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THE RENORMALIZED SOLUTIONS FOR TRANSPORT EQUATIONS WITH PARTICULAR FORMS

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Abstract. We prove the existence and uniqueness of renormalized solutions for transport equations with particular forms. As a direct application, we derive the high-order differentiability of the almost everywhere flow solutions of ordinary differential equations to initial values. Meantime, as an another application, the transport equation with a partial viscosity term, i.e. Fokker-Planck equations is also treated, and we obtain some regularity results.

1. Introduction

We study in this article the existence and uniqueness of solutions to a class of transport equations with irregular coefficients, namely the equation of the form

$$\partial_t u(t,x_1,x_2,x_3)+(b_1(x_1),b_2(x_1,x_2),b_3(x_1,x_2,x_3))\cdot\nabla u(t,x_1,x_2,x_3)=0, \quad (1.1)$$

with coefficients b_1, b_2 and b_3 that only have Sobolev (typically $W^{1,1}_{loc}$) regularity. Our work is a follow-up of Le. Bris and Lions' work [7], but with a slight extension. In our present work, we obtain some new existence and uniqueness results for renormalized solutions under weaker presumptions on $b=(b_1,b_2,b_3)$ and it is the main part of Section 2.

Section 3 is devoted to state the first application which aims at raising the art of the theory of solutions of ordinary differential equations with coefficients in Sobolev spaces to the level of the classical Cauchy-Lipschitz theory for the regular coefficients. We discuss it briefly below now.

Consider the following ordinary differential equation

$$\begin{cases} \dot{X}(t)=b(t,X), \\ X(0)=x, \end{cases} \quad (1.2)$$

where $x\in\mathbb{R}^N,\ b:\mathbb{R}^+\times\mathbb{R}^N\to\mathbb{R}^N$ (For simplicity, we assume the vector field b is time independent and $N=1$ now). It is well known in the Cauchy-Lipschitz theory that, once the existence and uniqueness of a solution (1.2) are proved, then under the additional assumptions that the first kth derivatives of vector field b with respect to the spatial variable exists and is continuous, one may prove that

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the solution $X(t,x)$ to (1.2) is kth derivable with respect to its initial datum x and that its first kth derivative satisfies the following ODE system:

$$\begin{cases} \frac{\partial}{\partial t}\frac{\partial X}{\partial x} = \frac{\partial b}{\partial X}(X)\frac{\partial X}{\partial x}, \\ \frac{\partial}{\partial t}\frac{\partial^2 X}{\partial x^2}(t,x) = \frac{\partial^2 b}{\partial X^2}(\frac{\partial X}{\partial x})^2 + \frac{\partial b}{\partial X}\frac{\partial^2 X}{\partial x^2}, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\cdots \\ \frac{\partial}{\partial t}\frac{\partial^k X}{\partial x^k} = \frac{\partial^k b}{\partial x^k}. \end{cases} \tag{1.3}$$

On the other hand, the renormalized theory for the linear transport equation below

$$\partial_t u + b \cdot \nabla u = 0, \text{ in } (0,T) \times \mathbb{R}^N, \tag{1.4}$$

has been allowed us to achieve (see [12]) the existence and uniqueness result for ODEs of type (1.2) with only locally $W^{1,1}$ vector field b (with a bounded divergence, say). The observation allows us to do so is the fact that a generalized almost every flow $X(t,x)$ associated to (1.3) is indeed related in a unique way to a renormalized group solution $u_0(X(t,x))$ to a transport equation of the type (1.4), u_0 being the initial value supplied with (1.4) (see[11]). We mention that several authors (Di Perna, Lions, Le Bris) have been given more than a formal sense to ODE system (1.3), exactly like in the framework of the Cauchy-Lipschitz theory. Of course, all this is loosely stated, and will be made precise in Section 3. It is sufficient to say for the time being that the vector $(X, \frac{\partial X}{\partial x}, \cdots \frac{\partial^k X}{\partial x^k})$ is a solution of the system (1.3), which is indeed of the form

$$\begin{cases} \dot{Y}_1(t,x) = c_1(t, Y_1), \\ \dot{Y}_2(t,x) = c_2(t, Y_1(t,x), Y_2(t,x)), \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \dot{Y}_{k+1}(t,x) = c_{k+1}(t, Y_1(t,x),\ \dots\ , Y_{k+1}(t,x)), \end{cases} \tag{1.5}$$

with $c_1 = b,\ c_2(t,x_1,x_2) = \frac{\partial b}{\partial x_1}x_2,\ c_3(t,x_1,x_2,x_3) = \frac{\partial^2 b}{\partial x_1^2}x_2^2 + \frac{\partial b}{\partial x_1}x_3, \cdots$

We remark that for $k = 1$, Le Bris and Lions have studied the $1th$ order differentiability of the flow of ODE (1.2) by virtue of transport equation (1.4) with the following particular form

$$\partial_t u(t,x) + (b_1(x_1), b_2(x_1,x_2)) \cdot \nabla u(t,x) = 0 \tag{1.6}$$

and they also get the $2nd$ order differentiability of the flow of ODE (1.2) under the additional presumption $b \in W^{2,1}$. Now in order to recover the kth order differentiability of the flow of ODE (1.2) to the level of classical Cauchy-Lipschitz theory, we study a particular class of transport equations namely, the transport equation with the following form

$$\partial_t u(t,x) + (b_1(x_1), b_2(x_1,x_2),\ \dots\ , b_{k+1}(x_1,x_2,\ \dots\ , x_{k+1})) \cdot \nabla u(t,x) = 0.$$

In particular, firstly, we argue the above equation for $k = 2$ and achieve the existence and uniqueness of the renormalized solutions under the weaker assumptions, then based upon the equivalence between transport equations and ordinary differential equations, the $2nd$ order differentiability of the flow of ODE (1.2) is obtained. Besides, we fulfil the kth order differentiability of the flow of ODE (1.2) with vector field b in $W^{k,1}_{loc}(\mathbb{R}^N)$ by analogy. It is noted that for $k = 2$, our present result plays the same role as [7] but we only presume the vector field b with locally $W^{2,1}$ regularity. In this point view, our result is a slight extension comparing to [7]. For more topics about the transport equations and its applications, one can pay his attention to [1-6], [9] and the references cited up there.

As another application, we discuss the transport equations with partial viscosity terms, namely the equation

$$\begin{cases} \partial_t u + (b_1(x_1), b_2(x_1,x_2), b_3(x)) \cdot \nabla u - \frac{1}{2}\Delta_{x_3} u(t,x) = 0, \text{ in } (0,T)\times\mathbb{R}^N, \\ u(t=0,) = u_0, \text{ in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

in Section 4, and we get some new regularity results of solutions.

2. Transport equations

This section is devoted to the statement and proof of existence and uniqueness result for solutions of (1.4) with a particular form and before opening our discussion, let us make precise the mathematical setting and introduce some notations.

As announced in the introduction, we consider the linear transport equation (1.4) with initial value u_0, namely

$$\begin{cases} \partial_t u + b\cdot\nabla u = 0, \text{ in } (0,T)\times\mathbb{R}^N, \\ u(t=0,\ \cdot) = u_0, \text{ in } \mathbb{R}^N, \end{cases}$$

where $T>0$ is a given real number.

For simplicity, we firstly consider b has the form

$$b(x) = (b_1(x_1), b_2(x_1,x_2), b_3(x_1,x_2,x_3)),$$

where

$$b_1:\ \mathbb{R}^{N_1}\to\mathbb{R}^{N_1},\ b_2:\ \mathbb{R}^{N_1+N_2}\to\mathbb{R}^{N_2},\ b_3:\ \mathbb{R}^{N}\to\mathbb{R}^{N_3},\ \sum_{i=1}^{3} N_i = N.$$

Denote by

$$\nabla = (\nabla_{x_1}, \nabla_{x_2}, \nabla_{x_3}),\quad \mathrm{div}_x = \mathrm{div}_{x_1} + \mathrm{div}_{x_2} + \mathrm{div}_{x_3}.$$

We make the following assumptions on the vector field:

$$\begin{cases} (H_1): b_1 = b_1(x_1) \in W^{1,1}_{x_1,loc}(\mathbb{R}^{N_1});\ (H_2): \frac{b_1(x_1)}{1+|x_1|} \in L^1_{x_1}(\mathbb{R}^{N_1}) + L^\infty_{x_1}(\mathbb{R}^{N_1}); \\ (H_3): \mathrm{div}_{x_1} b_1(x_1) = 0;\ (H_4):\ b_2 = b_2(x_1,x_2) \in L^1_{x_1,loc}(\mathbb{R}^{N_1}; W^{1,1}_{x_2,loc}(\mathbb{R}^{N_2})); \\ (H_5):\ \frac{b_2(x_1,x_2)}{1+|x_2|} \in L^1_{x_1,loc}(\mathbb{R}^{N_1}; L^1_{x_2}(\mathbb{R}^{N_2}) + L^\infty_{x_2}(\mathbb{R}^{N_2}));\ (H_6): \mathrm{div}_{x_2} b_2 = 0; \\ (H_7):\ b_3 = b_3(x_1,x_2,x_3) \in L^1_{x_1,x_2,loc}(\mathbb{R}^{N_1+N_2}; W^{1,1}_{x_3,loc}(\mathbb{R}^{N_3})); \\ (H_8): \frac{b_3}{1+|x_3|} \in L^1_{x_1,x_2,loc}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}) + L^\infty_{x_3}(\mathbb{R}^{N_3}));\ (H_9): \mathrm{div}_{x_3} b_3 = 0. \end{cases}$$

In view of (H_1) and (H_4), we rewrite above equation in a particular form

$$\partial_t u(t,x) + b_1(x_1)\cdot\nabla_{x_1} u + b_2(x_1,x_2)\cdot\nabla_{x_2} u + b_3(x)\cdot\nabla_{x_3} u = 0, \text{ in } (0,T)\times\mathbb{R}^N. \tag{2.1}$$

We are now in a position to state our existence and uniqueness result :

Theorem 2.1. Assume $(H_1)-(H_9)$, let

$$\begin{aligned} u_0(x) \in (L^1\cap L^\infty)(\mathbb{R}^N) \cap L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3})) \\ \cap L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3})), \end{aligned} \tag{2.2}$$

then there exists a unique

$$\begin{aligned} u(t,x) \in L^\infty([0,T]; (L^1\cap L^\infty)(\mathbb{R}^N))\ \cap L^\infty([0,T]; L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}))) \\ \cap L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))) \end{aligned} \tag{2.3}$$

solution of (2.1) corresponding to the initial value $u(t=0,\cdot)=u_0$.

Remark 2.1. Here we adapt the same notion of the solution to (2.1) as in [12]: we call u a solution of (2.1) with initial value u_0 if the following identity holds:

$$\int_0^T dt \int_{\mathbb{R}^N} dxu\varphi_t + \int_{\mathbb{R}^N} dxu_0\varphi(0,x)dx + \int_0^T dt \int_{\mathbb{R}^N} dxu\mathrm{div}(\varphi b) = 0,$$

for any $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$, where $\mathcal{D}([0,T)\times\mathbb{R}^N)$ denotes the set of all smooth functions on $[0,T)\times\mathbb{R}^N$ with compact supports. We also denote $\mathcal{D}_+([0,T)\times\mathbb{R}^N)$ the nonnegative functions in $\mathcal{D}([0,T)\times\mathbb{R}^N)$.

We divide the proof into two steps. The uniqueness being the central issue, we begin by verifying it. It is the consequence of the following two lemmas, the first one dealing with the regularization, the second one stating the uniqueness. Then, we demonstrate the existence part.

Lemma 2.1. We assume $(H_1)-(H_7)$. Let $u \in L^\infty([0,T];(L^1\cap L^\infty)(\mathbb{R}^N))$ be a solution of (2.1), that $\rho_{\alpha_1},\ \rho_{\alpha_2}$, and ρ_{α_3} be three regularization kernels in variables x_1, x_2 and x_3, respectively (i.e. $\rho_{\alpha_i}=\frac{1}{\alpha_i^{N_i}}\rho_i(\frac{\cdot}{\alpha_i}),\ \ \rho_i\in\mathcal{D}_+(\mathbb{R}^{N_i}),\ \int\rho_i=1$, for $i=$ 1, 2, 3), then $u_{\alpha_1,\alpha_2,\alpha_3}=((u*\rho_{\alpha_1})*\rho_{\alpha_2})*\rho_{\alpha_3}$ is a smooth(in x) solution of

$$\frac{\partial u_{\alpha_1,\alpha_2,\alpha_3}}{\partial t} + b\cdot\nabla u_{\alpha_1,\alpha_2,\alpha_3} = \varepsilon_{\alpha_1,\alpha_2,\alpha_3}, \tag{2.4}$$

with

$$\lim_{\alpha_3\to0}\lim_{\alpha_2\to0}\lim_{\alpha_1\to0}\varepsilon_{\alpha_1,\alpha_2,\alpha_3}=0 \ \text{ in } \ L^\infty([0,T];(L^1_{x,loc}\cap L^\infty_{x,loc})). \tag{2.5}$$

Proof. All the functional spaces used here are local, this is clearly enough. However, in order to lighten the notations, we default the subscript *loc*.

We first regularize in the x_3 variable by regularization kernel ρ_{α_3}, namely,

$$\frac{\partial(u*\rho_{\alpha_3})}{\partial t} + b_1\cdot\nabla_{x_1}(u*\rho_{\alpha_3}) + b_2\cdot\nabla_{x_2}(u*\rho_{\alpha_3}) + (b_3\cdot\nabla_{x_3}u)*\rho_{\alpha_3}=0, \tag{2.6}$$

where we used the fact b_1 and b_2 do not depend on x_3. Denote by

$$[b_3\cdot\nabla_{x_3},\rho_{\alpha_3}](u) = b_3\cdot\nabla_{x_3}(u*\rho_{\alpha_3}) - (b_3\cdot\nabla_{x_3}u)*\rho_{\alpha_3},$$

then we can write (2.6) as

$$\frac{\partial(u*\rho_{\alpha_3})}{\partial t} + b_1\cdot\nabla_{x_1}(u*\rho_{\alpha_3}) + b_2\cdot\nabla_{x_2}(u*\rho_{\alpha_3}) + b_3\cdot\nabla_{x_3}(u*\rho_{\alpha_3}) = [b_3\cdot\nabla_{x_3},\rho_{\alpha_3}](u)$$

by condition (H_7), It is easy to see that (for example see [12])

$$[b_3\cdot\nabla_{x_3},\rho_{\alpha_3}](u)\to 0 \ \text{ in } \ L^1_x \ , \text{ as } \ \alpha_3\to0.$$

Set $u_{\alpha_3}=u*\rho_{\alpha_3}$, and $\varepsilon_{\alpha_3}=[b_3\cdot\nabla_{x_3},\rho_{\alpha_3}](u)$, then

$$\frac{\partial u_{\alpha_3}}{\partial t} + b_1\cdot\nabla_{x_1}u_{\alpha_3} + b_2\cdot\nabla_{x_2}u_{\alpha_3} + b_3\cdot\nabla_{x_3}u_{\alpha_3} = \varepsilon_{\alpha_3}.$$

Next, by regularizing variable x_2, we obtain

$$\partial_t u_{\alpha_3,\alpha_2} + b_1\cdot\nabla_{x_1}u_{\alpha_3,\alpha_2} + (b_2\cdot\nabla_{x_2}u_{\alpha_3})*\rho_{\alpha_2} + (b_3\cdot\nabla_{x_3}u_{\alpha_3})*\rho_{\alpha_2} = (\varepsilon_{\alpha_3})*\rho_{\alpha_2}, \tag{2.7}$$

where $u_{\alpha_3,\alpha_2}=u_{\alpha_3}*\rho_{\alpha_2}=(u*\rho_{\alpha_3})*\rho_{\alpha_2}=(u*\rho_{\alpha_2})*\rho_{\alpha_3}=u_{\alpha_2,\alpha_3}$.

Notice that

$$(b_2\cdot\nabla_{x_2}u_{\alpha_3})*\rho_{\alpha_2} = b_2\cdot\nabla_{x_2}u_{\alpha_2,\alpha_3} - [b_2\cdot\nabla_{x_2},\rho_{\alpha_2}](u_{\alpha_3})$$

and

$$(b_3 \cdot \nabla_{x_3} u_{\alpha_3}) * \rho_{\alpha_2} = b_3 \cdot \nabla_{x_3} u_{\alpha_2,\alpha_3} - [b_3 \cdot \nabla_{x_3}, \rho_{\alpha_2}](u_{\alpha_3})$$

and conditions (H_4), It is clear that

$$[b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}](u_{\alpha_3}) \to 0 \text{ in } L^1_x \text{ as } \alpha_2 \to 0 \text{ for ixed } \alpha_3 > 0.$$

Hence (2.7) can be written as

$$\frac{\partial u_{\alpha_2,\alpha_3}}{\partial t} + b_1 \cdot \nabla_{x_1} u_{\alpha_2,\alpha_3} + b_2 \cdot \nabla_{x_2} u_{\alpha_2,\alpha_3} + b_3 \cdot \nabla_{x_3} u_{\alpha_2,\alpha_3} = \varepsilon_{\alpha_2,\alpha_3},$$

where $\varepsilon_{\alpha_2,\alpha_3} = \varepsilon_{\alpha_3} * \rho_{\alpha_2} + [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_2}](u_{\alpha_3}) + [b_3 \cdot \nabla_{x_3}, \rho_{\alpha_2}](u_{\alpha_3})$.

Duplicating above regularization procedure with variable x_1, if one denotes by $u_{\alpha_3,\alpha_2} * \rho_{\alpha_1}$ by $u_{\alpha_1,\alpha_2,\alpha_3}$, then it yields

$$\partial_t u_{\alpha_1,\alpha_2,\alpha_3} + b_1 \cdot \nabla_{x_1} u_{\alpha_1,\alpha_2,\alpha_3} + b_2 \cdot \nabla_{x_2} u_{\alpha_1,\alpha_2,\alpha_3} + b_3 \cdot \nabla_{x_3} u_{\alpha_1,\alpha_2,\alpha_3} = \varepsilon_{\alpha_1,\alpha_2,\alpha_3}, \quad (2.8)$$

where

$$\begin{aligned} \varepsilon_{\alpha_1,\alpha_2,\alpha_3} &= [b_1 \cdot \nabla_{x_1}, \rho_{\alpha_1}](u_{\alpha_2,\alpha_3}) + [b_2 \cdot \nabla_{x_2}, \rho_{\alpha_1}](u_{\alpha_2,\alpha_3}) \\ &+ [b_3 \cdot \nabla_{x_3}, \rho_{\alpha_1}](u_{\alpha_2,\alpha_3}) + \varepsilon_{\alpha_2,\alpha_3} * \rho_{\alpha_1}. \end{aligned}$$

Obviously for fixed $\alpha_2 > 0, \alpha_3 > 0$ by [12], we know the first three error terms tend to zero as α_1 goes to zero in L^1_x, and the last term goes to $\varepsilon_{\alpha_2,\alpha_3}$.

Similarly as $\alpha_3 > 0$ be fixed,

$$\varepsilon_{\alpha_2,\alpha_3} \to \varepsilon_{\alpha_3} \text{ in } L^1_x \text{ as } \alpha_2 \to 0,$$

Thus we proved our result for smooth (in x) solution of u, with

$$\lim_{\alpha_3\to 0} \lim_{\alpha_2\to 0} \lim_{\alpha_1\to 0} \varepsilon_{\alpha_1,\alpha_2,\alpha_3} = 0 \text{ in } L^\infty([0,T]; (L^1_{x,loc})(\mathbb{R}^N)).$$

On the other hand, observing the fact that: If u_n is a sequence in $L^1 \cap L^\infty$ (or $L^1_{loc} \cap L^\infty_{loc}$) and $u_n \to u$ in L^1(or L^1_{loc}) and u_n is uniformly bounded in L^∞ (or L^∞_{loc}), then $u \in L^\infty$ (or L^∞_{loc}), and $u_n \to u$ in L^∞(or L^∞_{loc}), we complete our proof.

To prove the uniqueness, we need another useful lemma.

Lemma 2.2. We assume $(H_1) - (H_9)$. Let

$$\begin{aligned} u(t,x) \in L^\infty([0,T]; (L^1 \cap L^\infty)(\mathbb{R}^N)) \cap L^\infty([0,T]; L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}))) \\ \cap\, L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))) \end{aligned}$$

be a nonnegative solution of (2.1) with the initial condition $u_0 = 0$, then $u \equiv 0$.

Proof. Let u be a nonnegative solution as claimed in the lemma. With obvious notations introduced in Lemma 2.1, we have

$$\begin{aligned} \varepsilon_{\alpha_1,\alpha_2,\alpha_3}(x) &= \partial_t u_{\alpha_1,\alpha_2,\alpha_3}(t,x) + b_1(x_1) \cdot \nabla_{x_1} u_{\alpha_1,\alpha_2,\alpha_3}(t,x) \\ &+ b_2(x_1,x_2) \cdot \nabla_{x_2} u_{\alpha_1,\alpha_2,\alpha_3}(t,x) + b_3(x) \cdot \nabla_{x_3} u_{\alpha_1,\alpha_2,\alpha_3}, \qquad (2.9) \end{aligned}$$

with

$$\lim_{\alpha_3\to 0} \lim_{\alpha_2\to 0} \lim_{\alpha_1\to 0} \varepsilon_{\alpha_1,\alpha_2,\alpha_3} = 0 \text{ in } L^\infty([0,T]; (L^1_{x,loc} \cap L^\infty_{x,loc})).$$

We introduce three cut-off functions: φ, ψ, ξ, with respect to each variable x_1, x_2 and x_3. For any natural numbers m, n, k, we denote them by

$$\varphi_n(x_1) = \varphi(\frac{x_1}{n}), \ \psi_m(x_2) = \psi(\frac{x_2}{m}), \ \xi_k(x_3) = \xi(\frac{x_3}{k}),$$

where

$$\varphi \in \mathcal{D}_+(\mathbb{R}^{N_1})\ ,\ \psi \in \mathcal{D}_+(\mathbb{R}^{N_2})\ ,\ \xi \in \mathcal{D}_+(\mathbb{R}^{N_3}),$$
$$\varphi,\ \psi,\ \xi = 1 \ \text{ on } \ |x_1| \leq 1,\ |x_2| \leq 1,\ |x_3| \leq 1\ ,$$

and

$$\varphi,\ \psi,\ \xi = 0 \ \text{ on } \ |x_1| \geq 2,\ |x_2| \geq 2,\ |x_3| \geq 2$$

respectively.

Firstly we multiply (2.9) by ξ_k and integrate over the x_3 space to derive

$$\begin{aligned}\int \xi_k \varepsilon_{\alpha_1,\alpha_2,\alpha_3} dx_3 \quad = \quad & \frac{\partial}{\partial t}\int_{\mathbb{R}^{N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k(x)dx_3 \\ & + b_1(x_1)\cdot\nabla_{x_1}\int_{\mathbb{R}^{N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k dx_3 \\ & + b_2(x_1,x_2)\cdot\nabla_{x_2}\int_{\mathbb{R}^{N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k dx_3 \\ & + \int_{\mathbb{R}^{N_3}} b_3(x_1,x_2,x_3)\cdot\nabla_{x_3} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k dx_3. \end{aligned} \tag{2.10}$$

By integration by parts formula, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{N_3}} b_3(x_1,x_2,x_3)\cdot\nabla_{x_3}\ u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k dx_3 \\ = \quad & -\int_{\mathbb{R}^{N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\ \frac{b_3(x_1,x_2,x_3)}{k}\cdot\nabla_{x_3}\ \xi(\frac{x_3}{k})dx_3 \\ = \quad & -\int_{\mathbb{R}^{N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\ \frac{1+|x_3|}{k}\frac{b_3(x_1,x_2,x_3)}{1+|x_3|}\cdot\nabla_{x_3}\xi(\frac{x_3}{k})dx_3, \end{aligned} \tag{2.11}$$

where we used the fact $\mathrm{div}_{x_3} b_3$=0.

Then multiplying ψ_m to (2.10), combining (2.11), we get

$$\begin{aligned} & \int_{\mathbb{R}^{N_2+N_3}} \xi_k\psi_m\varepsilon_{\alpha_1,\alpha_2,\alpha_3} dx_2dx_3 \\ = \quad & \frac{\partial}{\partial t}\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m dx_2dx_3 \\ & + b_1(x_1)\cdot\nabla_{x_1}\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m dx_2dx_3 \\ & + \int_{\mathbb{R}^{N_2}} b_2(x_1,x_2)\cdot\nabla_{x_2}\int_{\mathbb{R}^{N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m dx_3dx_2 \\ & -\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\frac{1+|x_3|}{k}\frac{b_3(x)}{1+|x_3|}\cdot\nabla_{x_3}\xi(\frac{x_3}{k})\psi_m dx_2dx_3. \end{aligned} \tag{2.12}$$

We treat the third term in the right hand side like the fourth term in (2.10), namely integration by parts, we conclude

$$\begin{aligned} & \int_{\mathbb{R}^{N_2+N_3}} \xi_k\psi_m\varepsilon_{\alpha_1,\alpha_2,\alpha_3} dx_2dx_3 \\ = \quad & \frac{\partial}{\partial t}\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m dx_2dx_3 \\ & + b_1(x_1)\cdot\nabla_{x_1}\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m dx_2dx_3 \end{aligned}$$

$$-\int_{\mathbb{R}^{N_2+N_3}} \frac{b_2(x_1,x_2)}{1+|x_2|}\cdot\nabla_{x_2}\psi(\frac{x_2}{m})\frac{1+|x_2|}{m}u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k dx_3dx_2$$
$$-\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\frac{1+|x_3|}{k}\frac{b_3(x_1,x_2,x_3)}{1+|x_3|}\cdot\nabla_{x_3}\xi(\frac{x_3}{k})\ \psi_m dx_2dx_3. \quad (2.13)$$

Repeat above calculations to the first variable, i.e., multiplying a cut-off function φ_n and integrating over the x_1 space, then by virtue of integration by parts formula, we fulfill

$$\begin{aligned}&\int \xi_k\psi_m\varphi_n\varepsilon_{\alpha_1,\alpha_2,\alpha_3}dx\\ = \ & \frac{d}{dt}\int_{\mathbb{R}^N} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m\varphi_n dx\\ &-\int_{\mathbb{R}^{N_1}} b_1(x_1)\cdot\nabla_{x_1}\int_{\mathbb{R}^{N_2+N_3}} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k\psi_m\varphi_n dx_2dx_3dx_1\\ &-\int_{\mathbb{R}^N}\frac{b_2(x_1,x_2)}{1+|x_2|}\cdot\nabla_{x_2}\psi(\frac{x_2}{m})\frac{1+|x_2|}{m}\varphi_n u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\xi_k dx\\ &-\int_{\mathbb{R}^N} u_{\alpha_1,\alpha_2,\alpha_3}(t,x)\varphi_n\psi_m\frac{1+|x_3|}{k}\frac{b_3(x_1,x_2,x_3)}{1+|x_3|}\cdot\nabla_{x_3}\xi(\frac{x_3}{k})dx. \qquad (2.14)\end{aligned}$$

By applying Lemma 2.1, one gains

$$\begin{aligned}0 \ = \ & \frac{d}{dt}\int_{\mathbb{R}^N} u\xi_k\psi_m\varphi_n dx-\int_{\mathbb{R}^{N_1}} b_1(x_1)\cdot\nabla_{x_1}\int_{\mathbb{R}^{N_2+N_3}} u\xi_k\psi_m\varphi_n dx_2dx_3dx_1\\ &-\int_{\mathbb{R}^N}\frac{b_2(x_1,x_2)}{1+|x_2|}\cdot\nabla_{x_2}\psi(\frac{x_2}{m})\frac{1+|x_2|}{m}\varphi_n u\xi_k dx\\ &-\int_{\mathbb{R}^N} u(t,x)\varphi_n\psi_m\frac{1+|x_3|}{k}\frac{b_3(x_1,x_2,x_3)}{1+|x_3|}\cdot\nabla_{x_3}\xi(\frac{x_3}{k})dx, \qquad (2.15)\end{aligned}$$

if one tends α_1 first, α_2 next, α_3 third to zero in turn, for fixed k, m and n.

Notice that

$$u(t,x)\in L^\infty([0,T];(L^1\cap L^\infty)(\mathbb{R}^N))\cap L^\infty([0,T];L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2};L^1_{x_3}(\mathbb{R}^{N_3})))$$

and

$$\frac{|b_3(x)|}{1+|x_3|}\in L^1_{x_1,x_2,loc}(\mathbb{R}^{N_1+N_2};L^1_{x_3}(\mathbb{R}^{N_3})+L^\infty_{x_3}(\mathbb{R}^{N_3}))$$

and u is nonnegative, for any fixed nature numbers n and m

$$u\varphi_n\psi_m\frac{1+|x_3|}{k}\frac{b_3(x_1,x_2,x_3)}{1+|x_3|}\cdot\nabla_{x_3}\xi(\frac{x_3}{k})\mathbf{1}_{k\leq|x_3|\leq 2k}\to 0, \text{ as } k \text{ goes to infinity.}$$

By applying the Lebesgue dominated convergence theorem, for fixed n and m, the fourth term tends to zero as k goes to infinity.

Now let us estimate the third term

$$\int_{\mathbb{R}^N}\frac{b_2(x_1,x_2)}{1+|x_2|}\cdot\nabla_{x_2}\psi(\frac{x_2}{m})\frac{1+|x_2|}{m}\varphi_n u\xi_k dx. \qquad (2.16)$$

Note that

$$\frac{|b_2(x_1,x_2)|}{1+|x_2|}\in L^1_{x_1}(\mathbb{R}^{N_1};L^1_{x_2}(\mathbb{R}^{N_2})+L^\infty_{x_2}(\mathbb{R}^{N_2})),$$

one can write $|\frac{b_2(x_1,x_2)}{1+|x_2|}|=b_2^1+b_2^2$, where

$$b_2^1\in L^1_{x_1,loc}(\mathbb{R}^{N_1};L^1_{x_2}(\mathbb{R}^{N_2})),\quad b_2^2\in L^1_{x_1,loc}(\mathbb{R}^{N_1};L^\infty_{x_2}(\mathbb{R}^{N_2})),$$

which implies

$$\varphi_n\left|\frac{b_2(x_1,x_2)}{1+|x_2|}\right| = \varphi_n b_2^1 + \varphi_n b_2^2 \in L^1_{x_1,x_2}(\mathbb{R}^{N_1+N_2}) + L^1_{x_1}(\mathbb{R}^{N_1}; L^\infty_{x_2}(\mathbb{R}^{N_2})).$$

On the other hand, for above fixed n,

$$\int_{\mathbb{R}^{N_1+N_3}} \frac{|b_2(x_1,x_2)|}{1+|x_2|}\varphi_n u dx_1 dx_3 \in L^1_{x_2}(\mathbb{R}^{N_2}),$$

for

$$u \in L^\infty([0,T];(L^1\cap L^\infty)(\mathbb{R}^N)) \cap L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3})))$$

and the integrand of (2.16) tends to zero as m goes to infinity, and u is non-negative. By making use of the dominated convergence theorem, we also get the third term vanishes as m goes to infinity.

The same discussion also implies the second term of (2.15) vanishes as n tends to infinity. Collecting the behavior of the last three terms, and using the Bochner theorem, we obtain with (2.15), as k first, m next, then n, go to infinity

$$\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\frac{d}{dt}\int_{\mathbb{R}^N} u\xi_k\psi_m\varphi_n dx = \frac{d}{dt}\int_{\mathbb{R}^N} u dx = 0.$$

As $u_0 = 0$, this yields $u = 0$, for all $t \geq 0$, since u is nonnegative and this concludes the proof.

Having proved Lemma 2.1 and Lemma 2.2, we now complete the proof of Theorem 2.1.

Proof of Theorem 2.1.

(Uniqueness) Assume for the time being that we have at hand two solutions u_1 and u_2 to (2.1) satisfying the regularity stated in Theorem 2.1 and sharing the same initial datum. By virtue of Lemma 2.1, the difference $u = u_1 - u_2$, satisfies

$$\frac{\partial u_{\alpha_1,\alpha_2,\alpha_3}}{\partial t} + b\cdot\nabla u_{\alpha_1,\alpha_2,\alpha_3} = \varepsilon_{\alpha_1,\alpha_2,\alpha_3},$$

with

$$\lim_{\alpha_3\to 0}\lim_{\alpha_2\to 0}\lim_{\alpha_1\to 0}\varepsilon_{\alpha_1,\alpha_2,\alpha_3} = 0 \ \text{ in } \ L^\infty([0,T];(L^1_{x,loc}\cap L^\infty_{x,loc})(\mathbb{R}^N)),$$

with obvious notations.

Multiplying $\beta'(u_{\alpha_1,\alpha_2,\alpha_3})$, for some function $\beta \in \mathcal{C}(\mathbb{R})$, β' bounded, we conclude

$$\frac{\partial \beta(u_{\alpha_1,\alpha_2,\alpha_3})}{\partial t} + b\cdot\nabla\beta(u_{\alpha_1,\alpha_2,\alpha_3}) = \varepsilon_{\alpha_1,\alpha_2,\alpha_3}\beta'(u_{\alpha_1,\alpha_2,\alpha_3}).$$

By letting α_1 first, next α_2, then α_3 go to zero, we obtain

$$\frac{\partial\beta(u)}{\partial t} + b\cdot\nabla\beta(u) = 0,$$

for such admissible function β.

By a tedious discussion argument, we can choose a sequence of admissible functions $\beta_k(x)$, such that $\beta_k(x) \to |x|$, as k goes to infinity. We end up with

$$\frac{\partial |u|}{\partial t} + b\cdot\nabla|u| = 0.$$

Applying Lemma 2.2, we know $u = 0$, and show the uniqueness.

(Existence) Existence in the functional space $L^\infty([0,T];(L^1\cap L^\infty)(\mathbb{R}^N))$ is given in a straightforward way by an application of proposition II.1 of [12], and we omit it. Thus what we should do now is to check the solution in

$$L^\infty([0,T];L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2};L^1_{x_3}(\mathbb{R}^{N_3})))\cap L^\infty([0,T];L^\infty_{x_1}(\mathbb{R}^{N_1};L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))).$$

Let $b_\varepsilon=\rho_\varepsilon * b, u_0^\varepsilon=\rho_\varepsilon * u_0$, where ρ_ε is a smooth kernel, and in particular, we choose ρ_ε as a product of three regularization kernels $\rho_{1,\varepsilon}(x_1),\ \rho_{2,\varepsilon}(x_2),\ \rho_{3,\varepsilon}(x_3)$, then we can rewrite b_ε as

$$b_\varepsilon=(b_1^\varepsilon,\ b_2^\varepsilon,\ b_3^\varepsilon)$$

where

$$b_1^\varepsilon=(b_1*\rho_{1,\varepsilon})*\rho_{2,\varepsilon}*\rho_{3,\varepsilon},\ b_2^\varepsilon=(b_2*\rho_{1,\varepsilon}*\rho_{2,\varepsilon})*\rho_{3,\varepsilon},\ b_3^\varepsilon=b_3*\rho_{1,\varepsilon}*\rho_{2,\varepsilon}*\rho_{3,\varepsilon}.$$

Consider regularized equation

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}+b_\varepsilon\cdot\nabla u_\varepsilon=0,\ \text{in } (0,T)\times\mathbb{R}^N,\\ u(t=0,\cdot)=u_0^\varepsilon,\ \text{in } \mathbb{R}^N, \end{cases} \tag{2.17}$$

it is clear that there is a unique solution u_ε to (2.17) which is smooth and

$$u_\varepsilon\in L^\infty([0,T];(L^1\cap L^\infty)(\mathbb{R}^N))$$

and $u_\varepsilon\to u$, as $\varepsilon\to 0$, since $u_0^\varepsilon\to u_0$, as $\varepsilon\to 0$. Hence if we multiply (2.17) by ξ_k, with clear notation, introduced in Lemma 2.2 and integrate it over x_3 space and by partial integration, we deduce

$$\begin{cases} \frac{\partial}{\partial t}\int_{\mathbb{R}^{N_3}} u_\varepsilon dx_3+(b_1^\varepsilon,b_2^\varepsilon)\cdot\nabla_{x_1,x_2}\int_{\mathbb{R}^{N_3}} u_\varepsilon dx_3=0,\ \text{in}\ \ (0,T)\times\mathbb{R}^{N_1+N_2},\\ \int_{\mathbb{R}^{N_3}} u_\varepsilon(t=0,\cdot)=\int_{\mathbb{R}^{N_3}} u_0^\varepsilon dx_3,\ \text{in}\ \ \mathbb{R}^{N_1+N_2}, \end{cases} \tag{2.18}$$

by virtue of letting k go to infinity.

By the Kružkov theorem for conservation laws equations, we know

$$\left\|\int_{\mathbb{R}^{N_3}} u_\varepsilon dx_3\right\|_{L^\infty_{x_1,x_2,t}}=\left\|\int_{\mathbb{R}^{N_3}} u_0^\varepsilon dx_3\right\|_{L^\infty_{x_1,x_2}}.$$

With the aid of taking ε go to zero and notice that $u_0\in L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2};L^1_{x_3}(\mathbb{R}^{N_3}))$, we deduce that

$$u(t,x)\in L^\infty([0,T];L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2};L^1_{x_3}(\mathbb{R}^{N_3}))).$$

Similarly, if we integrate (2.18) over x_2 space and use integration by parts formula, combining

$$u_0\in L^\infty_{x_1}(\mathbb{R}^{N_1};L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3})),\quad \text{and}\quad u_0^\varepsilon\to u_0,\quad \text{as}\quad \varepsilon\to 0,$$

we also get

$$u(t,x)\in L^\infty([0,T];L^\infty_{x_1}(\mathbb{R}^{N_1};L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))),$$

by letting ε go to zero. This completes the proof of Theorem 2.1.

Remark 2.2. For the sake of simplicity, we have chosen to present our result when the vector field b is assumed not to depend on the time variable, although our result also hold mutatis mutandis in the time dependent case $b=b(t,x)$ when we allow an L^1 dependent with respect to time. Then, assumptions $(H_1)-(H_9)$ are replaced by

$$\begin{cases} (H_{1'}) : b_1 = b_1(t,x_1) \in L^1([0,T]; W^{1,1}_{x_1,loc}(\mathbb{R}^{N_1})); \\ (H_{2'}) : \frac{b_1(x_1)}{1+|x_1|} \in L^1([0,T]; L^1_{x_1} + L^\infty_{x_1}); \\ (H_{4'}) : b_2 = b_2(t,x_1,x_2) \in L^1([0,T]; L^1_{x_1,loc}(\mathbb{R}^{N_1}; W^{1,1}_{x_2,loc}(\mathbb{R}^{N_2}))); \\ (H_{5'}) : \frac{b_2(t,x_1,x_2)}{1+|x_2|} \in L^1([0,T]; L^1_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2}(\mathbb{R}^{N_2}) + L^\infty_{x_2}(\mathbb{R}^{N_2}))); \\ (H_{7'}) : b_3 = \ b_3(t,x_1,x_2,x_3) \in L^1([0,T]; L^1_{x_1,x_2,loc}(\mathbb{R}^{N_1+N_2}; W^{1,1}_{x_3,loc}(\mathbb{R}^{N_3}))); \\ (H_{8'}) : \frac{b_3(t,x_1,x_2,x_3)}{1+|x_3|} \in L^1([0,T]; L^1_{x_1,x_2,loc}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}) + L^\infty_{x_3}(\mathbb{R}^{N_3}))). \end{cases}$$

Remark 2.3. Similarly, we can extend our result when the divergence is controlled in the L^∞ norm, namely

$$\begin{cases} (H_{3''}) : \mathrm{div}_{x_1} b_1 \in L^\infty_{x_1}(\mathbb{R}^{N_1}); \ (H_{6''}) : \ \mathrm{div}_{x_2} b_2 \in L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}); \\ (H_{9''}) : \ \mathrm{div}_{x_3} b_3 \in L^\infty_{x}(\mathbb{R}^{N}). \end{cases}$$

3. The high-order differentiability of the flow solutions with initial datums of ordinary differential equations

3.1 Preliminaries

We devote this short section to state our first application of the theory of renormalized solutions for linear transport equations to the type of ODE (1.2) in order to recover the art of the theory of solutions of ordinary differential equations with coefficients in Sobolev spaces to the level of the classical Cauchy-Lipschitz theory for the regular coefficients. For this aim, let us look back some notions and old results.

Remembering that in [12], firstly, Di Perna and Lions introduce

$$L^0 = \{u \in L; \ \text{measure}\{|u| > \lambda\} < \infty, \ \forall \ \lambda > 0\},$$

where L is the set of all measurable functions taking values in $\overline{R}$ and admissible functions we denote by

$$\mathcal{A} = \{\beta \in \mathcal{C}(\mathbb{R}); \ \beta \ \text{ is bounded and vanishing near zero}\},$$

then they give the notion of renormalized solutions for linear transport equations and by the equivalence between ODEs and the associated linear transport equations they show the existence and uniqueness of the almost everywhere flow solutions to the ordinary differential equations with vector field b belongs to L^1_{loc}.

Very similar to [12], in [7], Le Bris and Lions introduce

$$L^{0,0} = \{u \in L^0; \ \text{measure}\{x_2; \ |u(x_1,x_2)| > \delta\} < c_\delta(x_1) \in L^\infty_{x_1}, \ \ \forall \ \delta > 0\}$$

and the same admissible functions $\mathcal{A}$ as before, then they get the $1th$ order differentiability of the almost everywhere flow solution to initial values with vector field partially belongs to $W^{1,1}_{loc}$ and $2nd$ order differentiability in plus the $W^{2,1}$-regularity on vector field b.

But now we should make a slight change and plus some additional conditions in order to draw forth the almost everywhere flow solutions here encouraged by Le Bris and Lions. For this purpose, we present a new notation L_0 as follows.

We denote L_0 by the set

$$\{u \in L^0; \ \text{measure}\{x_3; \ |u(x_1,x_2,x_3)| > \delta \ \} < c_\delta(x_1,x_2) \in L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}),$$

$$\text{measure}\{(x_2, x_3);\ |u(x_1, x_2, x_3)| > \delta\} < c_\delta(x_1) \in L^\infty_{x_1}(\mathbb{R}^{N_1}), \quad \forall\ \delta > 0\}.$$

Then for any admissible function β vanishes on $[0, \delta]$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^{N_3}} |\beta(u(x_1, x_2, x_3))| dx_3 \\
= & \int_{\{x_3; |u(x_1,x_2,x_3)|>\delta\}} |\beta(u(x))| dx_3 + \int_{\{x_3; |u(x_1,x_2,x_3)|<\delta\}} |\beta(u(x))| dx_3 \\
\leq & \ \|\beta\|_{L^\infty} c_\delta(x_1, x_2) \in L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2})
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^{N_2+N_3}} |\beta(u(x_1, x_2, x_3))| dx_3 \\
= & \int_{\{(x_2,x_3); |u(x)|>\delta\}} |\beta(u(x))| dx_3 + \int_{\{(x_2,x_3); |u(x)|<\delta\}} |\beta(u(x))| dx_3 \\
\leq & \ \|\beta\|_{L^\infty} c_\delta(x_1) \in L^\infty_{x_1}(\mathbb{R}^{N_1}),
\end{aligned} \tag{3.2}$$

which hints

$$\beta(u) \in L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3})) \cap L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3})),$$

where L_0 as the subset of L^0, is equipped with the topology induced by that of L^0.

We therefore say that u is a renormalized solution of (2.1) supplied with an initial condition $u_0 \in L_0$ whenever $\beta(u)$ is a solution of (2.1) in the sense of Section 2, with initial datum $\beta(u_0)$.

From [13] (Proposition 1), we have the following fact :

Lemma 3.1. Consider the ordinary differential equation

$$\begin{cases} \dot{X}(t, x) = b(X(t, x)), \\ X(t = 0, x) = x, \end{cases} \tag{3.3}$$

where b is given by in (2.1). Then X is an almost everywhere flow solution of (3.3), which means that X satisfy (3.3), for almost all x, in the sense of distributions together with the following three properties:

$$\begin{cases} \text{(i)} X \in \mathcal{C}(\mathbb{R}; L^1(\mathbb{R}^N)); \\ \text{(ii)} \int \varphi(X(t, x)) dx = \int \varphi(x) dx, \quad \forall\ \varphi \in \mathcal{D}(\mathbb{R}^N),\ \forall\ t \in \mathbb{R}; \\ \text{(iii)} X(t + s, x) = X(t, X(s, x)),\ \forall\ t, s \in \mathbb{R},\ a.e.\ x \in \mathbb{R}^N; \end{cases} \tag{3.4}$$

iff $[S(t)u_0](x) = u_0(X(t, x))$ is a renormalized group solution of the following transport equation

$$\begin{cases} \frac{\partial}{\partial t} u(x) = b(x) \cdot \nabla u(x), \\ u(t = 0, x) = u_0(x), \end{cases} \tag{3.5}$$

that is

$$\begin{cases} \text{(i) } S(t)u_0 \in \mathcal{C}(\mathbb{R}; L^1(\mathbb{R}^N)); \\ \text{(ii) } S(t)\beta(u_0) = \beta(S(t)u_0),\ \forall\ \beta \in \mathcal{D}(\mathbb{R}^N),\ \forall\ u_0 \in L^\infty(\mathbb{R}^N),\ \forall\ t \in \mathbb{R}; \\ \text{(iii) } S(t) \text{ is linear}, \forall\ t \in \mathbb{R}; \\ \text{(iv) } S(t + s) = S(t) \circ S(s),\ \forall\ t, s \in \mathbb{R}; \\ \text{(v) } u(t, x) = [S(t)u_0](x) = u_0(X(t, x)) \text{ is a solution of (3.5)}, \end{cases}$$

where we shall say that X is a solution of (3.3) whenever for all $\forall\ \beta \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\begin{cases} \frac{\partial}{\partial t}\beta(X) = \nabla\beta(X(t,y)) \cdot b(X(t,y)), \\ \beta(X)(t=0,y) = \beta(y), \end{cases} \tag{3.6}$$

in the sense of distributions, where b satisfies $(H_{1'}) - (H_{9'})$.

Remark 3.1. It is to be remarked that the equivalence we have recalled above holds when the equation is set on the torus. A necessary and sufficient modification when one works on the whole space is to impose a convenient behavior at infinity for the vector field b. In our case $(H_{2'})$, $(H_{5'})$ and $(H_{8'})$ will play this role. Similarly the above result hold true if we replace L^1 by L^1_{loc} or L^0 or L^0_{loc} from [12] (P_{527}), but what we should modify is to take place of $\varphi \in \mathcal{D}(\mathbb{R}^N)$ by $\varphi \in \mathcal{A}$.

3.2 kth-order differentiability

We are now in a position to state our first result in this section. Since this result is only a corollary of above Lemma, we omit the proof.

We consider the ordinary differential equation (3.3) again, for the sake of simplicity we rewrite here in the case for a vector, denoted by c, which does not depend on time

$$\begin{cases} \dot{Y}(t,y) = c(Y(t,y)), \\ Y(t=0,y) = y. \end{cases} \tag{3.7}$$

Let us assume that c satisfies the following properties

$$(P_1) : c(y)\ \in W^{2,1}_{y,loc};\ (P_2) : \frac{c}{1+|y|} \in\ (L^1 + L^\infty)(\mathbb{R}^N);\ (P_3) : \mathrm{div}_y c = 0.$$

For some fixed $r \in \mathbb{R}^N$, by differentiating Y with respect to the initial datum y along the direction r, we obtain

$$\frac{\partial}{\partial t}(r \cdot \nabla_y Y)(t,y) = \nabla_y c(Y)(r \cdot \nabla_y Y)(t,y).$$

Grouping the two equation together we may write

$$\begin{cases} \dot{Y}(t,y) = c(Y(t,y)), \\ \dot{R}(t,y,r) = \nabla_y c(Y) R(t,y,r), \\ Y(t=0,y) = y, \\ R(t=0,y,r) = r, \end{cases} \tag{3.8}$$

where we have denoted by $R(t,y,r) = (r \cdot \nabla_y Y)(t,y)$.

Then we fix $(y',r') \in \mathbb{R}^{2N}$, and differentiate Y and R with respect to (y,r) along the direction (y',r'), we get

$$\begin{cases} \dot{Y}'(t,y,y') = \nabla_y c(Y(t,y)) Y'(t,y,y'), \\ \dot{R}'(t,y,r,y',r') = \nabla_y c(Y(t,y)) R' + \nabla^2_{y,y} c(Y(t,y)) \cdot (R, Y'(t,y,y')), \\ Y'(t=0,y,y') = y', \\ R'(t=0,y,r,y'r') = r', \end{cases} \tag{3.9}$$

where $Y' = y' \cdot \nabla_y Y,\ R' = R'(t,y,r,y'r') = (y',r') \cdot \nabla_{y,r} R(t,y,r)$.

Collecting (3.9) with (3.8), we obtain the closed system :

$$\begin{cases} \dot{Y}(t,y) = c(Y(t,y)), \\ \dot{R}(t,y,r) = \nabla_y c(Y) R(t,y,r), \\ \dot{Y'}(t,y,y') = \nabla_y c(Y(t,y)) Y'(t,y,y'), \\ \dot{R'}(t,y,r,y',r') = \nabla_y c(Y(t,y)) R' + \nabla^2_{y,y} c(Y(t,y)) \cdot (R(t,y,r), Y'), \\ Y(t=0,y) = y, \\ R(t=0,y,r) = r, \\ Y'(t=0,y,y') = y', \\ R'(t=0,y,r,y',r') = r'. \end{cases} \tag{3.10}$$

It is easy to see that this system of ordinary differential equations is associated with the following transport equation:

$$\frac{\partial}{\partial t} u - c(y) \cdot \nabla_y u - \nabla_y c(y) r \nabla_r u - \nabla_y c(y) y' \nabla_{y'} u$$
$$- (\nabla_y c(y) r' + \nabla^2_{y,y} c(Y(t,y)) \cdot (r, y')) \nabla_{r'} u = 0. \tag{3.11}$$

Set on a function $u = u(t, y, r, y', r')$. If we denote by

$b_1(x_1) = -c(x_1),\ b_2(x_1,x_2) = -\nabla_{x_1} c(x_1) x_2,\ b_3(x) = -\nabla_{x_1} c(x_1) x_3 - \nabla^2_{x_1,x_1} c(x_1) x_2,$

where $x_1 = y,\ x_2 = (r, y'),\ x_3 = r'$, then we can rewrite (3.11) as a compact form

$$\frac{\partial}{\partial t} u + b(x) \cdot \nabla_x u = 0,$$

where

$$b(x) = b(x_1, x_2, x_3) = (b_1(x_1), b_2(x_1, x_2), b_3(x_1, x_2, x_3)).$$

Obviously it is easy to check that vector field b satisfies conditions $(H_1) - (H_9)$. By Theorem 2.1 and Lemma 3.1, we get the $2nd$ order differentiability of the almost everywhere flow $Y(t, y)$ with y only under the assumption $b \in W^{2,1}_{loc}$, precisely speaking we have :

Theorem 3.2. We assume (P_1)–(P_3), then there exists a unique almost everywhere flow (Y, R, Y', R'), such that

• Y, R satisfy Theorem 4.1 in [7];

• Y' is continuous from $[0, T]$ to the set of functions of (y, y') that, for almost all y are $L^1_{y',loc}$, and for almost all y', $L_{y,loc}$, i.e. almost everywhere finite measurable functions of y;

• R' is continuous from $[0, T]$ to the set of functions of (y, r, y', r') that, for almost all y are $L^1_{r,y',r'loc}$, and for almost all r, y', r', $L_{y,loc}$;

• (Y, R, Y', R') satisfies the conservation property (3.4):(ii) of the Lebesgue measure in (y, r, y', r'), and the semigroup (3.4): (iii);

• (Y, R, Y', R') satisfies (3.10) in the sense of (3.6).

If we assume, in addition, $c(y) \in L^p + (1 + |y|)L^\infty,\ p \in [1, \infty]$, then

$$(Y, R, Y') \in (\mathcal{C}([0, T]; L^1_{y,loc}), \mathcal{C}([0, T]; L^1_{y,r,loc}), \mathcal{C}([0, T]; L^1_{y,y',loc}))$$

and

$$R' \in \mathcal{C}([0, T]; L^1_{y,r,y',r',loc})$$

Remark 3.2. The first item has been proved in [7] by Le Bris and Lions, now we spread out it only for the completeness. It time for us to present and prove our second result.

Theorem 3.3. Assume $bb(x) \in W^{k,1}_{loc}(\mathbb{R}^N)$, consider the following ordinary differential equation

$$\begin{cases} \dot{X}(t,x) = b(X(t,x)), \\ X(0) = x. \end{cases}$$

Assume, in addition, that $b(x) \in L^1 + (1+|x|)L^\infty(\mathbb{R}^N)$, $\operatorname{div} b = 0$, then there is a unique almost everywhere flow $X(t,x)$, such that $X(t,x) \in \mathcal{C}([0,T]; W^{k,1}_{loc}(\mathbb{R}^N))$.

Proof. We divide our proof into three steps. Since the proof is machinery and awkward and we have proved it for $k = 2$, in order to avoid unnecessary overlapping we now only present the train of thinking and omit some details.

Step 1. Construct the existence and uniqueness theorem of the solution for the following linear transport equation

$$\frac{\partial}{\partial t} u + b \cdot \nabla u = 0,$$

where

$$b(x) = b(x_1, x_2, \cdots, x_{k+1}) = (b_1(x_1), b_2(x_1, x_2), \ \cdots, \ b_{k+1}(x_1, x_2, \cdots, x_{k+1})),$$

with partial $W^{1,1}_{loc}$ regularity, by extending Lemma 2.2 and Lemma 2.3.

Step 2. Mimic Section 3.1 to define the renormalized solutions, by virtue of [12] and [7] we can get a equivalence between ordinary equation $\dot{X} = -b$ and the above linear transport equation.

Step 3. By step 1 and step 2 to get the conclusion .

Remark 3.3. (1) If we only presume $\frac{b}{1+|x|} \in L^1 + L^\infty$, one also obtains a relative weaker result to the almost everywhere flow, namely

$$X(t,x) \in \mathcal{C}([0,T]; L_{x,loc}), \ \ \nabla^m X(t,x) y_1 y_2 \cdots y_m \in \mathcal{C}([0,T]; L_{x,loc} \times L^1_{y_1,y_2,\cdots,y_m,loc}),$$

where $y_m \in \mathbb{R}^N$, and $1 \le m \le k$.

(2) If we replace

$$b(x) \in L^1 + (1+|x|)L^\infty(\mathbb{R}^N)$$

by

$$b(x) \in L^p + (1+|x|)L^\infty(\mathbb{R}^N),$$

for any $p \ge 1$, then above conclusion is also true.

(3) Besides, we have in addition that for almost all $x, y_1, y_2, \cdots, y_m \in \mathbb{R}^N$, $\nabla^m X(t,x) y_1 y_2 \cdots y_m \in \mathcal{C}^1([0,T])$.

4. Micro-Macro model

As an another application, in this section, we discuss a class of transport equations which have been argued in Section 2, with an additional viscosity term, i.e. the Fokker-Planck equations, with the particular form

$$\begin{cases} \partial_t u(t,x) + (b_1(x_1), b_2(x_1,x_2), b_3(x)) \cdot \nabla u - \frac{1}{2}\Delta_{x_3} u = 0, \ \text{in } (0,T) \times \mathbb{R}^N \\ u(t=0, \ \cdot, \ \cdot, \ \cdot) = u_0, \ \text{in } \ \mathbb{R}^N, \end{cases} \tag{4.1}$$

which is arising in the modeling of polymeric.

The strategy for giving a meaning to this equation is then a combination of the existence and uniqueness for solutions. Let us consider it now.

Firstly, it is noted that if the viscosity term vanishes, then Theorem 2.1 tells that if

$$u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N) \cap L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3})) \\ \cap L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3})),$$

there exists a unique

$$u(t,x) \in L^\infty([0,T]; (L^1 \cap L^\infty)(\mathbb{R}^N)) \ \cap L^\infty([0,T]; L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}))) \\ \cap L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))).$$

solution to (4.1), under the assumptions (H_1)–(H_9) on b. But, in present, we wish to the solution u have better properties under the existence of the viscosity term if the vector field b as before. Let us discuss it now.

Theorem 4.1. Let $b = (b_1, b_2, b_3)$ be as in Theorem 2.1 but replacing (H_7) by $(I_7) : b_3 = b_3(x) \in L^2_{x,loc}(\mathbb{R}^N)$. We presume, further, that

$$\frac{\partial b_3^i}{\partial x_3^j} + \frac{\partial b_3^j}{\partial x_3^i} \geq cId \ \text{ uniformly definite for } 1 \leq i,j \leq N_3, \qquad (Q_1)$$

in the sense of symmetric matrices, where b_3^j and x_3^j denote the jth component of vector field b_3 and variable x_3. Let

$$u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N) \cap L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3})) \\ \cap\ L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3})),$$

then there exists a unique

$$u(t,x) \in L^\infty([0,T]; (L^1 \cap L^\infty)(\mathbb{R}^N)) \cap L^\infty([0,T]; L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}))) \\ \cap L^\infty([0,T];\ L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))) \\ \cap L^2([0,T];\ L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^1_{x_3}(\mathbb{R}^{N_3}))).$$

solving (4.1) associated with the initial condition $u(t=0,\cdot) = u_0$. Besides, we also have:

$$t^{\frac{1}{2}} u \in L^2([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^2_{x_3}(\mathbb{R}^{N_3}))) \cap L^\infty([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^1_{x_3}(\mathbb{R}^{N_3})))$$

We divide the proof into two steps: the existence and uniqueness part, the regularity part. The regularity part is being the central issue, we demonstrate it last. We begin by showing the existence and uniqueness of solutions in L^2 space, then by regularization to get the L^1 regularity. Now let us give some details.

Proof. Step 1 $L^2 - theory$

• an a priori estimate for tentative solution u

Multiply u to the both hand sides of identity (4.1) and integrate over the spatial variable over $\mathbb{R}^N$, this procedure yields

$$\frac{1}{2}\frac{d}{dt}\int u^2 + \frac{1}{2}\int |\nabla_{x_3} u|^2 = 0, \qquad (4.2)$$

by virtue of integration by parts formula and conditions (H_3), (H_6) and (H_9).

By applying Gronwall's lemma, we deduced that

$$u(t,x) \in L^2([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^1_{x_3}(\mathbb{R}^{N_3}))) \cap L^\infty([0,T]; L^2(\mathbb{R}^N))$$

for $u_0 \in L^2(\mathbb{R}^N)$.

On the other hand, $u_0 \in L^\infty(\mathbb{R}^N)$, the maximum principle also implies $u \in L^\infty([0,T] \times \mathbb{R}^N)$, thus

$$u(t,x) \in L^\infty([0,T]; L^2 \cap L^\infty(\mathbb{R}^N)) \cap L^2([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^1_{x_3}(\mathbb{R}^{N_3}))).$$

• regularization and existence of L^2 weak solutions

Here we call u is a solution of (4.1) with initial value u_0 if the following identity holds:

$$\int_0^T dt \int_{\mathbb{R}^N} dx u \varphi_t + \int_{\mathbb{R}^N} u_0 \varphi(0,x) dx + \int_0^T dt \int_{\mathbb{R}^N} dx u \mathrm{div}(\varphi b) - \frac{1}{2} \int_0^T dt \int_{\mathbb{R}^N} dx u \Delta_{x_3} \varphi = 0, \tag{4.3}$$

for any $\varphi \in \mathcal{D}([0, \mathrm{T}) \times \mathbb{R}^N)$.

Approximate vector field b by $b^\varepsilon \in \mathcal{D}(\mathbb{R}^N)$, then use the first step, by virtue of compactness method, it is easy to get the existence and uniqueness and we skip it.

Step 2 regularity

Now in this part, we will prove the solution u indeed has good properties, precise speaking:

$$u(t,x) \in L^\infty([0,T]; (L^1 \cap L^\infty)(\mathbb{R}^N)) \cap L^\infty([0,T]; L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}))) \cap L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))).$$

For this purpose, firstly, let us check that

$$t^{\frac{1}{2}} u \in L^2([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^2_{x_3}(\mathbb{R}^{N_3}))) \cap L^\infty([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^1_{x_3}(\mathbb{R}^{N_3}))).$$

Denote u^i by $\frac{\partial}{\partial x_3^i} u$, then multiplying u^i by $t^{\frac{1}{2}}$ first and differentiating it with t next, we get

$$\partial_t (t^{\frac{1}{2}} u^i) = \frac{1}{2} t^{-\frac{1}{2}} u^i - b \cdot \nabla (t^{\frac{1}{2}} u^i) - \frac{\partial b_3^j}{\partial x_3^i} (t^{\frac{1}{2}} u^j) + \frac{1}{2} \Delta_{x_3} (t^{\frac{1}{2}} u^i). \tag{4.4}$$

By multiplying by $t^{\frac{1}{2}} u^i \xi_k$ and summing over i and integrating over the space, we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla_{x_3} (t^{\frac{1}{2}} u)|^2 = \frac{1}{2} \int |\nabla_{x_3} u|^2 - \int \frac{\partial b_3^j}{\partial x_3^i} (t^{\frac{1}{2}} u^i)(t^{\frac{1}{2}} u^j) - \frac{1}{2} \int |\nabla^2_{x_3,x_3} (t^{\frac{1}{2}} u)|^2 \tag{4.5}$$

by making use of integration by parts formula and letting k take to infinity, where the repeated the indices will be summed.

By virtue of (Q_1), it follows that

$$\frac{1}{2} \frac{d}{dt} \int |\nabla_{x_3} (t^{\frac{1}{2}} u)|^2 + \frac{1}{2} \int |\nabla^2_{x_3,x_3} (t^{\frac{1}{2}} u)|^2 \leq \frac{1}{2} \int |\nabla_{x_3} u|^2 + c \int |\nabla_{x_3} (t^{\frac{1}{2}} u)|^2. \tag{4.6}$$

The Grönwall lemma yields that

$$t^{\frac{1}{2}} u \in L^2([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^2_{x_3}(\mathbb{R}^{N_3}))) \cap L^\infty([0,T]; L^2_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; H^1_{x_3}(\mathbb{R}^{N_3}))).$$

The similar discussion argument also tells us that $u \in L^1$. Now, it is time for us to check the remains.

Choose regular kernel ρ_{α_3} as in Lemma 2.1 with space variable x_3, by regularizing equation

$$\partial_t u(x) + (b_1(x_1), b_2(x_1, x_2), b_3(x_1, x_2, x_3)) \cdot \nabla u(x) - \frac{1}{2}\Delta_{x_3} u(x) = 0$$

to variable x_3 by ρ_{α_3}, we derive

$$\begin{aligned} [b_3 \cdot \nabla_{x_3}, \rho_{\alpha_3}](u) - \frac{1}{2}[\Delta_{x_3}, \rho_{\alpha_3}](u) &= \partial_t u_{\alpha_3} + b_1(x_1) \cdot \nabla_{x_1} u_{\alpha_3} + b_2 \cdot \nabla_{x_2} u_{\alpha_3} \\ &\quad + b_3(x_1, x_2, x_3) \cdot \nabla_{x_3} u_{\alpha_3} - \frac{1}{2}\Delta_{x_3} u_{\alpha_3}, \quad (4.7) \end{aligned}$$

where

$$[b_3 \cdot \nabla_{x_3}, \rho_{\alpha_3}](u) = b_3 \cdot \nabla_{x_3} u_{\alpha_3} - (b_3 \cdot \nabla_{x_3} u) * \rho_{\alpha_3} =: \varepsilon^1_{\alpha_3},$$

$$\frac{1}{2}[\Delta_{x_3}, \rho_{\alpha_3}](u) = \frac{1}{2}\Delta_{x_3} u_{\alpha_3} - \frac{1}{2}\Delta_{x_3} u * \rho_{\alpha_3} =: \varepsilon^2_{\alpha_3}$$

and for simplicity we denote $u * \rho_{\alpha_3}$ by u_{α_3}.

It is clear that

$$\varepsilon^1_{\alpha_3} \to 0 \ \text{ in } \ L^\infty([0,T]; L^1_{loc}(\mathbb{R}^N)), \quad \text{as} \quad \alpha_3 \to 0,$$

$$\varepsilon^2_{\alpha_3} \to 0 \ \text{ in } \ L^\infty([0,T] \times \mathbb{R}^{N_1+N_2}; \dot{H}^{-1}(\mathbb{R}^{N_3})), \quad \text{as} \quad \alpha_3 \to 0.$$

Let ξ_k be a cut off function to variable x_3 as in Theorem 2.1, then multiply (4.7) by $\xi_k(x_3)$ and integrating over x_3 space, we derive

$$\begin{aligned} &\int_{\mathbb{R}^{N_3}} (\varepsilon^1_{\alpha_3} + \varepsilon^2_{\alpha_3})\xi_k(x_3) \\ = \ & \frac{\partial}{\partial t}\int_{\mathbb{R}^{N_3}} u_{\alpha_3}\xi_k(x_3) + b_1(x_1) \cdot \nabla_{x_1} \int_{\mathbb{R}^{N_3}} u_{\alpha_3}\xi_k(x_3) - \frac{1}{2}\int_{\mathbb{R}^{N_3}} \Delta_{x_3} u_{\alpha_3}\xi_k(x_3) \\ & + b_2(x_1, x_2) \cdot \nabla_{x_2} \int_{\mathbb{R}^{N_3}} u_{\alpha_3}\xi_k(x_3) + \int_{\mathbb{R}^{N_3}} b_3(x) \cdot \nabla_{x_3} u_{\alpha_3}\xi_k(x_3). \qquad (4.8) \end{aligned}$$

Let α_3 go to zero first, k tend to infinity next, by applying integration by parts formula, we deduce

$$\begin{cases} \partial_t v(t, x_1, x_2) + (b_1(x_1), b_2(x_1, x_2) \cdot \ \nabla_{x_1,x_2} v(t, x_1, x_2) = 0, \text{ in } (0,T) \times \mathbb{R}^{N_1+N_2}, \\ v(t = 0, \ \cdot, \ \cdot) = v_0, \text{ in } \mathbb{R}^{N_1+N_2}, \end{cases} \qquad (4.9)$$

where

$$v_0 = \int_{\mathbb{R}^{N_3}} u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^{N_1+N_2}) \cap L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2}(\mathbb{R}^{N_2})) \text{ and } v = \int_{\mathbb{R}^{N_3}} u(t,x)dx_3.$$

By Theorem 2.1(now $b_3 = 0$), we get

$$v(t, x_1, x_2) \in L^\infty([0,T]; (L^1 \cap L^\infty)(\mathbb{R}^{N_1+N_2})) \ \cap L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2}(\mathbb{R}^{N_2}))).$$

Equivalently,

$$\begin{aligned} u(t,x) \in L^\infty([0,T]; (L^1 \cap L^\infty)(\mathbb{R}^N)) \ &\cap L^\infty([0,T]; L^\infty_{x_1,x_2}(\mathbb{R}^{N_1+N_2}; L^1_{x_3}(\mathbb{R}^{N_3}))) \\ &\cap L^\infty([0,T]; L^\infty_{x_1}(\mathbb{R}^{N_1}; L^1_{x_2,x_3}(\mathbb{R}^{N_2+N_3}))). \end{aligned}$$

Remark 4.2. (1) Our result is also mutatis mutandis in the time dependent case $b = b(t,x)$ when we allow an L^1 dependent with respect to time. The concrete form one may see Remark 2.2 in Section 2, but replacing $(H_{7'})$ by $b_3 = b_3(t, x_1, x_2, x_3) \in L^1([0,T]; L^2_{loc}(\mathbb{R}^N))$;

(2) Similarly, in the case of divergence is controlled in the L^∞ norm but not vanish, the above conclusion is also valid.

(3) For more details in Fokker-Planck equations, one can consult [8], [10] and the references cited therein.

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