Electronic Journal of Mathematical Analysis and Applications
Vol. 2(1) Jan. 2014, pp. 72-84.
ISSN: 2090-792X (online)
http://fcag-egypt.com/Journals/EJMAA/

ON THE EXISTENCE, MULTIPLICITY, AND NONEXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

LI HONG BAI

Abstract. In this paper, we are concerned with the existence, multiplicity and nonexistence of positive ω-periodic solutions of the following systems of second order differential equations

$$\begin{cases} x''(t) + A(t,x)x(t) = \lambda F(x(t)), \quad t \in (0,\omega), \\ x(0) = x(\omega), \quad x'(0) = x'(\omega), \end{cases}$$

where $x = (x_1, \cdots, x_n)^T$, $A(t,x)$=diag$[a_1(t,x_1), \cdots, a_n(t,x_n)]$, $F(t,x) = [f_1(t,x), \cdots, f_n(t,x)]^T$, and $\lambda > 0$ is a positive parameter. The proof of our main results is based upon fixed point theorem.

1. Introduction and main results

Due to a wide range of applications in physics and engineering, second order periodic boundary value problems have been investigated by many authors, see([1]-[10]) and the references therein.

In [2], the following systems of second order singular differential equations

$$x''(t) + a(t)x(t) = f_1(t,x,y) \tag{1}$$

$$y''(t) + a(t)y(t) = f_2(t,x,y) \tag{2}$$

are studied. where the nonlinear terms $f_1(t,x,y)$, $f_2(t,x,y)$ have singularity near (0,0). Suppose that $a_i(t) \in \Lambda^+ \cup \Lambda^-$ with $i = 1,2$(see[1]). Under some additional assumptions about growth condition on the nonlinearity f, the authors obtained two different positive solutions of equation1 and 2.

In [3], the following systems of second order differential equations

$$x'' + m^2 x = \lambda G(t)F(x) \tag{3}$$

with the periodic boundary conditions

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi) \tag{4}$$

2000 *Mathematics Subject Classification.* 34A12.
Key words and phrases. Differential equations, stability, existence, uniqueness.
Submitted Nov. 4, 2012. Published July 1, 2013.

are considered, where $m \in (0, \frac{1}{2})$ is a constant, $x = (x_1, \cdots, x_n)^T$, $G(t)$=diag$[g_1(t), \cdots, g_n(t)]$, $F(t,x) = [f_1(x), \cdots, f_n(x)]^T$, and $\lambda > 0$ is a positive parameter, by means of the Krasnosel'skii fixed point theorem, the authors established the existence , multiplicity and nonexistence of positive periodic solutions of (3).

In [4] the following systems of second order differential equations are considered

$$x'' + A(t)x = \lambda f(t,x) \tag{5}$$

with the periodic boundary conditions

$$x(0) = x(1), \quad x'(0) = x'(1) \tag{6}$$

where $x = (x_1, \cdots, x_n)^T$, $A(t)$=diag$[a_1(t), \cdots, a_n(t)]$, $f(t,x) = [f_1(t,x), \cdots, f_n(t,x)]^T$, and $\lambda > 0$ is a positive parameter, the authors established the existence, multiplicity, nonexistence of positive periodic solutions of (5) by Krasnosel'skii fixed point theorem on the compression and expression of cones.

In fact, almost all papers about second order periodic boundary value problems, for example ([1]-[9], [11]-[13]) and the references therein, the coefficient $a(\cdot)$ appeared in equations dependent only on time t. But in practical applications, the coefficient a dependent on t and x is more meaningful.

Inspired by the above works, A natural question is what would happen if the coefficient $a(t)$ in [3] is replaced by $a(t,x)$?

In order to answer this question, By using fixed point index theory, we consider more general second order differential systems

$$x''(t) + A(t,x)x(t) = \lambda F(x(t)), \quad t \in (0, \omega) \tag{7}$$

with the periodic boundary conditions

$$x(0) = x(\omega), \quad x'(0) = x'(\omega) \tag{8}$$

where $x = (x_1, \cdots, x_n)^T$, $A(t,x)$=diag$[a_1(t,x_1), \cdots, a_n(t,x_n)]$, $F(t,x) = [f_1(t,x) \cdots, f_n(t,x)]^T$, and $\lambda > 0$ is a positive parameter. We shall show that the number of $\omega-$periodic solution of (7) can be determined by the asymptotic behaviors of $\frac{F(x)}{x}$ at zero and infinity.

Let us to fix some notations to be used in the following: Given $a \in L'(0,\omega)$, we write $a \succ 0$ if a ≥ 0, for a.e. $t \in [0,\omega]$, and it is positive in a set of positive measure. The usual L^p -norm is denoted by $\|\cdot\|_{L^p}$. The conjugate exponent of p is denoted by p^*, $\frac{1}{p} + \frac{1}{p^*} = 1$. $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$, and for any $u = (u_1, \cdots, u_n) \in \mathbb{R}$, $\|u\| = \sum_{i=1}^n |u_i|$.

Our assumptions for this paper are:

(H1) For $i = 1, \cdots n$, $a_i(\cdot,\cdot) \in C([0,\omega] \times \mathbb{R}_+, \mathbb{R}_+)$ is a $\omega-$periodic functions with respect to the first variable, and there exist two ω-periodic functions $a_1(\cdot)$, $a_2(\cdot) \in C(\mathbb{R}, \mathbb{R}^+)$, such that

$$0 < a_1(t) \leq \frac{a_i(t,u_i)u_i - a_i(t,v_i)v_i}{u_i - v_i} \leq a_2(t), \quad t \in \mathbb{R}, \quad u_i, v_i \in \mathbb{R} \text{ with } u_i \neq v_i. \tag{9}$$

Furthermore, $\|a_2\|_{L^p} < K(2p^*, \omega)$,

$$K(p^*,\ \omega) = \begin{cases} \frac{2\pi}{p^* \omega^{1+\frac{2}{p^*}}} (\frac{2}{2+p^*})^{1-\frac{2}{p^*}} (\frac{\Gamma(\frac{1}{p^*})}{\Gamma(\frac{1}{2}+\frac{1}{p^*})})^2, & \text{if} \quad 1 \leq p^* < \infty, \\ \frac{4}{\omega}, & \text{if} \quad p^* = \infty, \end{cases}$$

where Γ is the Gamma function, (See [1]).

(H2) $f_i(t,x) : [0,\omega] \times \mathbb{R}^n_+ \to [0,\infty)$ is continuous, $f_i(t,x) > 0$ if $\|x\| > 0$, $i = 1,\cdots,n$.

Remark 1. It is easy to see that (9) is weaker than the condition $0 < (a(t,u)u)_u < a_2(t), \|a_2\|_{L^p} < K(2p^*,\omega)$, which was used by Torres and Zhang in [10].

In order to state our results, we introduce the notations

$$f_i^0 = \lim_{u\to 0}\frac{f(u)}{\|u\|},\quad f_i^\infty = \lim_{u\to\infty}\frac{f(u)}{\|u\|},\ u\in\mathbb{R}^n_+,\quad i=1,\cdots,n,\ \text{uniformly in t},$$
$$F_0 = \max_{i=1,\cdots,n}\{f_i^0\},\quad F_\infty = \max_{i=1,\cdots,n}\{f_i^\infty\},$$
$$i_0 = \text{number of zeros in the set } \{F_0,\ F_\infty\},$$
$$i_\infty = \text{number of infinities in the set } \{F_0,\ F_\infty\}.$$

Our main results are:

Theorem 1. Assume that (H1) and (H2) hold.
(a) If $F_0 = 0$ and $F_\infty = \infty$, then for all $\lambda > 0$ equations (7) has one positive solution. s (b) If $F_0 = \infty$ and $F_\infty = 0$, then for all $\lambda > 0$ equations (7) has one positive solution.

Theorem 2. Assume that (H1) and (H2) hold.
(a) If $i_0 = 1$ or 2, then there exist $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ equations (7) has i_0 positive solutions.
(b) If $i_\infty = 1$ or 2, then there exist $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ equations (7) has i_∞ positive solutions.
(c) If $i_\infty = 0$, then there exist $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ equations (7) has no positive solution.
(d) If $i_0 = 0$, then there exist $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ equations (7) has no positive solution.

Theorem 3. Assume that (H1) , (H2) and (H3)hold and $i_0 = i_\infty = 0$. If
$$\frac{1}{\sigma P\max\{F_0,F_\infty\}} < \lambda < \frac{1}{N\min\{F_0,F_\infty\}},$$
equations (7) has one periodic solution.

Remark 3. In first-order case, by using fixed point index theory, Wang [14] obtained the existence, multiplicity, of positive periodic solutions, but in second-order case, relatively little results are obtained in the present literatures.

Remark 4. systems(1-11) are special cases of system (7), so our results generalize the corresponding results in ([2]-[4]).

2. Preliminaries

Lemma 1.([16], [17]). Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tu \neq u$ for $u \in \partial K_r = \{u \in K : \|u\| = r\}$.
(i) If $\|Tu\| > \|u\|$ for $u \in \partial K_r$, then
$$i(T, K_r, K) = 0.$$

(ii) If $\|Tu\| < \|u\|$ for $u \in \partial K_r$, then

$$i(T, K_r, K) = 1.$$

Definition 1. [10]. A function $\alpha \in C^2((0,\omega)) \cap C^1([0,\omega])$ is a lower solution of the periodic problem

$$u''(t) + a_i(t,u)u(t) = 0, \quad t \in (0,\omega) \tag{10}$$

with the periodic boundary conditions

$$u(0) = u(\omega), \; u'(0) = u'(\omega) \tag{11}$$

if

$$\alpha''(t) + a_i(t,\alpha)\alpha(t) \geq 0, \quad t \in (0,\omega) \tag{12}$$

A function $\beta \in C^2((0,\omega)) \cap C^1([0,\omega])$ is an upper solution of the periodic problem (10) if

$$\begin{cases} \beta''(t) + a_i(t,\beta)\beta(t) \leq 0, \quad t \in (0,\omega) \\ \beta(0) = \beta(\omega), \; \beta'(0) \leq \beta'(\omega). \end{cases}$$

Lemma 3. [10]. Let us assume the existence of a couple of lower and upper solutions α and β such that $\beta \leq \alpha$. Suppose that there exists a function $\phi \in L^1(0,\omega)$ such that $\phi \succ 0$ and

$$a(t,u)u - a(t,v)v \leq \phi(t)(v-u)$$

for a.e. $t \in (0,\omega)$ and all $\beta(t) \leq u \leq v \leq \alpha(t)$. Then, if

$$\|\phi\|_{L^p} \leq K(2p^*, \omega)$$

for some $p \in [1,\infty)$, equtions (10) has a solution $x \in [\beta, \alpha]$

Lemma 4. Assume (H1) holds, then for $h(\cdot) \in L^1[0,\omega]$ and $h(t) \geq 0$, second order differential equation

$$u''(t) + a(t,u)u(t) = h(t), \quad t \in (0,\omega) \tag{13}$$

has a unique positive solution.

Proof. From [1], we know that for $h(\cdot) \in L^1[0,\omega]$ and $h(t) \geq 0$ on $[0,\omega]$, the linear problem

$$u_1''(t) + a_1(t)u_1(t) = h(t), \quad t \in (0,\omega) \tag{14}$$

exists the unique positive solution

$$u_1(t) = \int_0^\omega G_1(t,s)h(s)ds, \quad t \in [0,\omega].$$

Similarly, the linear problem

$$u_2''(t) + a_2(t)u_2(t) = h(t), \quad t \in (0,\omega) \tag{15}$$

exists the unique positive solution

$$u_2(t) = \int_0^\omega G_2(t,s)h(s)ds, \quad t \in [0,\omega],$$

where $G_i(t,s), i = 1,2$ denote the Green's function of linear problem (14) and (15).From [1], we know that $G_i(t,s) > 0, t,s \in [0,\omega], i = 1,2$.

Next, we will check that u_1, u_2 are lower and upper solution of (13), respectively.

In fact, we have

$$\begin{cases} u_1''(t) + a(t, u_1)u_1(t) - h(t) \geq u_1''(t) + a_1(t)u_1(t) - h(t) = 0, \quad t \in (0, \omega), \\ u_1(0) = u_1(\omega),\ u_1'(0) \geq u_1(\omega). \end{cases}$$

So u_1 is a lower solution of (13), similarly, we can check u_2 is an upper solution of (13).

Combining (H1) and Lemma 3, we can conclude that there exist a solution u of ((13)), such that

$$u_2(t) \leq u(t) \leq u_1(t), \qquad t \in [0, \omega] \tag{16}$$

As for the uniqueness of the periodic solution, we note that if x, y are ω-periodic solution of ((13)), then $z = x - y$ is a ω-periodic function satisfying

$$z''(t) + a(t, x)x(t) - a(t, y)y(t) = 0, \qquad t \in (0, \omega),$$

according to (H1), there exists ω-periodic function $c(\cdot) \in C(\mathbb{R}, [0, \infty))$ and $a_1(t) \leq c(t) \leq a_2(t)$ for all $t \in [0, \omega]$, such that z also satisfying

$$z''(t) + c(t)z = 0, \qquad t \in (0, \omega) \tag{17}$$

So, $\|c\|_{L^p} < K(2p^*, \omega)$. By the comparison result for eigenvalue (see Theorem 4.2 of [10]), we get equation (17) has only the trivial ω-periodic solution, which implies that $x \equiv y$. $\square$

Hence, the Green's function of (10) exists, and let $G_u(t, s)$ be the Green's function of (10), let us estimate the range of $G_u(t, s)$.

Lemma 5. Assume (H1) holds, then

$$G_2(t, s) \leq G_u(t, s) \leq G_1(t, s), \quad (t, s) \in [0, \omega] \times [0, \omega].$$

Proof.. We get

$$\int_0^\omega G_2(t, s)h(s)ds \leq \int_0^\omega G_u(t, s)h(s)ds \leq \int_0^\omega G_1(t, s)h(s)ds,\ t \in [0, \omega].$$

Next, we only show that $G_u(t, s) \leq G_1(t, s)$. The other case can be treated similarly.

Suppose on the contrary that there exists $(t_0, s_0) \in [0, \omega] \times (0, \omega)$, such that $G_u(t_0, s_0) > G_1(t_0, s_0)$.

Let

$$h(t) = \begin{cases} 0, \quad 0 \leq t \leq s_0 - \epsilon, \\ t - s_0 + \epsilon, \quad s_0 - \epsilon \leq t \leq s_0, \\ s_0 + \epsilon - t, \quad s_0 \leq t \leq s_0 + \epsilon, \\ 0, \quad s_0 + \epsilon \leq t \leq \omega. \end{cases}$$

Then, $h \in L^1[0, \omega]$ and $h(t) \geq 0$ in $[0, \omega]$.

By the continuity of $G_u(t, s)$ and $G_1(t, s)$ with respect to s, then there exists $\epsilon > 0$, such that

$$G_u(t_0, s) - G_1(t_0, s) > 0, \quad s \in (s_0 - \epsilon, s_0 + \epsilon).$$

Thus

$$\begin{aligned} u(t_0)-u_1(t_0) &= \int_0^\omega (G_u(t_0,s)-G_1(t_0,s))h(s)ds \\ &= \int_0^{s_0-\epsilon} (G_u(t_0,s)-G_1(t_0,s))h(s)ds + \int_{s_0-\epsilon}^{s_0+\epsilon} (G_u(t_0,s)-G_1(t_0,s))h(s)ds \\ &+ \int_{s_0+\epsilon}^{\omega} (G_u(t_0,s)-G_1(t_0,s))h(s)ds > 0, \end{aligned}$$

this contradicts.

when $s_0 = 0$ or ω, by the similar method, we can get contradiction.

□

Let

$$M = \max_{t,s\in[0,\omega]} G_1(t,s), \quad m = \min_{t,s\in[0,\omega]} G_2(t,s) \tag{18}$$

so, we have $0 < m \le G_u(t,s) \le M, \quad t,s \in [0,\omega]$.

Let X be the Banach space $C[0,\omega]^n$, with $\|u\| = \sum_{i=1}^n \sup_{t\in[0,\omega]} |u(t)|,\ u = (u_1,\cdots,\ u_n) \in X$, for $u \in X$ or $\mathbb{R}_+^n$, $\|u\|$ denotes the norm of u in X or $\mathbb{R}_+^n$, respectively.

$$A_i = \min_{0\le t,s\le\omega} G_i(t,s) > 0, \quad B_i = \max_{0\le t,s\le\omega} G_i(t,s) > 0, \quad 1 > \sigma_i = \frac{A_i}{B_i} > 0.$$

Define K be a cone in X by

$$K = \{u = (u_1,\cdots,u_n) \in X :\ u_i(t) \ge 0, t\in[0,\omega],\ \min_{t\in[0,\omega]} u_i(t) \ge \sigma_i \sup_{t\in[0,\omega]} |u_i(t)|,\ i = 1,\cdots,n\}.$$

Let the map $T_\lambda : K \to X$ be a map with the componenuts $(T_\lambda^1,\cdots,T_\lambda^n)$, which are defined by

$$T_\lambda^i u(t) = \int_0^\omega \lambda G_u^i(t,s) f_i(s,u(s))ds, \quad 0\le t\le\omega,\ i = 1,\cdots,n.$$

Lemma 6. Assume that (H1) and (H2) hold. Then $T_\lambda(K) : K \to K$ is compact and continuous.

Proof. It is not difficult to check that for $u \in K$, we have

$$\begin{aligned} T_\lambda^i u(t) &= \lambda \int_0^\omega G_u^i(t,s) f_i(s,u(s))ds \\ &\ge \lambda \frac{A_i}{B_i} \int_0^\omega G_u^i(t,s) f(t,u(t))ds \\ &\ge \lambda\sigma_i \int_0^\omega \max_{0\le t,s\le\omega} G_u^i(t,s) f(t,u(s))ds \\ &\ge \sigma_i \sup_{0\le t\le\omega} T_\lambda^i u(t), \end{aligned}$$

Consequently, we get $T_\lambda^i(K) \to K,\ i = 1,\cdots,n,$ and it is easy to show that $T_\lambda : K \to K$ is compact and continuous. □

Obviously, the equations (7) is equivalent to the fixed point problem of T_λ in K, Let

$$A = \min_{1\le i\le n} A_i,\ B = \max_{1\le i\le n} B_i,\ \sigma = \min_{1\le i\le n} \sigma_i,\ P = \sigma A,\ N = nB$$

Lemma 7. Assume that (H1)-(H2) hold. Let $u = ((u_1(t), \cdots, u_n(t))) \in K$, and $\eta > 0$,if there exists a component f_i of f such that

$$f_i(t, u(t)) \ge \eta \sum_{i=1}^{n} u_i(t), \quad t \in [0, \omega],$$

then $\|T_\lambda u\| \ge \lambda P\eta\|u\|$.

Proof. From the definition of $T_\lambda u$,it follow that

$$\begin{aligned} T^i_\lambda u(t) &= \lambda \int_0^\omega G^i_u(t,s) f_i(s, u(s)) ds \\ &\ge \lambda \frac{A_i}{B_i} \int_0^\omega G^i_u(t,s) f(t, u(t)) ds \\ &\ge \lambda\sigma_i \int_0^\omega \max_{0\le t,s\le\omega} G^i_u(t,s) f(t, u(s)) ds \\ &\ge \sigma_i \sup_{0\le t\le 1} T^i_\lambda u(t), \end{aligned}$$

Lemma 8. Assume that (H1) and (H2) hold. For any $r > 0, u = (u_1(t), \cdots, u_n(t)) \in \partial\Omega_r$, if there exists $\epsilon > 0$, such that

$$f_i(t, u(t)) \le \epsilon \sum_{i=1}^{n} u(t), \qquad t \in [0, \omega],\ i = 1, \cdots, n,$$

then

$$||T_\lambda u|| \le \lambda\epsilon N||u||.$$

Proof. From the definition of T, for $u \in \partial\Omega_r$, we have

$$\begin{aligned} \|T_\lambda u\| &\le \lambda \sum_{i=1}^{n} \int_0^\omega f(s, u(s)) ds \\ &\le \lambda \sum_{i=1}^{n} B_i \int_0^\omega \epsilon \sum_{i=1}^{n} u_i(s) ds \\ &\le \lambda \sum_{i=1}^{n} B_i ||u|| \le \lambda\epsilon N\|u\|. \end{aligned}$$

Lemma 9. Assume that (H1)-(H2) hold. If $u \in \partial\Omega_r,\ r > 0$, then

$$||T_\lambda u|| \ge \lambda \frac{P}{\sigma} m(r).$$

where $m_r = \min\{f_i(t, u) : u \in \mathbb{R}^n_+,\ \sigma r \le \|u\| \le r,\ t \in [0, \omega],\ i = 1, \cdots, n\} > 0$

Proof. Since $f(u(t)) \ge m(r)$ for $t \in [0, \omega]$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 7. □

Lemma 10. Assume that (H1), (H2), and (H3) hold. If $u \in \partial\Omega_r,\ r > 0$, then

$$||T_\lambda u|| \ge \lambda N M(r).$$

where $M_r = \min\{f_i(t,u) : u \in \mathbb{R}^n_+,\ \sigma r \le \|u\| \le r,\ t \in [0,\omega],\ i = 1,\cdots,n\} > 0$

Proof. Since $f(u(t)) \le M(r)$ for $t \in [0,\omega]$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 8. □

3. Proof of Theorem 1

Proof. Part (a). $F_0 = 0$ implies that $f_i^0 = 0,\ i = 1,\cdots,n$. So we can choose a number $r_1 > 0$ such that

$$f_i(t,u) \le \varepsilon\|u\|,\ u \in \mathbb{R}^n_+,\ \|u\| \le r_1,\ i = 1,\cdots,n,\ t \in [0,\omega],$$

where the constant $\epsilon > 0$ satisfies

$$\lambda\varepsilon N < 1$$

Therefore by Lemma 7 we have

$$\|T_\lambda u\| \le \|u\|,\ u \in \partial\Omega_{r_1} \tag{19}$$

On the other hand, since $F_\infty = \infty$, there is a component f_i of f such that $f_i^\infty = \infty$. Therefore, there exists a constant $H > r_1 > 0$ such that

$$f_i(t,u) \ge \eta\|u\|,\ u = (u_1,\cdots,u_n) \in \mathbb{R}^n_+,\ \|u\| \ge H,\ t \in [0,\omega],$$

where $\eta > 0$ is chosen such that

$$\lambda P\eta > 1.$$

Set $r_2 = \max\{2r_1, \frac{H}{\sigma}\}$, If $u = (u_1,\cdots,u_n) \in \partial\Omega_{r_2}$, then

$$\min_{0\le t\le\omega} \sum_{i=1}^{n} u_i(t) \ge \sigma\|u\| = \sigma r_2 \ge H,$$

which implies that

$$f_i(t,u(t)) \ge \eta u_i(t),\ t \in [0,\omega],$$

It follows from Lemma 6 that

$$\|T_\lambda u\| \ge \lambda P\eta\|u\| > \|u\|,\ u \in \partial\Omega r_2 \tag{20}$$

Now according to (19) and (20), by Lemma 1 we get the fixed point of T_λ on $\Omega_{r_2}\backslash\Omega_{r_1}$.

Part (b). If $F_0 = \infty$, there is a component f_i of f such that $f_i^0 = \infty$. Therefore there exists $r_1 > 0$ such that

$$f_i(t,u) \ge \eta\|u\|,\ u = (u_1,\cdots,u_n) \in \mathbb{R}^n_+,\ \|u\| \le r_1,\ t \in [0,\omega],$$

where $\eta > 0$ is chosen such that

$$\lambda P\eta > 1,$$

If $u = (u_1,\cdots,u_n) \in \partial\Omega_{r_1}$

$$f_i(t,u) \ge \eta\sum_{i=1}^{n} u_i(t),\ t \in [0,\omega],$$

which implies

$$\|T_\lambda u\| \ge \lambda P\eta\|u\| > \|u\|,\ u \in \partial\Omega_{r_1} \tag{21}$$

Next we determine Ω_{r_2}, Since $F_\infty = 0$, we have $f_i^\infty = 0,\ i = 1,\cdots,n$. Therefore, there exists $H > r_1 > 0$, such that

$$f_i(t,u) \le \epsilon\|u\|,\ u = (u_1,\cdots,u_n) \in \mathbb{R}^n_+,\ \|u\| \ge H,\ t \in [0,\omega].$$

where ϵ satisfies

$$\lambda\epsilon N < 1$$

Set $r_2 = \max\{2r_1, \frac{H}{\sigma}\}$. If $u = (u_1, \cdots, u_n) \in \partial\Omega_{r_2}$, then

$$\min_{0\le t\le\omega} \sum_{i=1}^{n} u_i(t) \ge \sigma\|u\| = \sigma r_2 \ge H,\ u \in \partial\Omega_{r_2},\ t \in [0,\omega],$$

which implies that

$$f_i(t, u(t)) \le \epsilon \sum_{i=1}^{n} u_i(t),\ t \in [0,\omega]$$

Thus, by Lemma 7, the following inequality

$$\|T_\lambda u\| \le \lambda N\epsilon\|u\| < \|u\|,\ u \in \partial\Omega_{r_2} \tag{22}$$

holds.

Now according to (21) and (22), by Lemma 1 we get the fixed point of T_λ on $\Omega_{r_2}\backslash\Omega_{r_1}$.

This method can be applied to the case that $f_i(t,x)$ has a singularity near $\mathbf{0}$. We assume

$(H_2') : f_i(t,x) : [0,\omega] \times \mathbb{R}^n_+\backslash\{0\} \to (0,+\infty)$ is continuous, and has a singularity near $\mathbf{0}$, $i = 1, \cdots, n$.

Then we can obtain the analogous result.

Corollary 1. Assume that (H1)and (H_2') hold. If $F_\infty = 0$, then equations (7) has one positive solution.

Proof. In fact, since $f_i(t,u),\ i = 1, \cdots, n$ have singularity near u=0, then

$$\lim_{\|u\|\to 0} \frac{f_i(t,u)}{\|u\|} = \infty, \quad i = 1, \cdots, n,$$

that is $F_0 = \infty$.By part(b) of Theorem 1 we have finished the proof.

4. Proof of Theorem 2

Let $r_1 = 1$, according to Lemma 8, we have

$$\|T_\lambda u\| > \|u\|,\ u \in \partial\Omega_{r_1},\ \lambda > \lambda_0 := \frac{\sigma}{m_1 P}$$

If $F_0 = 0$, then $f_i^\infty = 0,\ i = 1, \cdots, n$. Therefore, there exists $r_2 < r_1$ such that

$$f_i(t,u) \le \epsilon\|u\|,\ u = (u_1, \cdots, u_n) \in \mathbb{R}^n_+,\ \|u\| \le r_2,\ t \in [0,\omega].$$

where the constant ϵ satisfies

$$\lambda\epsilon N < 1,$$

If $u = (u_1, \cdots, u_n) \in \partial\Omega_{r_2}$, then

$$f_i(t, u(t)) \le \epsilon \sum_{i=i}^{n} u_i(t),\ t \in [0,\omega]$$

which implies, according to Lemma 7, that

$$\|T_\lambda u\| \le \lambda N\epsilon\|u\| < \|u\|, \quad u \in \partial\Omega_{r_2} \tag{23}$$

If $F_\infty = 0$, we have $f_i^\infty = 0,\ i = 1, \cdots, n$. Therefore, there exists $H > r_1 > 0$, such that

$$f_i(t,u) \le \epsilon\|u\|,\ u = (u_1, \cdots, u_n) \in \mathbb{R}^n_+,\ \|u\| \ge H,\ t \in [0,\omega].$$

where ϵ satisfies

$$\lambda\epsilon N < 1.$$

Set $r_2 = \max\{2r_1, \frac{H}{\sigma}\}$. If $u = (u_1, \cdots, u_n) \in \partial\Omega_{r_2}$, then $\min_{0\le t\le\omega} \sum_{i=1}^{n} u_i(t) \ge \sigma\|u\| = \sigma r_2 \ge H, u \in \partial\Omega_{r_2},\ t \in [0, \omega]$ which implies that

$$f_i(t, u(t)) \le \epsilon \sum_{i=1}^{n} u_i(t), t \in [0, \omega].$$

Thus, by Lemma 7, the following inequality

$$\|T_\lambda u\| \le \lambda N\epsilon\|u\|,\ u \in \partial\Omega_{r_2} \tag{24}$$

holds.

Thus, in view of Lemma 1, T_λ has a fixed point on $\Omega_{r_1}\backslash\Omega_{r_2}$ or $\Omega_{r_1}\backslash\Omega_{r_2}$ according to $F_0 = 0$, or $F_0 = 0$ or $F_\infty = 0$ respectively . Consequently, equations (7) has one positive solution for $\lambda > \lambda_0$

If $F_0 = F_\infty = 0$, it's easy to see from the above proof that T_λ has a fixed point $u_1(t) \in \Omega_{r_1}\backslash\Omega_{r_2}$ and a fixed point $u_2(t) \in \Omega_{r_3}\backslash\Omega_{r_1}$, such that

$$r_2 < \|u_1\| < r_1 < \|u_2\| < r_3.$$

Therefor equations (7) has two different positive solutions

Let $r_1 = 1$, according to Lemma 9 ,we have

$$\|T_\lambda u\| < \|u\|, u \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0 := \frac{1}{NM_1} \tag{25}$$

If $F_0 = \infty$ there is a component f_i of f such that $f_i^0 = \infty$. Therefore,there exists $r_2 < r_1$ such that

$$f_i(t, u) \ge \eta\|u\|,\ u = (u_1, \cdots, u_n) \in \mathbb{R}_+^\ltimes, \|u\| \le r_1, t \in [0, \omega].$$

where $\eta > 0$ is chosen such that

$$\lambda P\eta > 1$$

$u = (u_1, \cdots, u_n) \in \partial\Omega_{r_1}$

$$f_i(t, u) \ge \eta \sum_i = 1^n u_i(t),\ t \in [0, \omega],$$

which implies

$$\|T_\lambda u\| \ge \lambda P\eta\|u\| > \|u\|,\ u \in \partial\Omega_{r_2} \tag{26}$$

If $F_\infty = \infty$, there is a component f_i of f such that $f_i^\infty = \infty$ Therefore ,there exists a constant $H > r_1 > 0$ such that

$$f_i(t, u) \ge \eta\|u\|,\ u = (u_1, \cdots, u_n) \in \mathbb{R}_+^n, \|u\| \ge H, t \in [0, \omega]$$

where $\eta > 0$ is chosen such that

$$\lambda P\eta > 1.$$

Set $r_2 = \max\{2r_1, \frac{H}{\sigma}\}$, If$u = (u_1, \cdots, u_n) \in \partial\Omega_{r_2}$, then

$$\min_{0\le t\le 1} \sum_i = 1^n u_i(t) \ge \sigma\|u\| = \sigma r_2 \ge H$$

which implies that

$$f_i(t, u(t)) \ge \eta u_i(t),\ t \in [0, \omega]$$

It follows from Lemma 6 that

$$\|T_\lambda u\| \ge \lambda P\eta\|u\| > \|u\|,\ u \in \partial\Omega r_2 \tag{27}$$

thus, in view of Lemma 4,T_λ has a fixed point on $\Omega_{r_1}\backslash\Omega_{r_2}$ or$\Omega_{r_1}\backslash\Omega_{r_2}$ If $F_0 = F_\infty = 0$, it's easy to see from the above proof that T_λ has a fixed point $u_1(t) \in \Omega_{r_1}\backslash\Omega_{r_2}$ and a fixed point $u_2(t) \in \Omega_{r_3}\backslash\Omega_{r_1}$,such that

$$r_2 < \|u_1\| < r_1 < \|u_2\| < r_3.$$

Therefore equations (7) has two different positive solutions

Since $F_0 < \infty$ and $F_\infty < \infty$ then $f_i^0 < \infty, f^\infty < \infty,\ i = 1, \cdots, n$. It is easy to see that there exists an ϵ such that

$$f_i(t, u) \le \epsilon\|u\|, u \in \mathbb{R}_+^n, t \in [0, 1], i = 1, \cdots, n.$$

Assume that $v(t) = (v_1(t), \cdots, v_n(t))$ is one positive solution of equations (7) .We will show this leads to a contradiction for $0 < \lambda < \lambda_0 := \frac{1}{N\epsilon N}$. In fact, for $0 < \lambda < \lambda_0$,since $T_\lambda v(t) = v(t)$, we find $\|v\| = \|T_\lambda v\| = \sum_{i=1}^n \max_{0\le t\le 1} T_\lambda^i v(t) \le \sum_{i=1}^n \lambda B_i \epsilon\|v\| < \|v\|$.

Since $F_0 > 0$ and $F_\infty > 0$,there exist two components f_i and f_j of f and positive numbers c_1, c_2, r_1 and r_2 such that $r_1 < r_2$ and

$$f_i(t, u) \ge c_1\|u\|,\ u \in \mathbb{R}_+^n, \|u\| \le r_1,$$

$$f_j(t, u) \ge c_1\|u\|,\ u \in \mathbb{R}_+^n, \|u\| \le r_2,$$

Let

$$\eta = \min\{c_1, c_2, \min\{\frac{f_j(t, u)}{\|u\|} : u \in \mathbb{R}_+^n\}, r_1\sigma \le \|u\| \le r_2\} > 0.$$

Thus ,we have

$$f_i(t, u) \ge \eta\|u\|, u \in \mathbb{R}_+^n, \|u\| \le r_1 \tag{28}$$

$$f_i(t, u) \ge \eta\|u\|, u \in \mathbb{R}_+^n, \|u\| \ge \sigma r_1 \tag{29}$$

Assume that $v(t) = (v_1(t), \cdots, v_n(t))$ is one positive solution of equations (7), we will show this leads to a contradiction for $\lambda > \lambda_0 := \frac{1}{P\eta}$ In fact ,if $\|v\| \le r_1$, according to (28) ,we have

$$f_i(t, v(t)) \ge \eta \sum_i = 1^n v_i(t),\ t \in [0, \omega].$$

On the other hand, if $\|v\| > r_1$,then

$$\min_{0\le t\le\omega} \sum_{i=1}^n v_i(t) \ge \sigma\|v\| > \sigma r_1$$

which ,together with (29), implies that

$$f_i(t, v(t)) \ge \eta \sum_i = 1^n v_i(t),\ t \in [0, \omega].$$

Since $T_\lambda v(t) = v(t)$ for $t \in [0, \omega]$, it follows from Lemma 6 for $\lambda > \lambda_0$, that

$$\|v\| = \|T_\lambda v\| \ge \lambda P\eta\|v\| > \|v\|,$$

which is a contradiction. Similar to Corollary 1,we have

Assume that (H1)and (H_1') hold

(a) If $0 < F_\infty < \infty$,then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ equations (7) has one positive solution.

(b) If $F_\infty = \infty$ then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ equations (7) has one positive solution.

Without loss of generality, we may assume that $F_\infty \geq F_0$, the other case $F_\infty < F_0$ can be treated similarly. If λ satisfies

$$\frac{1}{\sigma P F_\infty} < \lambda < \frac{1}{N F_0}$$

then there exists an $0 < \epsilon < F_\infty$ such that

$$\frac{1}{\sigma P (F_\infty - \epsilon)} < \lambda < \frac{1}{N (F_0 + \epsilon)}$$

now ,according an $0 < \epsilon < F_0$, there exists $r_1 > 0$ such that

$$f_i(t,u) \leq (, u \in \mathbb{R}^n_+,\ \|u\| \leq r_1,\ i = 1, \cdots, n,\ t \in [0, \omega]$$

Thus,

$$f_i(t,u(t)) \leq (F_0 + \epsilon)\|u\|,\ u \in \partial\Omega_{r_1},\ i = 1, \cdots, n, t \in [0, \omega]$$

Therefore ,we have, by Lemma 7 ,that

$$\|T_\lambda u\| \leq \lambda N (F_0 + \epsilon)\|u\| < \|u\|,\ u \in \partial\Omega_{r_1} \tag{30}$$

On the other hand, there is some i such that $f_i^\infty = F_\infty > 0$. Thus, there exists $H > r_1 > 0$ such that

$$f_i(t,u) \geq (F_\infty - \epsilon)\|u\|,\ u = (u_1, \cdots, u_n) \in \mathbb{R}^n_+,\ \|u\| \geq H, t \in [0, \omega].$$

Set $r_2 = \{2r_1, \frac{H}{\sigma}\}$, Then it follows that

$$\min_{0 \leq t \leq 1} \sum_{j=1}^{n} u_j(t) \geq \sigma\|u\| = \sigma r_2 \geq H,\ u = (u_1, \cdots, u_n) \in \partial\Omega_{r_2},$$

which implies that

$$f_i(t,u(t)) \geq (F_\infty - \epsilon) \sum_{j=1}^{n} u_j(t),\ u \in \partial\Omega_{r_2},$$

In view of Lemma 6 ,we have

$$\|T_\lambda u\| \geq \lambda\sigma P (F_\infty - \epsilon)\|u\|,\ u \in \partial\Omega_{r_2} \tag{31}$$

Hence ,according to (30) and (31), by Lemma 1, T_λ has a fixed point on $\Omega_{r_2} \backslash \Omega_{r_1}$, Consequently, equations (7) has one positive solution. □

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LI HONG BAI
Gansu Construction Vocational Technical College
E-mail address: bailihong0305@126.com