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SOME RESULTS ON $L-$ORDER, $L-$HYPER ORDER AND L^*-ORDER, L^*-HYPER ORDER OF ENTIRE FUNCTIONS DEPENDING ON THE GROWTH OF CENTRAL INDEX

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Abstract. In this paper we discuss $L-$order($L-$lower order), $L-$hyper order($L-$hyper lower order) and L^*-order(L^*-lower order), L^*-hyper order (L^*-hyper lower order) of entire functions with respect to central index and use these to estimate the growth of composite entire functions.

1. Introduction, Definitions and Notations

Let

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

be an entire function in the complex plane $\mathbb{C}$. Let $M(r,f)=\max\limits_{|z|=r}|f(z)|$ denotes the maximum modulus of f on $|z|=r$ and $\mu(r,f)=\max\limits_{n\geq 0}|a_n|\,r^n$ denotes the maximum term of f on $|z|=r$. The central index $\nu(r,f)$ is the greatest exponent m such that $|a_m|\,r^m=\mu(r,f)$. We note that $\nu(r,f)$ is real, non-decreasing function of r.

For $0\leq r<R$,

$$\mu(r,f)\leq M(r,f)\leq \frac{R}{R-r}\mu(r,f)\ \ \{cf.\,[8]\}$$

and

$$\left|a_{\nu(r,f)}\right|r^{\nu(r,f)}=\mu(r,f).$$

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [2, 3, 9, 10]).

The order ρ_f , lower order λ_f and hyper order $\overline{\rho}_f$, hyper lower order $\overline{\lambda}_f$ of an entire function f are defined as follows:

$$\rho_f \;=\; \limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r},\ \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \qquad (1)$$

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and

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r},\ \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \qquad (2)$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Somasundaram and Thamizharasi [7] introduced the notions of $L-$order and $L-$lower order for entire functions, where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a, on the basis of maximum modulus $M(r,f)$ as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}. \qquad (3)$$

Similarly, one can define the $L-$hyper order and $L-$hyper lower order of an entire function f by

$$\overline{\rho}_f^L = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log[rL(r)]} \text{ and } \overline{\lambda}_f^L = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log[rL(r)]}. \qquad (4)$$

The more generalised concept of $L-$order ($L-$lower order) defined by Somasundaram and Thamizharasi [7] are L^*-order (L^*-lower order). Their definitions are as follows:

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}. \qquad (5)$$

Similarly, one can define the L^*-hyper order and L^*-hyper lower order of an entire function f by

$$\overline{\rho}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \overline{\lambda}_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log\left[re^{L(r)}\right]}. \qquad (6)$$

In this paper using the notion of central index, we intend to establish some results relating to the growth properties of composite entire functions on the basis of $L-$order ($L-$lower order), $L-$hyper order ($L-$hyper lower order) and L^*-order (L^*-lower order), L^*-hyper order (L^*-hyper lower order), where $L \equiv L(r)$ is a slowly changing function.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1] and [4, *Theorems* 1.9 and 1.10, *or* 11, Satz 4.3 and 4.4]) Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function, $\mu(r,f)$ be the maximum term, i.e., $\mu(r,f) = \max_{n\geq 0} |a_n| r^n$ and $\nu(r,f)$ be the central index of f. Then

(i) For $a_0 \neq 0$,

$$\log \mu(r,f) = \log|a_0| + \int_0^r \frac{\nu(t,f)}{t} dt,$$

(ii) For $r < R$,

$$M(r,f) < \mu(r,f)\left\{\nu(R,f) + \frac{R}{R-r}\right\}.$$

Lemma 2 [1, 4, 5, 6] If $f(z)$ be an entire function of order ρ_f and $\nu(r,f)$ be the central index of $f(z)$, then

$$\limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r} = \rho_f.$$

Analogously, one can easily show that for lower order λ_f

$$\liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r} = \lambda_f.$$

Lemma 3 Let $f(z)$ be an entire function with $L-$order ρ_f^L and $L-$lower order λ_f^L. If $\nu(r,f)$ be the central index of f, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log\left[rL\left(r\right)\right]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log\left[rL\left(r\right)\right]},$$

where $L \equiv L\left(r\right)$ is a slowly changing function.

Proof. Set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Without loss of generality, we can assume that $|a_0| \neq 0$. By (i) of Lemma 1 we have

$$\log \mu(2r,f) = \log|a_0| + \int_0^{2r} \frac{\nu(t,f)}{t}dt \geq \nu(r,f)\log 2.$$

Using the Cauchy's Inequality, it is easy to see that $\mu(2r,f) \leq M(2r,f)$. Hence

$$\nu(r,f)\log 2 \leq \log M(2r,f) + C,$$

where $C > 0$ is a suitable constant. By this and (3), we get

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log\left[rL\left(r\right)\right]} &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(2r,f)}{\log\left[2rL\left(2r\right)\right]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[rL\left(r\right)\right]} = \rho_f^L. \end{aligned} \tag{7}$$

On the other hand, by (ii) of Lemma 1, we have

$$M\left(r,f\right) < \mu(r,f)\left\{\nu(2r,f)+2\right\} = \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \left\{\nu(2r,f)+2\right\}.$$

Since $\{|a_n|\}$ is a bounded sequence, we have

$$\begin{aligned} \log M(r,f) &\leq \nu(r,f)\log r + \log \nu(2r,f) + C_1 \\ \Rightarrow \log^{[2]} M(r,f) &\leq \log \nu(r,f) + \log^{[2]} \nu(2r,f) + \log^{[2]} r + C_2 \\ \Rightarrow \log^{[2]} M(r,f) &\leq \log \nu(2r,f)\left\{1 + \frac{\log^{[2]} \nu(2r,f)}{\log \nu(2r,f)}\right\} + \log^{[2]} r + C_3, \end{aligned}$$

where $C_j > 0$ with $j \in \{1, 2, 3\}$ are suitable constants. By this and (3), we get

$$\begin{aligned}\rho_f^L &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]} \\ &\leq \limsup_{r\to\infty} \frac{\log \nu(2r,f)}{\log [2rL(2r)]} \\ &= \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log [rL(r)]}. \qquad (8)\end{aligned}$$

From (7) and (8), it follows that

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log [rL(r)]}.$$

Similarly, one can show that

$$\lambda_f^L = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log [rL(r)]}.$$

□

Lemma 4 Let $f(z)$ be an entire function with $L-$hyper order $\overline{\rho}_f^L$ and $L-$hyper lower order $\overline{\lambda}_f^L$. If $\nu(r,f)$ be the central index of f, then

$$\overline{\rho}_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log [rL(r)]} \text{ and } \overline{\lambda}_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log [rL(r)]},$$

where $L \equiv L(r)$ is a slowly changing function.

Proof. Set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Without loss of generality, we can assume that $|a_0| \neq 0$. By (i) of Lemma 1, we have

$$\log \mu(2r,f) = \log |a_0| + \int_0^{2r} \frac{\nu(t,f)}{t} dt \geq \nu(r,f) \log 2.$$

Using the Cauchy's Inequality, it is easy to see that $\mu(2r,f) \leq M(2r,f)$. Hence

$$\nu(r,f) \log 2 \leq \log M(2r,f) + C,$$

where $C > 0$ is a suitable constant. By the above inequality and (4), we get

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log [rL(r)]} &\leq \limsup_{r\to\infty} \frac{\log^{[3]} M(2r,f)}{\log [2rL(2r)]} \\ &= \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log [rL(r)]} = \overline{\rho}_f^L. \qquad (9)\end{aligned}$$

On the other hand, by (ii) of Lemma 1 we have

$$M(r,f) < \mu(r,f) \{\nu(2r,f) + 2\} = \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \{\nu(2r,f) + 2\}.$$

Since $\{|a_n|\}$ is a bounded sequence, we have

$$\begin{aligned} \log M(r,f) &\le \nu(r,f)\log r + \log\nu(2r,f) + C_1 \\ \Rightarrow \log^{[3]} M(r,f) &\le \log^{[2]}\nu(r,f) + \log^{[3]}\nu(2r,f) + \log^{[3]} r + C_2 \\ \Rightarrow \log^{[3]} M(r,f) &\le \log^{[2]}\nu(2r,f)\left[1 + \frac{\log^{[3]}\nu(2r,f)}{\log^{[2]}\nu(2r,f)}\right] + \log^{3]} r + C_3, \end{aligned}$$

where $C_j > 0$ with $j \in \{1,2,3\}$ are suitable constants. By this and (4), we get

$$\begin{aligned} \overline{\rho}_f^L &= \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log[rL(r)]} \\ &\le \limsup_{r\to\infty} \frac{\log^{[2]}\nu(2r,f)}{\log[2rL(2r)]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]}\nu(r,f)}{\log[rL(r)]}. \end{aligned} \tag{10}$$

From (9) and (10), it follows that

$$\overline{\rho}_f^L = \limsup_{r\to\infty} \frac{\log^{[2]}\nu(r,f)}{\log[rL(r)]}.$$

Similarly, we can verify that

$$\overline{\lambda}_f^L = \liminf_{r\to\infty} \frac{\log^{[2]}\nu(r,f)}{\log[rL(r)]}.$$

□

Proceeding similarly as in Lemma 3, one can easily prove the following lemma:

Lemma 5 Let $f(z)$ be an entire function with L^*- order $\rho_f^{L^*}$ and L^*-lower order $\lambda_f^{L^*}$. If $\nu(r,f)$ be the central index of f, then

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log\nu(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log\nu(r,f)}{\log\left[re^{L(r)}\right]},$$

where $L \equiv L(r)$ is a slowly changing function.

Proceeding similarly as in Lemma 4, one can easily prove the following lemma:

Lemma 6 Let $f(z)$ be an entire function with L^*-hyper order $\overline{\rho}_f^{L^*}$ and L^*-hyper lower order $\overline{\lambda}_f^{L^*}$. If $\nu(r,f)$ be the central index of f, then

$$\overline{\rho}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]}\nu(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \overline{\lambda}_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]}\nu(r,f)}{\log\left[re^{L(r)}\right]},$$

where $L \equiv L(r)$ is a slowly changing function.

3. Statement and Proof of main Theorems

In this section we present the main results of the paper.

Theorem 1 Let f and g be two entire functions. Also let $0 < \lambda_{fog}^L \le \rho_{fog}^L < \infty$ and $0 < \lambda_g^L \le \rho_g^L < \infty$. Then

$$\frac{\lambda_{fog}^L}{\rho_g^L} \le \liminf_{r\to\infty} \frac{\log\nu(r,fog)}{\log\nu(r,g)} \le \min\left\{\frac{\lambda_{fog}^L}{\lambda_g^L}, \frac{\rho_{fog}^L}{\rho_g^L}\right\}$$

$$\leq \max\left\{\frac{\lambda^L_{fog}}{\lambda^L_g}, \frac{\rho^L_{fog}}{\rho^L_g}\right\} \leq \limsup_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\rho^L_{fog}}{\lambda^L_g}.$$

Proof. Using Lemma 3 for the entire function g, for arbitrary positive ε and for all sufficiently large values of r we have

$$\log \nu(r, g) \leq (\rho^L_g + \varepsilon) \log [rL(r)] \tag{11}$$

and

$$\log \nu(r, g) \geq (\lambda^L_g - \varepsilon) \log [rL(r)]. \tag{12}$$

Also for a sequence of values of r tending to infinity, we get

$$\log \nu(r, g) \leq (\lambda^L_g + \varepsilon) \log [rL(r)] \tag{13}$$

and

$$\log \nu(r, g) \geq (\rho^L_g - \varepsilon) \log [rL(r)]. \tag{14}$$

Again using Lemma 3 for the composite entire function fog, for arbitrary positive ε and for all sufficiently large values of r we have

$$\log \nu(r, fog) \leq (\rho^L_{fog} + \varepsilon) \log [rL(r)] \tag{15}$$

and

$$\log \nu(r, fog) \geq (\lambda^L_{fog} - \varepsilon) \log [rL(r)]. \tag{16}$$

Also for a sequence of values of r tending to infinity, we get

$$\log \nu(r, fog) \leq (\lambda^L_{fog} + \varepsilon) \log [rL(r)] \tag{17}$$

and

$$\log \nu(r, fog) \geq (\rho^L_{fog} - \varepsilon) \log [rL(r)]. \tag{18}$$

Now from (11) and (16) it follows for all sufficiently large values of r that

$$\frac{\log \nu(r, fog)}{\log \nu(r, g)} \geq \frac{\lambda^L_{fog} - \varepsilon}{\rho^L_g + \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \geq \frac{\lambda^L_{fog}}{\rho^L_g}. \tag{19}$$

Again combining (12) and (17), we get for a sequence of values of r tending to infinity

$$\frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\lambda^L_{fog} + \varepsilon}{\lambda^L_g - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\lambda^L_{fog}}{\lambda^L_g}. \tag{20}$$

Similarly from (14) and (15) it follows for a sequence of values of r tending to infinity

$$\frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\rho^L_{fog} + \varepsilon}{\rho^L_g - \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\rho^L_{fog}}{\rho^L_g}. \tag{21}$$

Now combining (19), (20) and (21) we get

$$\frac{\lambda_{fog}^L}{\rho_g^L} \leq \liminf_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \min\left\{\frac{\lambda_{fog}^L}{\lambda_g^L}, \frac{\rho_{fog}^L}{\rho_g^L}\right\}. \tag{22}$$

Now from (13) and (16), for a sequence of values of r tending to infinity we obtain

$$\frac{\log \nu(r, fog)}{\log \nu(r, g)} \geq \frac{\lambda_{fog}^L - \varepsilon}{\lambda_g^L + \varepsilon}.$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \geq \frac{\lambda_{fog}^L}{\lambda_g^L}. \tag{23}$$

Again from (12) and (15) it follows that for all sufficiently large values of r

$$\frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\rho_{fog}^L + \varepsilon}{\lambda_g^L - \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\rho_{fog}^L}{\lambda_g^L}. \tag{24}$$

Similarly combining (11) and (18) we get for a sequence of values of r tending to infinity

$$\frac{\log \nu(r, fog)}{\log \nu(r, g)} \geq \frac{\rho_{fog}^L - \varepsilon}{\rho_g^L + \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\limsup_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \geq \frac{\rho_{fog}^L}{\rho_g^L}. \tag{25}$$

Therefore combining (23), (24) and (25) we get that

$$\max\left\{\frac{\lambda_{fog}^L}{\lambda_g^L}, \frac{\rho_{fog}^L}{\rho_g^L}\right\} \leq \limsup_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, g)} \leq \frac{\rho_{fog}^L}{\lambda_g^L}. \tag{26}$$

Thus the theorem follows from (22) and (26). □

Remark 1 If we take $0 < \lambda_f^L \leq \rho_f^L < \infty$ instead of $0 < \lambda_g^L \leq \rho_g^L < \infty$ and the other conditions remain the same then also Theorem 1 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 2 Let f and g be two entire functions. Also let $0 < \lambda_{fog}^L \leq \rho_{fog}^L < \infty$ and $0 < \lambda_f^L \leq \rho_f^L < \infty$. Then

$$\frac{\lambda_{fog}^L}{\rho_f^L} \leq \liminf_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, f)} \leq \min\left\{\frac{\lambda_{fog}^L}{\lambda_f^L}, \frac{\rho_{fog}^L}{\rho_f^L}\right\}$$

$$\leq \max\left\{\frac{\lambda_{fog}^L}{\lambda_f^L}, \frac{\rho_{fog}^L}{\rho_f^L}\right\} \leq \limsup_{r\to\infty} \frac{\log \nu(r, fog)}{\log \nu(r, f)} \leq \frac{\rho_{fog}^L}{\lambda_f^L}.$$

Proof. Proof is similar to Theorem 1 and so omitted. □

Extending the notion we can prove the following theorem using $L-$hyper order($L-$hyper lower order).

Theorem 3 Let f and g be two entire functions. Also let $0 < \overline{\lambda}_{fog}^{L} \leq \overline{\rho}_{fog}^{L} < \infty$ and $0 < \overline{\lambda}_{g}^{L} \leq \overline{\rho}_{g}^{L} < \infty$. Then

$$\frac{\overline{\lambda}_{fog}^{L}}{\overline{\rho}_{g}^{L}} \leq \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \min\left\{\frac{\overline{\lambda}_{fog}^{L}}{\overline{\lambda}^{L}g}, \frac{\overline{\rho}_{fog}}{\overline{\rho}_{g}^{L}}\right\}$$

$$\leq \max\left\{\frac{\overline{\lambda}_{fog}^{L}}{\overline{\lambda}^{L}g}, \frac{\overline{\rho}_{fog}}{\overline{\rho}_{g}^{L}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}_{fog}^{L}}{\overline{\lambda}^{L}g}.$$

Proof. Using Lemma 4 for the entire function g we have for arbitrary positive ε and for all sufficiently large values of r

$$\log^{[2]} \nu(r, g) \leq (\overline{\rho}_{g}^{L} + \varepsilon) \log [rL(r)] \tag{27}$$

and

$$\log^{[2]} \nu(r, g) \geq (\overline{\lambda}^{L}g - \varepsilon) \log [rL(r)]. \tag{28}$$

Also for a sequence of values of r tending to infinity, we get

$$\log^{[2]} \nu(r, g) \leq (\overline{\lambda}^{L}g + \varepsilon) \log [rL(r)] \tag{29}$$

and

$$\log^{[2]} \nu(r, g) \geq (\overline{\rho}_{g}^{L} - \varepsilon) \log [rL(r)]. \tag{30}$$

Again using Lemma 4 for the composite entire function fog we have for arbitrary positive ε and for all sufficiently large values of r

$$\log^{[2]} \nu(r, fog) \leq (\overline{\rho}_{fog}^{L} + \varepsilon) \log [rL(r)] \tag{31}$$

and

$$\log^{[2]} \nu(r, fog) \geq (\overline{\lambda}_{fog}^{L} - \varepsilon) \log [rL(r)]. \tag{32}$$

Again for a sequence of values of r tending to infinity, we get

$$\log^{[2]} \nu(r, fog) \leq (\overline{\lambda}_{fog}^{L} + \varepsilon) \log [rL(r)] \tag{33}$$

and

$$\log^{[2]} \nu(r, fog) \geq (\overline{\rho}_{fog}^{L} - \varepsilon) \log [rL(r)]. \tag{34}$$

Now from (27) and (32) it follows that for all sufficiently large values of r

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \geq \frac{\overline{\lambda}_{fog}^{L} - \varepsilon}{\overline{\rho}_{g}^{L} + \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \geq \frac{\overline{\lambda}_{fog}^{L}}{\overline{\rho}_{g}^{L}}. \tag{35}$$

Again combining (28) and (33), we get for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\lambda}_{fog}^{L} + \varepsilon}{\overline{\lambda}_{g}^{L} - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\lambda}^L_{fog}}{\overline{\lambda}^L_g}. \tag{36}$$

Similarly from (30) and (31) it follows that for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}^L_{fog} + \varepsilon}{\overline{\rho}^L_g - \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}^L_{fog}}{\overline{\rho}^L_g}. \tag{37}$$

Now combining (35), (36) and (37) we get

$$\frac{\overline{\lambda}^L_{fog}}{\overline{\rho}^L_g} \leq \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \min\left\{\frac{\overline{\lambda}^L_{fog}}{\overline{\lambda}^L_g}, \frac{\overline{\rho}^L_{fog}}{\overline{\rho}^L_g}\right\}. \tag{38}$$

Now from (29) and (32) we obtain for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \geq \frac{\overline{\lambda}^L_{fog} - \varepsilon}{\overline{\lambda}^L_g + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get

$$\limsup_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \geq \frac{\overline{\lambda}^L_{fog}}{\overline{\lambda}^L_g}. \tag{39}$$

Again from (28) and (31), it follows for all sufficiently large values of r

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}^L_{fog} + \varepsilon}{\overline{\lambda}^L_g - \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}^L_{fog}}{\overline{\lambda}^L_g}. \tag{40}$$

Similarly combining (27) and (34) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \geq \frac{\overline{\rho}^L_{fog} - \varepsilon}{\overline{\rho}^L_g + \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\limsup_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \geq \frac{\overline{\rho}^L_{fog}}{\overline{\rho}^L_g}. \tag{41}$$

Therefore combining (39), (40) and (41) we get

$$\max\left\{\frac{\overline{\lambda}^L_{fog}}{\overline{\lambda}^L g}, \frac{\overline{\rho}_{fog}}{\overline{\rho}^L_g}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}^L_{fog}}{\overline{\lambda}^L_g}. \tag{42}$$

Thus the theorem follows from (38) and (42). □

Remark 2 If we take $0<\overline{\lambda}_f^L\leq\overline{\rho}_f^L<\infty$ instead of $0<\overline{\lambda}_g^L\leq\overline{\rho}_g^L<\infty$ and the other conditions remain the same then also Theorem 3 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 4 Let f and g be two entire functions. Also let $0<\overline{\lambda}_{fog}^L\leq\overline{\rho}_{fog}^L<\infty$ and $0<\overline{\lambda}_f^L\leq\overline{\rho}_f^L<\infty$. Then

$$\frac{\overline{\lambda}_{fog}^L}{\overline{\rho}_f^L}\leq\liminf_{r\to\infty}\frac{\log^{[2]}\nu(r,fog)}{\log^{[2]}\nu(r,f)}\leq\min\left\{\frac{\overline{\lambda}_{fog}^L}{\overline{\lambda}_f^L},\frac{\overline{\rho}_{fog}^L}{\overline{\rho}_f^L}\right\}$$

$$\leq\max\left\{\frac{\overline{\lambda}_{fog}^L}{\overline{\lambda}_f^L},\frac{\overline{\rho}_{fog}^L}{\overline{\rho}_f^L}\right\}\leq\limsup_{r\to\infty}\frac{\log^{[2]}\nu(r,fog)}{\log^{[2]}\nu(r,f)}\leq\frac{\overline{\rho}_{fog}^L}{\overline{\lambda}_f^L}.$$

Proof. Proof is similar to Theorem 3 and so omitted. □

In the line of Theorem 1, one can prove the following theorem:

Theorem 5 Let f and g be two entire functions. Also let $0<\lambda_{fog}^{L^*}\leq\rho_{fog}^{L^*}<\infty$ and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$. Then

$$\frac{\lambda_{fog}^{L^*}}{\rho_g^{L^*}}\leq\liminf_{r\to\infty}\frac{\log\nu(r,fog)}{\log\nu(r,g)}\leq\min\left\{\frac{\lambda_{fog}^{L^*}}{\lambda_g^{L^*}},\frac{\rho_{fog}^{L^*}}{\rho_g^{L^*}}\right\}$$

$$\leq\max\left\{\frac{\lambda_{fog}^{L^*}}{\lambda_g^{L^*}},\frac{\rho_{fog}^{L^*}}{\rho_g^{L^*}}\right\}\leq\limsup_{r\to\infty}\frac{\log\nu(r,fog)}{\log\nu(r,g)}\leq\frac{\rho_{fog}^{L^*}}{\lambda_g^{L^*}}.$$

Remark 3 If we take $0<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$ instead of $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and the other conditions remain the same then also Theorem 5 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 6 Let f and g be two entire functions. Also let $0<\lambda_{fog}^{L^*}\leq\rho_{fog}^{L^*}<\infty$ and $0<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$. Then

$$\frac{\lambda_{fog}^{L^*}}{\rho_f^{L^*}}\leq\liminf_{r\to\infty}\frac{\log\nu(r,fog)}{\log\nu(r,f)}\leq\min\left\{\frac{\lambda_{fog}^{L^*}}{\lambda_f^{L^*}},\frac{\rho_{fog}^{L^*}}{\rho_f^{L^*}}\right\}$$

$$\leq\max\left\{\frac{\lambda_{fog}^{L^*}}{\lambda_f^{L^*}},\frac{\rho_{fog}^{L^*}}{\rho_f^{L^*}}\right\}\leq\limsup_{r\to\infty}\frac{\log\nu(r,fog)}{\log\nu(r,f)}\leq\frac{\rho_{fog}^{L^*}}{\lambda_f^{L^*}}.$$

In the line of Theorem 3, one can prove the following theorem:

Theorem 7 Let f and g be two entire functions. Also let $0<\overline{\lambda}_{fog}^{L^*}\leq\overline{\rho}_{fog}^{L^*}<\infty$ and $0<\overline{\lambda}_g^{L^*}\leq\overline{\rho}_g^{L^*}<\infty$. Then

$$\frac{\overline{\lambda}_{fog}^{L^*}}{\overline{\rho}_g^{L^*}}\leq\liminf_{r\to\infty}\frac{\log^{[2]}\nu(r,fog)}{\log^{[2]}\nu(r,g)}\leq\min\left\{\frac{\overline{\lambda}_{fog}^{L^*}}{\overline{\lambda}_g^{L^*}},\frac{\overline{\rho}_{fog}^{L^*}}{\overline{\rho}_g^{L^*}}\right\}$$

$$\leq\max\left\{\frac{\overline{\lambda}_{fog}^{L^*}}{\overline{\lambda}_g^{L^*}},\frac{\overline{\rho}_{fog}^{L^*}}{\overline{\rho}_g^{L^*}}\right\}\leq\limsup_{r\to\infty}\frac{\log^{[2]}\nu(r,fog)}{\log^{[2]}\nu(r,g)}\leq\frac{\overline{\rho}_{fog}^{L^*}}{\overline{\lambda}_g^{L^*}}.$$

Remark 4 If we take $0<\overline{\lambda}_f^{L^*}\leq\overline{\rho}_f^{L^*}<\infty$ instead of $0<\overline{\lambda}_g^{L^*}\leq\overline{\rho}_g^{L^*}<\infty$ and the other conditions remain the same then also Theorem 7 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 8 Let f and g be two entire functions. Also let $0 < \overline{\lambda}_{fog}^{L^*} \leq \overline{\rho}_{fog}^{L^*} < \infty$ and $0 < \overline{\lambda}_{f}^{L^*} \leq \overline{\rho}_{f}^{L^*} < \infty$. Then

$$\frac{\overline{\lambda}_{fog}^{L^*}}{\overline{\rho}_{f}^{L^*}} \leq \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \min\left\{\frac{\overline{\lambda}_{fog}^{L^*}}{\overline{\lambda}_{f}^{L^*}}, \frac{\overline{\rho}_{fog}^{L^*}}{\overline{\rho}_{f}^{L^*}}\right\}$$

$$\leq \max\left\{\frac{\overline{\lambda}_{fog}^{L^*}}{\overline{\lambda}_{f}^{L^*}}, \frac{\overline{\rho}_{fog}^{L^*}}{\overline{\rho}_{f}^{L^*}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r, fog)}{\log^{[2]} \nu(r, g)} \leq \frac{\overline{\rho}_{fog}^{L^*}}{\overline{\lambda}_{f}^{L^*}}.$$

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