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ON FINITE SUM OF G-FRAMES AND NEAR EXACT G-FRAMES

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Abstract. Finite sum of g-frames in Hilbert spaces has been defined and studied. A necessary and sufficient condition for the finite sum of g-frames to be a g-frame for a Hilbert space has been given. Also, a sufficient condition for the stability of finite sum of g-frames and a sufficient condition for the stability of finite sum of g-Bessel sequences to be a g-frame for a Hilbert space have been given. Further, near exact g-frames in Hilbert space have been defined and studied. Finally, a sufficient condition for a g-frame to be a near exact g-frame has been given.

1. Introduction

In 1952, Duffin and Schaeffer [5] introduced frames for Hilbert spaces while addressing some difficult problems arising from the theory of non-harmonic Fourier series. Today, frames have been widely used in signal processing, data compression, sampling theory and many other fields.

Recently, Sun [13] introduced a g-frame and a g-Riesz bases in a Hilbert space and obtained some results for g-frames and g-Riesz bases. He also observed that frame of subspaces (fusion frames) introduced by Casazza and Kutyniok [2] is a particular case of g-frame in a Hilbert space. Also, a system of bounded quasi-projectors introduced by Fornasier [7] is a particular case of g-frame in a Hilbert space.

In the present paper, we study finite sum of g-frames in Hilbert spaces and observe that a finite sum of g-frames may not be a g-frame for a Hilbert space. A necessary and sufficient condition for the finite sum of g-frames to be a g-frame for a Hilbert space has been given. Also, a sufficient condition for the stability of finite sum of g-frames and a sufficient condition for the stability of finite sum of g-Bessel sequences to be a g-frame for a Hilbert space have been given. Further, near exact g-frame in Hilbert space has been defined and studied . Finally, a sufficient condition for a g-frame to be a near exact g-frame for a Hilbert space has been given.

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2. Preliminaries

Throughout this paper, H is a Hilbert space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{H_n\}_{n\in\mathbb{N}}$ is a sequence of Hilbert spaces over $\mathbb{K}$. $B(H, H_n)$ is the collection of all bounded linear operators from H into H_n.

Definition 1 A sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ is called a *frame* for H if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \text{for all } \ x \in H\,.$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the frame $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2 [13] A sequence $\{\Lambda_n \in B(H, H_n)\}_{n\in\mathbb{N}}$ is called a *generalized frame* or simply a *g-frame* for H with respect to $\{H_n\}_{n\in\mathbb{N}}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{n\in\mathbb{N}} \|\Lambda_n(x)\|^2 \leq B\|x\|^2, \quad \text{for all } \ x \in H\,. \tag{1}$$

The positive constants A and B, respectively, are called the lower and upper frame bounds of the g-frame $\{\Lambda_n\}_{n\in\mathbb{N}}$. The g-frame $\{\Lambda_n\}_{n\in\mathbb{N}}$ is called *a tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. The sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is called a g-Bessel *sequence* for H with respect to $\{H_n\}_{n\in\mathbb{N}}$ with bound B if $\{\Lambda_n\}_{n\in\mathbb{N}}$ satisfies the right hand side of the inequality (1).

Definition 3 [10] A sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is called a g-frame sequence for H, if it is a g-frame for $\overline{span}\{\Lambda_n^*(H_n)\}_{n\in\mathbb{N}}$.

Notation. For each sequence $\{H_n\}_{n\in\mathbb{N}}$, define $\left(\sum\limits_{n\in\mathbb{N}} \oplus H_n\right)_{\ell_2}$ by

$$\left(\sum_{n\in\mathbb{N}} \oplus H_n\right)_{\ell_2} = \left\{\{a_n\}_{n\in\mathbb{N}} : a_n \in H_n, n \in \mathbb{N} \ \text{ and } \sum_{n\in\mathbb{N}} \|a_n\|^2 < \infty\right\}$$

with the inner product defined by $\langle\{a_n\}, \{b_n\}\rangle = \sum\limits_{n\in\mathbb{N}} \langle a_n, b_n\rangle$.

It is clear that $\left(\sum\limits_{n\in\mathbb{N}} \oplus H_n\right)_{\ell_2}$ is a Hilbert space with pointwise operations.

The following results which are referred in this paper are listed in the form of lemmas.

Lemma 1[10] $\{\Lambda_n \in B(H, H_n)\}_{n\in\mathbb{N}}$ is a g-Bessel sequences for H with respect to $\{H_n\}_{n\in\mathbb{N}}$ if and only if

$$T : \{x_n\}_{n\in\mathbb{N}} \to \sum_{n\in\mathbb{N}} \Lambda_n^*(x_n)$$

is well-defined and bounded mapping from $\left(\sum\limits_{n\in\mathbb{N}} \oplus H_n\right)_{\ell_2}$ to H.

Lemma 2[10] If $\{\Lambda_n \in B(H, H_n)\}_{n\in\mathbb{N}}$ is a g-frame for H, then $\overline{span}\{\Lambda_n^*(H_n)\}_{n\in\mathbb{N}} = H$.

3. Finite Sum of G-Frames

Let $\{\Lambda_{i,n} \in B(H, H_n)\}_{n\in\mathbb{N}}$, $i = 1, 2, \cdots, k$ be g-frames for H. Consider the sequence $\{\sum\limits_{i=1}^{k} \Lambda_{i,n} \in B(H, H_n)\}_{n\in\mathbb{N}}$. Then $\{\sum\limits_{i=1}^{k} \Lambda_{i,n}\}_{n\in\mathbb{N}}$ may not be a g-frame for H. In this direction, we give the following examples:

Example 1 Let $\{\Lambda_{i,n} \in B(H, H_n)\}_{n\in\mathbb{N}},\ i = 1, 2, \cdots, k$ be g-frames for H. If for some $1 \le p \le k$,

$$\Lambda_{i,n}(x) = \Lambda_{p,n}(x), \text{ for all } x \in H,\ i = 1, 2, \cdots, k \text{ and } n \in \mathbb{N}.$$

Then $\{\sum_{i=1}^{k} \Lambda_{i,n}(x)\} = \{k\Lambda_{p,n}(x)\},\ n \in \mathbb{N}$. Therefore $\{\sum_{i=1}^{k} \Lambda_{i,n}(x)\}_{n\in\mathbb{N}}$ is a g-frame for H.

Example 2 Let $\{\Lambda_{1,n} \in B(H, H_n)\}_{n\in\mathbb{N}}$ and $\{\Lambda_{2,n} \in B(H, H_n)\}_{n\in\mathbb{N}}$ be two g-frames for H such that

$$\Lambda_{1,n}(x) = -\Lambda_{2,n}(x), \text{ for all } x \in H \text{ and } n \in \mathbb{N}.$$

Then $\{\sum_{i=1}^{2} \Lambda_{i,n}(x)\} = \{0\}$, which is not a g-frame for H.

In view of Examples 1 and 2, we give a necessary and sufficient condition for the finite sum of g-frames to be a g-frame.

Theorem 1 Let $\{\Lambda_{i,n} \in B(H, H_n)\}_{n\in\mathbb{N}},\ i = 1, 2, \cdots, k$ be g-frames for H. Let $\{\alpha_i\},\ i = 1, 2, \cdots, k$ be any scalars. Then $\{\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}\}$ is a g-frame for H if and only if there exists $\beta > 0$ and some $p \in \{1, 2, \cdots, k\}$ such that

$$\beta \sum_{n\in\mathbb{N}} \|\Lambda_{p,n}(x)\|^2 \le \sum_{n\in\mathbb{N}} \|\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}(x)\|^2, \quad x \in H.$$

Proof. For each $1 \le p \le k$, let A_p and B_p be the bounds of the g-frame $\{\Lambda_{p,n}\}$. Let $\beta > 0$ be a constant satisfying

$$\beta \sum_{n\in\mathbb{N}} \|\Lambda_{p,n}(x)\|^2 \le \sum_{n\in\mathbb{N}} \|\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}(x)\|^2, \quad x \in H.$$

Then

$$\beta A_p\|x\|^2 \le \sum_{n\in\mathbb{N}} \|\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}(x)\|^2, \quad x \in H.$$

For any $x \in H$, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} \|\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}(x)\|^2 &\le \sum_{n\in\mathbb{N}} k\bigg(\sum_{i=1}^{k} \|\alpha_i\Lambda_{i,n}(x)\|^2\bigg)\\ &= k\sum_{i=1}^{k}\bigg(|\alpha_i|^2 \sum_{n\in\mathbb{N}} \|\Lambda_{i,n}(x)\|^2\bigg)\\ &\le k(\max|\alpha_i|^2)\bigg(\sum_{i=1}^{k} B_i\bigg)\|x\|^2.\end{aligned}$$

Hence $\{\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}\}_{n\in\mathbb{N}}$ is a g-frame for H with bounds βA_p and $k(\max|\alpha_i|^2)(\sum_{i=1}^{k} B_i)$.

Conversely, let $\{\sum_{i=1}^{k} \alpha_i\Lambda_{i,n}\}_{n\in\mathbb{N}}$ be a g-frame for H with bounds $A,\ B$ and let for any $p \in \{1, 2, \cdots, k\}$, $\{\Lambda_{p,n}\}_{n\in\mathbb{N}}$ be a g-frame for H with bounds A_p and B_p. Then, for any $x \in H$, $p \in \{1, 2, \cdots, k\}$, we have

$$A_p\|x\|^2 \le \sum_{n\in\mathbb{N}} \|\Lambda_{p,n}(x)\|^2 \le B_p\|x\|^2.$$

This gives

$$\frac{1}{B_p}\sum_{n\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2 \le \|x\|^2, \quad x\in H.$$

Also, we have

$$A\|x\|^2 \le \sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\alpha_i\Lambda_{i,n}(x)\|^2 \le B\|x\|^2, \quad x\in H.$$

So,

$$\|x\|^2 \le \frac{1}{A}\sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\alpha_i\Lambda_{i,n}(x)\|^2, \quad x\in H.$$

Hence

$$\frac{A}{B_p}\sum_{n\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2 \le \sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\alpha_i\Lambda_{i,n}(x)\|^2, \quad x\in H.$$

Write $\frac{A}{B_p}=\beta$. Then

$$\beta\sum_{n\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2 \le \sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\alpha_i\Lambda_{i,n}(x)\|^2, \quad x\in H.$$

Next, we give a sufficient condition for the stability of finite sum of g-frames.

Theorem 2 For $i=1,2,\cdots,k$, let $\{\Lambda_{i,n}\in B(H,H_n)\}_{n\in\mathbb{N}}$ be g-frames for H, $\{\Theta_{i,n}\in B(H,H_n)\}_{n\in\mathbb{N}}$ be any sequence. Let $L:\left(\sum_{n\in\mathbb{N}}\oplus H_n\right)_{\ell_2}\to\left(\sum_{n\in\mathbb{N}}\oplus H_n\right)_{\ell_2}$ be a bounded linear operator such that $L\left(\{\sum_{i=1}^{k}\Theta_{i,n}(x)\}_{n\in\mathbb{N}}\right)=\{\Lambda_{p,n}(x)\}_{n\in\mathbb{N}}$, for some $p\in\{1,2,\cdots,k\}$. If there exists a non-negative constant λ such that

$$\sum_{n\in\mathbb{N}}\|(\Lambda_{i,n}-\Theta_{i,n})(x)\|^2 \le \lambda\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2, \quad x\in H \text{ and } i=1,2,\cdots,k.$$

Then $\{\sum_{i=1}^{k}\Theta_{i,n}\}_{n\in\mathbb{N}}$ is a g-frame for H.

Proof. For any $x\in H$, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\Theta_{i,n}(x)\|^2 &= \sum_{n\in\mathbb{N}}\|\left(\sum_{i=1}^{k}(\Theta_{i,n}-\Lambda_{i,n})(x)\right)+\sum_{i=1}^{k}\Lambda_{i,n}(x)\|^2\\ &\le 2k\sum_{i=1}^{k}\left(\sum_{n\in\mathbb{N}}\|(\Theta_{i,n}-\Lambda_{i,n})(x)\|^2+\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2\right)\\ &\le 2k\sum_{i=1}^{k}\left(\lambda\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2+\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2\right)\\ &\le 2k(1+\lambda)\left(\sum_{i=1}^{k}B_i\right)\|x\|^2.\end{aligned}$$

where, B_i is an upper bound for $\{\Lambda_{i,n}\}_{n\in\mathbb{N}}$, $i=1,2,....,k$.

Also, for each $x \in H$, we have

$$\left\| L\left(\left\{\sum_{i=1}^{k}\Theta_{i,n}(x)\right\}\right)\right\|^2 = \|\{\Lambda_{p,n}(x)\}\|^2$$
$$= \sum_{i\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2$$

Therefore, we get

$$A_p\|x\|^2 \leq \sum_{n\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2 \leq \|L\|^2\sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\Theta_{i,n}(x)\|^2, \quad x\in H,$$

where A_p is a lower bound of the g-frame $\{\Lambda_{p,n}\}_{n\in\mathbb{N}}$.
This gives

$$\frac{A_p}{\|L\|^2} \leq \sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\Theta_{i,n}(x)\|^2, \quad x\in H.$$

Hence $\{\sum_{i=1}^{k}\Theta_{i,n}\}_{n\in\mathbb{N}}$ is a g-frame for H.

Now, we give another sufficient condition for the stability of finite sum of g-Bessel sequences to be a g-frame for H.

Theorem 3 For $i=1,2,\cdots,k$, let $\{\Lambda_{i,n}\in B(H,H_n)\}_{n\in\mathbb{N}}$ be g-Bessel sequences in H with coefficient mapping $T_i : H \to \left(\sum_{n\in\mathbb{N}}\oplus H_n\right)_{\ell_2}$ given by $T_i(x)=\{\Lambda_{i,n}(x)\}_{n\in\mathbb{N}}$, $x\in\ H,\ i=1,2,\cdots,k$ and let $\{\Theta_{i,n}\in B(H,H_n)\}_{n\in\mathbb{N}}$ be any sequence such that $\sum_{n\in\mathbb{N}}\|(\Lambda_{i,n}-\Theta_{i,n}(x))\|^2 \leq \lambda\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2,\quad x\in H,\ \lambda\geq 0$ and $\ i=1,2,\cdots,k$.
If for some $p\in\{1,2,\cdots,k\}$, there exists $A_p>0$ such

$$\sum_{n\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2 \geq A_p\|x\|^2, \quad x\in H$$

and

$$A_p > \left(2(k-1)\sum_{\substack{i=1\\ i\neq p}}^{k}\|T_i\|^2 + 4\lambda k\sum_{i=1}^{k}\|T_i\|^2\right).$$

Then $\{\sum_{i=1}^{k}\Theta_{i,n}\}_{n\in\mathbb{N}}$ is a g-frame for H.

Proof. For each $x\in H$, we have

$$\sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\Theta_{i,n}(x)\|^2 = \sum_{n\in\mathbb{N}}\|\left(\sum_{i=1}^{k}(\Theta_{i,n}-\Lambda_{i,n})(x)\right)+\sum_{i=1}^{k}\Lambda_{i,n}(x)\|^2$$
$$\geq \frac{1}{2}\left(\frac{1}{2}\left(\sum_{n\in\mathbb{N}}\|\Lambda_{p,n}(x)\|^2 - 2\sum_{n\in\mathbb{N}}\|\sum_{\substack{i=1\\ i\neq p}}^{k}\Lambda_{i,n}(x)\|^2\right)\right.$$
$$\left. - 2k\sum_{i=1}^{k}\sum_{n\in\mathbb{N}}\|(\Theta_{i,n}-\Lambda_{i,n})(x)\|^2\right)$$

$$
\begin{aligned}
&\geq \frac{1}{4}\bigg(A_p\|x\|^2 - 2(k-1)\sum_{\substack{i=1\\ i\neq p}}^{k}\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2\bigg)\\
&\qquad\qquad\qquad\qquad - k\lambda\sum_{i=1}^{k}\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2\\
&= \frac{1}{4}\bigg(A_p\|x\|^2 - 2(k-1)\sum_{\substack{i=1\\ i\neq p}}^{k}\|T_i(x)\|^2\bigg) - k\lambda\sum_{i=1}^{k}\|T_i(x)\|^2\\
&= \frac{1}{4}\bigg(A_p - 2(k-1)\sum_{\substack{i=1\\ i\neq p}}^{k}\|T_i\|^2 - 4\lambda k\sum_{i=1}^{k}\|T_i\|^2\bigg)\|x\|^2.
\end{aligned}
$$

Also, for each $x \in H$, we have

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\Theta_{i,n}(x)\|^2 &= \sum_{n\in\mathbb{N}}\|\bigg(\sum_{i=1}^{k}(\Theta_{i,n}-\Lambda_{i,n})(x)\bigg) + \sum_{i=1}^{k}\Lambda_{i,n}(x)\|^2\\
&\leq 2\bigg(\sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}(\Theta_{i,n}-\Lambda_{i,n})(x)\|^2 + \sum_{n\in\mathbb{N}}\|\sum_{i=1}^{k}\Lambda_{i,n}(x)\|^2\bigg)\\
&\leq 2k\lambda\sum_{i=1}^{k}\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2 + 2k\sum_{i=1}^{k}\sum_{n\in\mathbb{N}}\|\Lambda_{i,n}(x)\|^2\\
&\leq 2k(1+\lambda)\bigg(\sum_{i=1}^{k}B_i\bigg)\|x\|^2
\end{aligned}
$$

where B_i is a g-Bessel bound for $\{\Lambda_{i,n}\}_{n\in\mathbb{N}}$. Hence $\{\sum_{i=1}^{k}\Theta_{i,n}\}_{n\in\mathbb{N}}$ is a g-frame for H.

4. Near Exact G-Frames

We begin this section with the following definition of near exact g-frame:

Definition 4 A g-frame $\{\Lambda_i \in B(H, H_i)\}_{i\in I}$ for H is said to be near-exact if it can be made exact by removing finitely many members from it.

To show the existence of near exact g-frame, we give the following example.

Example 3 Let $\{e_n\}$ be an orthonormal basis for H. Define a sequence $\{x_n\} \subset H$ by

$$
\begin{cases} x_i = e_i, & i = 1, 2, \cdots, n\\ x_i = e_{i-n}, & i = n+1, n+2, \cdots \end{cases}
$$

For each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ by

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H.$$

Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame for H with respect to $\mathbb{C}$. Indeed, we have

$$\|x\|^2 \leq \sum_{i\in\mathbb{N}}\|\Lambda_i(x)\|^2 \leq 2\|x\|^2, \quad x \in H.$$

Further, by removing first n members from $\{\Lambda_i\}_{i\in\mathbb{N}}$, we obtain $\{\Lambda_i\}_{i=n+1}^{\infty}$ which is an exact g-frame for H.

Example 4 Let $\{e_n\}$ be an orthonormal basis for H. Define a sequence $\{x_n\} \subset H$ by

$$x_{2n-1} = x_{2n} = e_n, \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, define $\Lambda_n : H \to \mathbb{C}$ by

$$\Lambda_n(x) = \langle x, x_n \rangle, \quad x \in H.$$

Then $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a g-frame for H with respect to $\mathbb{C}$. Indeed, we have

$$\sum_{n\in\mathbb{N}} \|\Lambda_n(x)\|^2 = 2\|x\|^2, \quad x \in H.$$

But, after removing finitely many elements from $\{\Lambda_n\}_{n\in\mathbb{N}}$, it is an not exact g-frame for H.

Next, we give a sufficient condition for a g-frame to be a near exact g-frame.

Theorem 4 Let H be a separable Hilbert space and $\{\Lambda_i \in B(H, H_i)\}_{i\in I}$ be a g-frame for H. Then $\{\Lambda_i\}_{i\in I}$ is near exact if every infinite sequences $\{\sigma(k)\}_{k=1}^{\infty}$ of positive integers such that

$$\overline{span}\{\Lambda_i^*(H_i) : i \in I\}_{i\neq\sigma(1),\sigma(2),\cdots} \neq H \tag{2}$$

where Λ_i^* is the adjoint operator of Λ_i.

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ is not near exact. Then $\{\Lambda_i\}_{i\in I}$ is not an exact g-frame. Therefore, there exist a positive integer $\sigma(1)$ such that $\{\Lambda_i\}_{\substack{i\neq\sigma(1)\\ i\in I}}$ is a g-frame for H with respect to $\{H_i\}_{\substack{i\neq\sigma(1)\\ i\in I}}$. Then, by Lemma 2,

$$\Lambda_{\sigma(1)}^*(x_{\sigma(1)}) \in \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i\neq\sigma(1)\\ i\in I}} = H, \quad x_{\sigma(1)} \in H_{\sigma(1)}. \tag{3}$$

Since H is separable, there exists a positive integer $n_1 \geq \sigma(1)$ such that

$$\text{dist}\left(\Lambda_{\sigma(1)}^*(x_{\sigma(1)}), \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=1\\ i\neq\sigma(1)}}^{n_1}\right) < \frac{1}{2}, \quad x_{\sigma(1)} \in H_{\sigma(1)}.$$

By assumption, $\{\Lambda_i\}_{\substack{i=n_1+1\\ i\neq\sigma(1)}}^{\infty}$ is not exact. Therefore, there exist a positive integer $\sigma(2) \geq n_1 + 1$ such that

$$\begin{aligned}\Lambda_{\sigma(2)}^*(x_{\sigma(2)}) &\in \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=n_1+1\\ i\neq\sigma(1),\ \sigma(2)}}^{\infty}\\ &= \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=n_1+1\\ i\neq\sigma(1)}}^{\infty}, x_{\sigma(2)} \in H_{\sigma(2)}\end{aligned} \tag{4}$$

and a positive integer $n_2 \geq \sigma(2)$ such that

$$\text{dist}\left(\Lambda_{\sigma(2)}^*(x_{\sigma(2)}), \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=1\\ i\neq\sigma(1),\ \sigma(2)}}^{n_2}\right) < \frac{1}{4}, \quad x_{\sigma(2)} \in H_{\sigma(2)}.$$

Further, since

$$\overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=1\\ i\neq\sigma(1)}}^{n_1} + \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=n_1+1\\ i\neq\sigma(1),\ \sigma(2)}}^{\infty} \subseteq \overline{span}\{\Lambda_i^*(H_i)\}_{i\neq\sigma(1),\ \sigma(2)}$$

$$\overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=1\\ i\neq\sigma(1)}}^{n_1} + \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=n_1+1\\ i\neq\sigma(1)}}^{\infty} \subseteq \overline{span}\{\Lambda_i^*(H_i)\}_{i\neq\sigma(1),\ \sigma(2)} \qquad \text{(using(4))}$$

$$\overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i\neq\sigma(1)\\ i\in I}} \subseteq \overline{span}\{\Lambda_i^*(H_i)\}_{i\neq\sigma(1),\ \sigma(2)}.$$

So, we have

$$\overline{span}\{\Lambda_i^*(H_i)\}_{i\neq\sigma(1),\ \sigma(2)} = H. \qquad \text{(using(3))}$$

Continuing in this way, we get a sequence of indices $\{\sigma(k)\}_{k=1}^{\infty}$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ with $n_{k-1} + 1 \leq \sigma(k) \leq n_k$ and $n_o = 0$ such that

$$\text{dist}\left(\Lambda_{\sigma(k)}^*(x_{\sigma(k)}), \overline{span}\{\Lambda_i^*(H_i)\}_{\substack{i=1\\ i\neq\sigma(1),\ \sigma(2),\cdots,\ \sigma(k)}}^{n_k}\right) < \frac{1}{2^k}, \quad x_{\sigma(k)} \in H_{\sigma(k)}$$

and

$$\overline{span}\{\Lambda_i^*(H_i) : i \in I\}_{i \neq \sigma(1),\sigma(2),\cdots,\sigma(k)} = H.$$

Thus, we get a sequence of indices $\{\sigma(k)\}_{k=1}^{\infty}$ such that

$$\overline{span}\{\Lambda_i^*(H_i) : i \in I\}_{i \neq \sigma(1),\sigma(2),\cdots} = H.$$

This is a contradiction.
Hence $\{\Lambda_i \in B(H, H_i)\}_{i \in I}$ is a near exact g-frame for H.
Remark 1 The condition in Theorem 4 is not necessary. (Example 3).

References

[1] M.S. Asgari and A. Khosravi: Frames and bases of subspaces in Hilbert spaces, J. Math. Anal. Appl., **308**, 541-553 (2005).

[2] P.G. Casazza and G. Kutyniok: Frames of subspaces,in: Wavelets, Frames and Operator theory, Contemp. Math., **345**, Providence, RI, 87–113 (2004)

[3] O. Christensen: An Introduction to Frames and Riesz Bases, Birkhauser (2002).

[4] I. Daubechies, A. Grossman and Y. Meyer: Painless non-orthogonal expansions, J. Math. Phys., **27**, 1271–1283 (1986).

[5] R.J. Duffin and A.C. Schaeffer: A class of non-harmonic Fourier series, Trans. Amer. Math. Soc., **72**, 341–366 (1952).

[6] P.K. Jain, S.K. Kaushik and Nisha Gupta: On near exact Banach frames in Banach spaces, Bull. Austral. Math. Soc., **18**, 335–342 (2008)

[7] M. Fornasier: Quasi-orthogonal decompositions of structured frames, J. Math. Anal. Appl., **289**, 180–199 (2004).

[8] S.K. Kaushik, Ghanshyam Singh and Virender: A note on frame sequences in Hilbert space, Int. J. Contemp. Math. Sciences, **3** (16), 791–799 (2008).

[9] S.K. Kaushik, Ghanshyam Singh and Virender: A note on block sequences in Hilbert spaces, Glasnik Matematicki, **43**(63), 387- 395, (2008).

[10] A. Najati, M.H. Faroughi and A. Rahimi: G-frames and stability of G-frames in Hilbert spaces, Methods of Functional Analysis and Topology, **14**(3), 271–286 (2008).

[11] Renu Chugh, S.K. Sharma and Shashank Goel: A note on g-frame sequences, Int. J. Pure and Appl. Mathematics, **70**(6), 797-805, (2011).

[12] Renu Chugh and Shashank Goel: On sum of g-frames in Hilbert spaces, Jordanian Journal of Mathematics and Statistics(JJMS), **5** (2), 115–124 (2012).

[13] W. Sun: G-frame and g-Riesz base, J. Math. Anal. Appl., **322**(1), 437–452 (2006).

[14] W. Sun: Stability of g-frames, J. Math. Anal. Appl., **326**(2), 858–868 (2007).

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