Main Results.

4.1. **Approximate problem.**

For $n>0$, we define approximations of b, H, f and u_0. First set

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r. \tag{4.1}$$

b_n is a Carathéodory function and satisfies (3.2). There exist $\lambda_n > 0$ and functions $A_n \in L^\infty(\Omega)$ and $B_n \in L^{p(\cdot)}(\Omega)$ such that

$$\lambda_n \leq \frac{\partial b_n(x,s)}{\partial s} \leq A_n(x) \ \text{ and } \ \Big|D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\Big| \leq B_n(x) \ a.e. \text{ in } \Omega,\ s\in\mathbb{R}.$$

Next, set

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}.$$

$$\text{Note that } |H_n(x,t,s,\xi)| \leq |H(x,t,s,\xi)|$$
$$\text{and } |H_n(x,t,s,\xi)| \leq n \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N$$

and select f_n, u_{0n} and b_n. So that

$$f_n \in L^{p'(.)}(Q) \text{ and } f_n \to f \text{ a.e. in } Q, \text{ strongly in } L^1(Q) \text{ as } n \to \infty, \tag{4.2}$$

$$u_{0n} \in D(\Omega), \quad \|b_n(x, u_{0n})\|_{L^1(\Omega)} \leq \| b_n(x, u_0)\|_{L^1(\Omega)}, \tag{4.3}$$

$$b_n(x, u_{0n}) \to b(x, u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{4.4}$$

Let us now consider the approximate problem

$$(\mathcal{P}_n)\begin{cases} \frac{\partial b_n(x,u_n)}{\partial t} - div(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = f_n \quad \text{in } D'(Q),\\ b_n(x,u_n) \mid_{t=0}= b_n(x,u_{0n}) \quad \text{in } \Omega,\\ u_n = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \quad u_n \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)). \end{cases}$$

Theorem 4.1. *Let $f_n \in L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega,\omega^*))$, $p(\cdot) \in C_+(\overline{\Omega})$ for fixed n, the approximate problem $(\mathcal{P}_n)$ has at least one weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$.*

Proof.
We define the operator $L_n : L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \to L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega,\omega^*))$ by $\Big\langle L_n u, v\Big\rangle = \int_Q \frac{\partial b_n(x,u)}{\partial t} v dxdt = \int_Q \frac{\partial b_n(x,u)}{\partial u}\frac{\partial u}{\partial t} v dxdt \quad \forall u,v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$, then,

$$\begin{aligned}\Big|\Big\langle L_n u, v\Big\rangle\Big| &\leq \Big|\int_0^T\int_\Omega A_n(x)\frac{\partial u}{\partial t} v dxdt\Big| = \Big|\int_0^T\int_\Omega A_n(x)\frac{\partial u}{\partial t}\omega^{-\frac{1}{p(x)}} v\omega^{\frac{1}{p(x)}} dxdt\Big|\\
&\leq \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{L^{p'(x)}(\Omega,\omega^*)}\|v\|_{L^{p(x)}(\Omega,w)}dt\\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{W^{-1,p'(.)}(\Omega,\omega^*)}\|v\|_{W_0^{1,p(x)}(\Omega,\omega)}dt \qquad (4.5)\\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\|\frac{\partial u}{\partial t}\|_{L^{p'^-}(0,T,W^{-1,p'(.)}(\Omega,\omega^*))}\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))}\\
&\leq C_1\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))}.\end{aligned}$$

We define the operator $G_n : L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \to L^{p^-}(0,T,W^{-1,p'(.)}(\Omega,\omega^*))$

$$by, \quad \Big\langle G_n u, v\Big\rangle = \int_Q H_n(x,t,u,\nabla u) v dxdt \quad \forall u,v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)).$$

Thanks to the Hölder inequality, we have that for $u, v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$

$$\begin{aligned}&\int_Q H_n(x,t,u,\nabla u)vdxdt \leq \Big|\int_0^T\int_\Omega H_n(x,t,u,\nabla u)vdxdt\Big|\\
&\leq \Big|\int_0^T\int_\Omega H_n(x,t,u,\nabla u)\omega^{-\frac{1}{p(x)}} v\omega^{\frac{1}{p(x)}}dxdt\Big|\\
&\leq \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\int_0^T\Big(\int_\Omega\Big|H_n(x,t,u,\nabla u)\Big|^{p'(x)}\omega^{-\frac{p'(x)}{p(x)}}dx\Big)^\theta\|v\|_{L^{p(x)}(\Omega,\omega)}dt\\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\int_0^T n^{\theta p'^+}\Big(\int_\Omega \omega^{-\frac{p'(x)}{p(x)}}dx\Big)^\theta\|v\|_{W_0^{1,p(x)}(\Omega,\omega)}dt\\
&\leq C_2\|v\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}. \qquad (4.6)\end{aligned}$$

with $\theta = \begin{cases} 1/p'^- & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} > 1\\ 1/p'^+ & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} \leq 1.\end{cases}$

Lemma 4.2. *Let $B_n : L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \to L^{p'^-}(0,T,W^{-1,p'(.)}(\Omega,\omega^*))$.*
The operator $B_n = A + G_n$ is
a) coercive

b) pseudo-monotone
c) bounded and demi continuous.

Proof. **a)** For the coercivity, we have for any $u \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$

$$\Big\langle B_n u, u\Big\rangle = \Big\langle G_n u, u\Big\rangle + \Big\langle Au, u\Big\rangle$$
$$\Rightarrow \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle = \Big\langle Au, u\Big\rangle$$
$$\textit{then},\ \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle = \int_Q a(x,t,u,\nabla u)\nabla u dxdt$$
$$= \int_0^T \int_\Omega a(x,t,u,\nabla u)\nabla u dxdt$$
$$\geq \int_0^T \alpha(\int_\Omega |\nabla u|^{p(x)}\omega(x)dx)dt \quad \text{(using (3.5))}$$
$$\geq \alpha\|\nabla u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \geq \beta\|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))},$$

which is due to Poincaré inequality with

$$\delta = \begin{cases} p^- & if\ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} > 1 \\ p^+ & if\ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \leq 1, \end{cases}$$

$$\textit{hence},\ \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle \geq \beta\|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}$$
$$\textit{then},\ \Big\langle B_n u, u\Big\rangle \geq \beta\|\ u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} - C_2\|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}$$

then, we have

$$\frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}} \geq \beta\|u\|^{\delta-1}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,w))} - C_2 \to +\infty$$
$$\Rightarrow \frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}} \to +\infty \quad as\ \|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \to +\infty$$

then, B_n is coercive.
b)It remains to show that B_n is pseudo-monotone.
Let $(u_k)_k$ a sequence in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ such that

$$\begin{aligned} & u_k \rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \\ & L_n u_k \rightharpoonup L_n u \ \text{ in } \ L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega,\omega^*)) \\ & \lim_{k\to\infty}\sup \Big\langle B_n u_k, u_k - u\Big\rangle \leq 0 \end{aligned} \tag{4.7}$$

that, we have prove that

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))\ and\ \langle B_n u_k, u_k\rangle \to \langle B_n u, u\rangle.$$

By the definition of the operator L_n defined in definition 2.12, we obtain that u_k is bounded in $W_0^{1,p(\cdot)}(\Omega,\omega)$ and since $W_0^{1,p(\cdot)}(\Omega,\omega) \hookrightarrow L^{p'(\cdot)}(\Omega)$,
then $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$, then the growth condition (3.3) $(a(x,t,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(\cdot)}(Q,\omega^*))^N$ therefore, there exists a function $\varphi \in (L^{p'(\cdot)}(Q,\omega^*))^N$ such that

$$a(x,t,u_k,\nabla u_k) \rightharpoonup \varphi\ as\ k \to +\infty. \tag{4.8}$$

Similarly, using condition (3.6), $\Big(H_n(x,t,u_k,\nabla u_k)\Big)_k$ is bounded in $L^1(Q)$, then there exists a function $\psi_n \in L^1(Q)$ such that:

$$H_n(x,t,u_k,\nabla u_k) \to \psi_n \text{ in } L^1(Q) \text{ as } k \to +\infty. \tag{4.9}$$

$$\begin{aligned}\lim_{k\to\infty}\Big\langle B_n u_k, u_k\Big\rangle &= lim_{k\to\infty}\Big[\Big\langle G_n u_k, u_k\Big\rangle + \Big\langle Au_k, u_k\Big\rangle\Big] \\ =\lim_{k\to\infty}\Big[&\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt + \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big] \\ &= \int_Q \varphi\nabla u_k dxdt + \int_Q \psi_n u_k dxdt \end{aligned} \tag{4.10}$$

using (4.7) and (4.10), we obtain

$$\begin{aligned}\lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k\Big\rangle &= \lim_{k\to\infty}\sup\Big\{\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \\ &+ \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big\} \\ &\leq \int_Q \varphi\nabla u dxdt + \int_Q \psi_n u dxdt \end{aligned} \tag{4.11}$$

thanks to (4.9), we have:

$$\int_Q H_n(x,t,u_k,\nabla u_k)dxdt \to \int_Q \psi_n dxdt. \tag{4.12}$$

therefore,

$$\lim_{k\to\infty}\sup\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k \leq \int_Q \varphi\nabla u dxdt \tag{4.13}$$

on the other hand, using (3.4), we have

$$\int_Q \Big[a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u)\Big](\nabla u_k - \nabla u)dxdt \geq 0. \tag{4.14}$$

Then,

$$\begin{aligned}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt &\geq -\int_Q a(x,t,u_k,\nabla u)\nabla u dxdt \\ &+\int_Q a(x,t,u_k,\nabla u_k)\nabla u dxdt \\ &+\int_Q a(x,t,u_k,\nabla u)\nabla u_k dxdt \end{aligned}$$

and by (4.8), we get

$$\lim_{k\to\infty}\inf\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \geq \int_Q \varphi\nabla u dxdt,$$

this implies, thanks to (4.13) that

$$\lim_{k\to\infty}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt = \int_Q \varphi\nabla u dxdt. \tag{4.15}$$

Now, by(4.15), we can obtain

$$\lim_{k\to\infty}\int_Q a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u))(\nabla u_k - \nabla u)dxdt = 0.$$

In view of the Lemma 2.10, we obtain

$$\begin{array}{rcl} u_k & \to & u \quad in \quad L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)),\\ \nabla u_k & \to & \nabla u \quad a.e.\ in \quad Q. \end{array}$$

Then,

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u) \quad in \quad (L^{p'(\cdot)}(Q,\omega^*))^N,$$
$$H_n(x,t,u_k,\nabla u_k) \rightharpoonup H_n(x,t,u,\nabla u) \quad in \quad L^1(Q),$$

we deduce that

$$Au_k \rightharpoonup Au \quad in \quad (L^{p'^-}(Q,\omega^*))^N$$

and

$$G_n u_k \rightharpoonup G_n u \quad in\ L^1(Q),$$

which implies

$$B_n u_k \rightharpoonup B_n u \quad in\ L^{p'^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$$

and

$$\Big\langle B_n u_k, u_k \Big\rangle \to \Big\langle B_n u, u \Big\rangle$$

completing the proof of assertion(b).
c) Using $H\ddot{o}lder's$ inequality and the growth condition (3.3), we can show that the operator A is bounded, and by using (4.6), we conclude that B_n is bounded. For to show that B_n is demicontinuous.
Let $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and prove that:

$$\Big\langle B_n u_k, \psi \Big\rangle \to \langle B_n u, \psi\rangle \quad for\ all\ \psi \in\ L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)).$$

Since $a(x,t,u_k,\nabla u_k) \to a(x,t,u,\nabla u)$ as $k\to\infty$ a.e. in Q. Then, by the growth condition (3.3) and Lemma 2.7

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u)\ in\ (L^{p'(\cdot)}(Q,\omega^*))^N$$

and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$, $\langle Au_k,\varphi\rangle \to \langle Au,\varphi\rangle$ as $k\to\infty$
similarly, $G_n u_k \to G_n u$ as $k\to\infty$ a.e. in Q, then by the (3.6) and Lemma 2.7 $G_n u_k \rightharpoonup G_n u$ in $L^{p'(\cdot)}(Q,\omega^*)$ and for all $\phi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$,
$\Big\langle G_n u_k, \phi\Big\rangle \to \Big\langle G_n u,\phi\Big\rangle$ as $k\to\infty$ which implies B_n is demi continuous.
In view of Theorem 4.1, there exists at least one weak solution $u_n \in L^{p^-}(0;T;W_0^{1,p(\cdot)}(\Omega,\omega))$ of the problem (P_n).(See [14].)

4.2. **A Priori Estimates.**

Proposition 4.3. *Let u_n a solution of the approximate problem (P_n). Then, there exists a constant C(which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \le kC \quad \forall k>0.$$

Proof.
Let $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \cap\ L^\infty(Q)$, with $\varphi > 0$. Choosing $v=\exp(G(u_n))\varphi$ as a test function in $(\mathcal{P}_n)$, where

$$G(s) = \int_0^s (\frac{g(r)}{\alpha})dr,$$

(the function g appears in (3.6)), we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla(\exp(G(u_n))\varphi)dxdt$$
$$+\int_Q H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\varphi dxdt = \int_Q f_n \exp(G(u_n))\varphi dxdt.$$

In view of (3.6), we obtain

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi dxdt$$
$$+\int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt \le \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt$$
$$+\int_Q f_n \exp(G(u_n))\varphi dxdt + \int_Q g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))\varphi dxdt.$$

By using (3.5), we obtain

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt$$
$$\le \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt \tag{4.16}$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))\cap L^\infty(Q), \quad \text{with } \varphi > 0.$
On the other hand, taking $v = \exp(-G(u_n))\varphi$ as a test function in $(\mathcal{P}_n)$, we deduce as in (4.16) that

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi dxdt$$
$$+\int_Q \gamma(x,t)\exp(-G(u_n))\varphi dxdt \ge \int_Q f_n\exp(-G(u_n))\varphi dxdt \tag{4.17}$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))\cap L^\infty(Q)$, with $\varphi > 0$.
Letting $\varphi = T_k(u_n)^+\chi_{(0,\tau)}$ for every $\tau\in[0,T]$, in (4.16), we have

$$\int_\Omega B^n_{k,G}(x,u_n(\tau))dx + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$
$$\le \int_{Q^\tau}\gamma(x,t)\exp(G(u_n))T_k(u_n)^+dxdt + \int_{Q^\tau} f_n\exp(G(u_n))T_k(u_n)^+dxdt$$
$$+\int_\Omega B^n_{k,G}(x,u_{0n})dx, \tag{4.18}$$

where,

$$B^n_{k,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}T_k(s)^+\exp(G(s))ds.$$

Due to the definition of $B^n_{k,G}$ and $|G(u_n)| \le \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})$, we have

$$0 \le \int_\Omega B^n_{k,G}(x,u_{0n})dx \le k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\|b(.,u_0\|_{L^1(\Omega)}. \tag{4.19}$$

Using (4.19), $B^n_{k,G}(x,u_n)\ge 0$, we obtain

$$\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$
$$\le k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big].$$

Thanks to (3.5), we have

$$\alpha\int_{Q^\tau}|\nabla T_k(u_n)^+|^{p(x)}\omega(x)\exp(G(u_n))dxdt \le k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}$$
$$+\|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]. \tag{4.20}$$

Let us observe that if we take: $\varphi = \rho(u_n) = \int_0^{u_n} g(s)\chi_{\{s>0\}}ds$ in (4.16) and use (3.5), we obtain

$$\int_\Omega \Big[B_g^n(x,u_n)\Big]_0^T dx + \alpha\int_Q |\nabla u_n|^{p(x)}\omega(x)g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt$$
$$\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],$$

where

$$B_g^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho(s)\exp(G(s))ds,$$

which implies, using $B_g^n(x,r) \geq 0$, we obtain

$$\alpha\int_{\{u_n>0\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt$$
$$\leq \|g\|_\infty \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]$$

$$then, \quad \int_{\{u_n>0\}} g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))dxdt \leq C_3.$$

Similarly, taking $\varphi = \int_{u_n}^0 g(s)\chi_{\{s<0\}}ds$ as a test function in (4.17), we conclude that

$$\int_{\{u_n<0\}} g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))dxdt \leq C_4.$$

Consequently,

$$\int_Q g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))dxdt \leq C_5. \tag{4.21}$$

Above, $C_1, ..., C_5$ are constants independent of n, we deduce that

$$\int_Q |\nabla T_k(u_n)^+|^{p(x)}\omega(x)dxdt \leq k\ C_6. \tag{4.22}$$

Similarly to (4.22), we take $\varphi = T_k(u_n)^-\chi(0,\tau)$ in (4.17) to deduce that

$$\int_Q |\nabla T_k(u_n)^-|^{p(x)}\omega(x)dxdt \leq k\ C_7. \tag{4.23}$$

Combining (4.22), (4.23) and Remark 2.3, we conclude that

$$\int_0^T \min\Big\{\|T_k(u_n)\|^{p^+}_{W_0^{1,p(\cdot)}(\Omega,\omega)}, \|T_k(u_n)\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega,\omega)}\Big\}dt \leq \rho(\nabla T_k(u_n)) \leq kC_8.$$
$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \leq\ k\ C_8. \tag{4.24}$$

Where C_6, C_7, C_8 are constants independent of n. Thus, $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ independently of n for any $k > 0$. Then, we deduce from (4.18), (4.19) and (4.24) that

$$\int_\Omega B_{k,G}^n(x,u_n(\tau))dx \leq\ kC. \tag{4.25}$$

4.3. **Almost everywhere convergence of the gradients.** Now, we turn to proving the almost everywhere convergence of u_n and $b_n(x,u_n)$. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that: $g_k(s) = \begin{cases} s & if\ |s| \le \frac{k}{2} \\ k & if\ |s| \ge k. \end{cases}$
Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\frac{\partial B_k^n(x,u_n)}{\partial t} - div(a(x,t,u_n,\nabla u_n)g_k'(u_n)) + a(x,t,u_n,\nabla u_n)g_k''(u_n)\nabla u_n$$
$$+H_n(x,t,u_n,\nabla u_n)g_k'(u_n) = f_n g_k'(u_n), \tag{4.26}$$

where

$$B_k^n(x,z) = \int_0^z \frac{\partial b_n(x,s))}{\partial s} g_k'(s)ds.$$

As a consequence of (4.24), we deduce that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$. Due to the properties of g_k and (3.2), we conclude that $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$, which implies that $g_k(u_n)$ is compact in $L^1(Q)$.
Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that u_n converges almost everywhere to some measurable function v in Q. Thus by using the same argument as in [7], [8], [9], we can show the following lemma.

Lemma 4.4. *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$ then,*

$$\begin{array}{ccccc} u_n & \to & u & a.e. & in\ Q. \\ b_n(x,u_n) & \to & b(x,u) & a.e. & in\ Q. \end{array}$$

We can deduce from (4.24) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$$

which implies, by using (3.3), that for all $k>0$ there exists $\varphi_k \in (L^{p'(\cdot)}(Q,\omega^*))^N$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k \quad in \quad \left(L^{p'(\cdot)}(Q,\omega^*)\right)^N.$$

Remark 4.5. *$b(.,u)$ it belongs to $L^\infty(0,T;L^1(\Omega))$.*

Proof.
Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$ passing to $\liminf$ in (4.25) as $n\to\infty$, we obtain

$$\frac{1}{k}\int_\Omega B_{k,G}(x,u(\tau))dx \le C, \text{ for a.e. } \quad \tau \text{ in } \quad [0,\tau].$$

Due to the definition of $B_{k,G}(x,s)$ and the fact that $\frac{1}{k}B_{k,G}(x,s)$ converge pointwise to $\int_0^u sgn(s)\frac{\partial b(x,s)}{\partial s}\exp(G(s))ds \ge |b(x,u)|$ as $k\to\infty$, it follows that $b(.,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

Lemma 4.6. *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{4.27}$$

Proof.
Set $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (4.16), this function is admissible since $\varphi \in$

$L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and $\varphi \geq 0$. Then, we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\alpha_m(u_n)dxdt$$
$$+\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt$$
$$\leq \int_Q |\gamma(x,t)|\exp(G(u_n))\alpha_m(u_n)dxdt + \int_Q |f_n|\exp(G(u_n))\alpha_m(u_n)dxdt.$$

This gives, by setting

$$B^m_{n,G}(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(G(s))\alpha_m(s)ds,$$

and by Young's Inequality,

$$\int_\Omega B^m_{n,G}(x,u_n)(T)dx + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt$$
$$\leq \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}}[|\gamma|+|f_n|+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]dxdt.$$

Since $B^m_{n,G}(x,u_n)(T)>0$ and use (3.5), we obtain

$$\alpha\int_{\{m\leq u_n\leq m+1\}} |\nabla u_n)|^{p(x)}\exp(G(u_n))\nabla u_n dxdt$$
$$\leq \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}}|\gamma|+|f_n|dxdt+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]. \tag{4.28}$$

Taking $\varphi=\rho_m(u_n)=\int_0^T g(s)\chi_{\{s>m\}}ds$ as a test function in (4.16), we obtain

$$\Big[\int_\Omega B^m_{m,n}(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))g(u_n)\nabla u_n\chi_{\{u_n>m\}}dxdt$$
$$\leq \Big(\int_m^\infty g(s)\chi_{\{u_n>m\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}\Big],$$

where $B^m_{m,n}(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))ds$ which implies,
since $B^m_{m,n}(x,r)\geq 0$, by (3.5) and Young's Inequality

$$\alpha\int_{\{u_n>m\}}|\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt\leq$$
$$\Big(\int_m^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]. \tag{4.29}$$

Using (4.29) and the strong convergence of f_n in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$, $\gamma\in L^1(\Omega)$, $g\in L^1(\mathbb{R})$, by Lebesgue's theorem, passing to limit in (4.28), we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{4.30}$$

On the other hand, taking $\varphi=T_1(u_n-T_m(u_n))^-$ as a test function in (4.17) and reasoning as in the proof (4.30), we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{4.31}$$

By using (4.30) and (4.31), we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq |u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{4.32}$$

To this end, we prove the strong convergence of truncation of $T_k(u_n)$ that we will use the following function of one real variable s, which is define as where $m>k$,

$$h_m(s)=\begin{cases}1 & if \quad |s|\le m\\ 0 & if \quad |s|>m+1\\ m+1+|s| & if \quad m\le |s|\le m+1.\end{cases}$$

Let $\psi_i\in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.
Set $w^i_\mu=(T_k(u))_\mu+e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t}=\mu(T_k(u)-w^i_\mu),\quad w^i_\mu(0)=T_k(\psi_i),\quad |w^i_\mu|\le k, \tag{4.33}$$

$$w^i_\mu\to T_k(u)\quad \text{in } L^{p^-}(0,T;W^{1,p(\cdot)}_0(\Omega,\omega))\quad \text{as } \mu\to\infty. \tag{4.34}$$

The very definition of the sequence w^i_μ makes it possible to establish the following lemma.

Lemma 4.7. *(See[9, 2].) For $k\ge 0$, we have*

$$\int_{\{T_k(u_n)-w^i_\mu\ge 0\}}\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-w^i_\mu)h_m(u_n)dxdt\ge\varepsilon(n,m,\mu,i).$$

Proposition 4.8. *The subsequence of u_n solution of problem $(\mathcal{P}_n)$ satisfies for any $k\ge 0$ following assertion:*

$$\lim_{n\to\infty}\int_Q\Big[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dxdt=0.$$

Proof.
For $m>k$, let $\varphi=(T_k(u_n)-w^i_\mu)^+h_m(u_n)\in L^{p^-}(0,T;W^{1,p(\cdot)}_0(\Omega,\omega))\cap L^\infty(Q)$ and $\varphi\ge 0$. If we take this function in (4.16), we obtain

$$\begin{aligned}&\int_{\{T_k(u_n)-w^i_\mu\ge 0\}}\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-w^i_\mu)h_m(u_n)dxdt\\&+\int_{\{T_k(u_n)-w^i_\mu\ge 0\}}a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n)-w^i_\mu)h_m(u_n)dxdt\\&-\int_{\{m\le u_n\le m+1\}}\exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-w^i_\mu)^+dxdt\\&\le\int_Q(f_n+\gamma)\exp(G(u_n))(T_k(u_n)-w^i_\mu)^+h_m(u_n)dxdt\end{aligned}\tag{4.35}$$

Observe that,

$$\begin{aligned}&\Big|\int_{\{m\le u_n\le m+1\}}\exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-w^i_\mu)^+dxdt\Big|\\&\quad\le 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{\{m\le u_n\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_ndxdt.\end{aligned}$$

Tanks to (4.27) the third and fourth integrals on the right hand side tend to zero as n and m tend to infinity and by Lebesgue's theorem, we deduce that the right hand side converges to zero as n, m and μ tend to infinity . Since $\Big(T_k(u_n)-w^i_\mu\Big)^+h_m(u_n)\rightharpoonup\Big(T_k(u)-w^i_\mu\Big)^+h_m(u)$ in $L^\infty(Q)$ as $n\to\infty$ and strongly in $L^{p^-}(0,T;W^{1,p(\cdot)}_0(\Omega,\omega))$ and $(T_k(u_n)-w^i_\mu)^+h_m(u_n)\rightharpoonup 0$ in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W^{1,p(\cdot)}_0(\Omega,\omega))$ as $\mu\to\infty$, it follows that the first and second integrals on the right-hand side of (4.35) converge to zeros as $n,\ m,\ \mu\to\infty$, using [3] Lemma 4.7 and Lemma 2.10, the proof of Proposition 4.8 is complete. Thanks to the Lemma 2.10, we have

$$T_k(u_n)\to T_k(u)\text{ strongly in } L^{p^-}(0,T;W^{1,p(\cdot)}_0(\Omega,\omega)),\quad\forall k \tag{4.36}$$

and $\nabla u_n \to \nabla u$ a.e. in Q, which implies that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \ \ in \ (L^{p'(\cdot)}(Q,\omega^*))^N. \qquad (4.37)$$

4.4. **Equi-Integrability of the non Linearity Sequence.**

Proposition 4.9. *Let u_n be a solution of problem $(\mathcal{P}_n)$. Then $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ strongly in $L^1(Q)$.*

Proof. By using Vitali's theorem. Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ a.e. in Q, considering now, $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as a test function in (4.16), we obtain

$$\Big[\int_\Omega B_h^n(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$\le \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))ds$,
which implies, in view of $B_h^n(x,r) \ge 0$ and (3.5)

$$\alpha\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt$$
$$\le \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (4.17), we conclude that: $\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0$.
Consequently, $\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0$.
Which implies, for h large enough and for a subset E of Q,

$$\lim_{measE\to 0}\int_E |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt \le \|g\|_\infty \lim_{measE\to 0}\int_E |\nabla T_h u_n|^{p(x)}\omega(x)dxdt$$
$$+\int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt,$$

so $g(u_n)|\nabla u_n|^{p(x)}\omega(x)$ is equi-integrable. Thus we have shown that

$$g(u_n)|\nabla u_n|^{p(x)}(x)\omega(x) \to g(u)|\nabla u|^{p(x)}(x)\omega(x) \text{ stongly in } L^1(Q).$$

Consequently, by using (3.6), we conclude that

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \ \text{ strongly } \ \text{in } L^1(Q). \qquad (4.38)$$

4.5. **Concluding the proof of Theorem 3.3.**

a) Proof that u satisfies (3.8). For any fixed $m \geq 0$, we have

$$\begin{aligned}\int_{\{m\leq|u_n|\leq m+1\}} & a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &= \int_Q a(x,t,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big]dxdt \\ &= \int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n) \\ &\quad - \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt.\end{aligned}$$

According to (4.36) and (4.37), one can pass to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} & a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u) \\ &\quad - \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt \\ &= \int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt. \qquad (4.39)\end{aligned}$$

Taking the limit as $m \to \infty$ in (4.39) and using the estimate (4.27), shows that u satisfies (3.8).

b) Proof that u satisfies (3.9)

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support. Let $M > 0$ such that supp$(S')\subset [-M, M]$. Pointwise multiplication of the approximate problem $(\mathcal{P}_n)$ by $S'(u_n)$, leads to

$$\begin{aligned}\frac{\partial B_S^n(x,u_n)}{\partial t} - div\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big] + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \\ + H_n(x,t,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) \text{ in } D'(Q). \qquad (4.40)\end{aligned}$$

In what follows, we pass to the limit in (4.40) as n tends to ∞.

• Limit of $\frac{\partial B_S^n(x,u_n)}{\partial t}$.

Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $B_S^n(x,u_n)$ converge to $B_S(x,u)$ a.e. in Q and L^∞ weakly

$$Then, \quad \frac{\partial B_S^n(x,u_n)}{\partial t} \to \frac{\partial B_S(x,u)}{\partial t} \quad \text{in } D'(Q), \text{ as } n \to \infty.$$

• Limit of $-div\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big]$.

Since supp$(S')\subset [-M, M]$, we have, for $n \geq M$

$$S'(u_n)a(x,t,u_n,\nabla u_n) = S'(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n)) \; a.e. \text{ in } Q.$$

The pointwise convergence of u_n to u and (4.37) and the boundedness of S' yied, as $n \to \infty$,

$$S'(u_n)a(x,t,u_n,\nabla u_n) \rightharpoonup S'(u)a(x,t,T_M(u),\nabla T_M(u)) \text{ in } (L^{p'(\cdot)}(Q,\omega^*))^N \qquad (4.41)$$

as $n \to \infty$,

$S'(u)a(x,t,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in equation

(3.9).
• Limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$.
Consider the "energy" term
$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Q.
The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (4.37) as $n \to \infty$ and the boundedness of S'' yield

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) \text{ in } L^1(Q). \quad (4.42)$$

Recall that $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M((u)) = S''(u)a(x,t,u,\nabla u)\nabla u$ a.e. in Q.
• Limit of $S'(u_n)H_n(x,t,u_n,\nabla u_n)$. From supp$(S')\subset [-M,M]$ and (4.38), we have

$$S'(u_n)H_n(x,t,u_n,\nabla u_n) \to S'(u)H(x,t,u,\nabla u) \textit{ strongly in } L^1(Q) \textit{ as } n \to \infty. \quad (4.43)$$

• Limit of $S'(u_n)f_n$. Since $u_n \to u$ a.e. in Q,
we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$, as $n \to \infty$.
As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (4.40) and to conclude that u satisfies (3.9).

c) Proof that u satisfies (3.10)

S is bounded and $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly by (4.40), we have $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$.
As a consequence, an Aubin type Lemma (see, e.g, [18] implies that $B_S^n(x,u_n)$ lies in a compact set in $C^0([0,T],L^1(\Omega))$.
It follows that on the hand, $B_S^n(x,u_n)\mid_{t=0}= B_S^n(x,u_0^n)$ converge to $B_S(x,u)\mid_{t=0}$ strongly in $L^1(\Omega)$ implies that: $B_S(x,u)\mid_{t=0}= B_S(x,u_0)$ in Ω.
As a conclusion, the proof of Theorem 3.3 is complete.

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