

RESULTS ON VALUE DISTRIBUTION OF L-FUNCTIONS RELATED TO BRÜCK CONJECTURE

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ABSTRACT. Relying on Nevanlinna theory and the properties of L -functions in the extended selberg class, we mainly study the uniqueness problems on L -functions related to Brück conjecture. This extend the result due to Q. C. Zhang.[33]

1. INTRODUCTION

The Riemann hypothesis as one of the millenium problems has been given a lot of attention by mathematical workers for a long time. Selberg guessed that the Riemann hypothesis is also true for L -functions in the selberg class. Such an L -function based on Riemann zeta function as the prototype is defined to be a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

of a complex variable $s = \sigma + it$ satisfying the following axioms:

- (i) Ramanujan hypothesis. $a(n) \ll n^\epsilon$ for every $\epsilon > 0$.
- (ii) Analytic continuation. there is a non-negative integer m such that $(s-1)^m L(s)$ is an entire function of finite order.
- (iii) Functional equation. L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1 - \bar{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^k \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re \nu_j \geq 0$ and $|\omega| = 1$.

- (iv) Euler product. $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^\theta$ for some $\theta < \frac{1}{2}$.

2010 *Mathematics Subject Classification.* Primary 30D35.

Key words and phrases. Value Distribution theory, L -functions, Entire function, Meromorphic function, Sharing values, Differential Polynomial.

Submitted March 11, 2019. Revised Aug. 07, 2019.

Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in \mathbb{C} . Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in \mathbb{C} must be identically equal if they share five values IM, and one must be a Möbius transform of the other if they share four values CM;

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane \mathbb{C} . We denote by $N_k(r, \frac{1}{f-a})$ the counting function for zeros of $f - a$ with multiplicity $\leq k$, and by $\overline{N}_k(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $N_{(k)}(r, \frac{1}{f-a})$ be the counting function for zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k)}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted.

Let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q . We denote by $\overline{N}_L(r, a; f)$ the counting function of those a -points of f and g where $p > q \geq 1$, by $N_E^1(r, a; f)$ the counting function of those a -points of f and g where $p = q = 1$ and by $\overline{N}_E^2(r, a; f)$ the counting function of those a -points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^1(r, a; g)$, $\overline{N}_E^2(r, a; g)$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share the value a with weight k then z_0 is an a -point of f with multiplicity $m(\leq k)$ if and only if it is an a -point of g with multiplicity $m(\leq k)$ and z_0 is an a -point of f with multiplicity $m(> k)$ if and only if it is an a -point of g with multiplicity $n(> k)$, where m is not necessarily equal to n . We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In connection to find the relation between an entire function with its derivative when they share one value CM, in 1996, in this direction the following famous conjecture was proposed by Brück[5]:

Conjecture : Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a nonzero constant.

Brück himself proved the conjecture for $a = 0$ and for $a = 1$, he showed that under the assumption $N(r, 0; f') = S(r, f)$ the conjecture was true.

Theorem A. [5] Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r, 0; f') = S(r, f)$, then $\frac{f'-1}{f-1}$ is a nonzero constant.

However for entire function of finite order, Yang[30] removed the supposition $N(r, 0; f') = 0$ and obtained the following result.

Theorem B. [30] Let f be a non-constant entire function of finite order and let

$a(\neq 0)$ be a finite constant. If $f, f^{(k)}$ share the value a CM, then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.

In 2005, Zhang[33] further extended the results of Lahiri-sarkar to a small function and proved the following result for IM sharing.

Theorem C. [33] Let f be a non-constant meromorphic function and $k(\geq 1)$, and $l \geq 0$ be integer. Also let $a \equiv a(z)(\neq 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $(f)^{(k)} - a$ share $(0, l)$.

If $l \geq 2$ and

$$2\bar{N}(r, f) + N_2\left(r, 0; (f)^{(k)}\right) + N_2\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

or, if $l = 1$

$$2\bar{N}(r, f) + N_2\left(r, 0; (f)^{(k)}\right) + 2\bar{N}\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

or, $l = 0$ and

$$4\bar{N}(r, f) + 3N_2\left(r, 0; (f)^{(k)}\right) + 2\bar{N}\left(r, 0; \left(\frac{f}{a}\right)'\right) \leq (\lambda + o(1))T\left(r, (f)^{(k)}\right)$$

for $r \in I$, where $0 < \lambda < 1$, and I is a set of infinite linear measure, then $\frac{f^{(k)} - a}{(f) - a} = c$, for some constant $c \in \mathbb{C} - \{0\}$.

Now it is natural to ask the following question which is the motivation of the paper.

Question. Can Brück type conclusion be obtained when f and $f^{(k)}$ is replaced by $[f^n]^{(k)}$ and $[L^n]^{(k)}$ in Theorem C.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in \mathbb{C} . We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \quad (1)$$

Lemma 1.[25] Let f be a non-constant meromorphic function and let $a_n(z)(\neq 0), a_{n-1}(z), \dots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \dots, n$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.[27] Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\bar{N}(r, \infty, f) + O(\log(T(r, f)) + \log r),$$

as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.

Lemma 3.[33] Let f be a nonconstant meromorphic function, and p, k be positive integers. Then

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \quad (2)$$

$$N_p(r, 0; f^{(k)}) \leq k\bar{N}(r, \infty, f) + N_{p+k}(r, 0; f) + S(r, f). \quad (3)$$

Lemma 4.[4] Let F and G share $(1, l)$ and $\bar{N}(r, \infty; F) = \bar{N}(r, \infty; G)$ and $H \neq 0$, where F, G and H are defined as earlier. Then

$$N(r, \infty, H) \leq \bar{N}(r, \infty; F) + \bar{N}(r, 0; F| \geq 2) + \bar{N}(r, 0; G| \geq 2) + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G') + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + S(r, F).$$

Lemma 5.[4] If F and G share $(1, l)$, then

$$\bar{N}_L(r, 1; F) \leq \frac{1}{2}\bar{N}(r, \infty; F) + \frac{1}{2}\bar{N}(r, 0; F) + S(r, F) \text{ when } l \geq 1,$$

$$\bar{N}_L(r, 1; F) \leq \bar{N}(r, \infty; F) + \bar{N}(r, 0; F) + S(r, F) \text{ when } l = 0.$$

Lemma 6.[4] Let F and G share $(1, l)$ and $H \neq 0$. Then

$$\bar{N}(r, 1, F) + \bar{N}(r, 1, G) \leq N(r, \infty; H) + \bar{N}_E^{(2)}(r, 1; F) + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) + S(r, F).$$

Now we have the following theorems

Theorem 1. Let f be a non-constant meromorphic function in C , let L be a nonconstant L -function and let $n (> k + 2)$ and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(a(z), l)$, where $a(z) (\neq 0, \infty)$ be a meromorphic function. If $l \geq 2$ and

$$(k + 2)\Theta(\infty, f) + \delta_{k+2}(0, f^n) + \delta_{k+2}(0, L^n) \leq k + 4 - n, \quad (4)$$

or $l = 1$ and

$$(k + 2)\Theta(\infty, f) + \delta_{k+2}(0, f^n) + \frac{3}{2}\delta_{k+2}(0, L^n) \leq k + \frac{9}{2} - n, \quad (5)$$

or $l = 0$ and

$$(2k + 3)\Theta(\infty, f) + 2\delta_{k+2}(0, f^n) + 3\delta_{k+2}(0, L^n) \leq 2k + 8 - n, \quad (6)$$

then $\frac{[L^n]^{(k)} - a}{[f^n]^{(k)} - a} = c$.

Proof. Suppose that d is the degree of L . Then $d = 2 \sum_{i=1}^k \lambda_j$, where k and λ_j are respectively the positive integer and the positive real number in the axiom (iii) of the definition of L -function.

Then we have that

$$T(r, L) = \frac{d}{\Pi} r \log r + O(r)$$

(cf. [8], P.150). Clearly, f and L are transcendental meromorphic functions (cf. [22], P.43). Note that an L -function at most has one pole $z = 1$ in the complex plane.

Let $F = \frac{(f^n)^{(k)}}{a(z)}$ and $G = \frac{(L^n)^{(k)}}{a(z)}$. Then $F - 1 = \frac{(f^n)^{(k)} - a}{a}$, $G - 1 = \frac{(L^n)^{(k)} - a}{a}$. Since $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share (a, l) it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$.

Now we consider the following cases.

Case 1. Let $H \neq 0$.

Subcase 1.1. Assume $l \geq 1$. Using the second fundamental Theorem and Lemmas

4, 6 we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq \bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + \bar{N}(r, 0; F) + \bar{N}(r, 0; G) + N(r, H) + \bar{N}_E^{(2)}(r, 1; F) \\ &\quad + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) - \bar{N}_0(r, 0; F') - \bar{N}_0(r, 0; G') + S(r, F) \\ &\leq 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \bar{N}_E^{(2)}(r, 1; F) + 2\bar{N}_L(r, 1; F) \\ &\quad + 2\bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) + S(r, F). \end{aligned} \quad (7)$$

Subcase-1.1.1. Next assume $l \geq 2$. Now by using the inequality (7) and Lemma 3, we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \bar{N}_E^{(2)}(r, 1; F) + 2\bar{N}_L(r, 1; F) \\ &\quad + 2\bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) + S(r, F) \\ &\leq 3\bar{N}(r, \infty; f) + N_2(r, 0; F) + N_2(r, 0; G) + N(r, 1; F) + S(r, f) \\ T(r, G) &\leq 3\bar{N}(r, \infty; f) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_2(r, 0; G) + S(r, f) \\ &\leq 3\bar{N}(r, \infty; f^n) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f^n) + N_2(r, 0; (L^n)^{(k)}) + S(r, f) \\ T(r, L^n) &\leq 3\bar{N}(r, \infty; f) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; L^n) + k\bar{N}(r, L) + S(r, f) \\ &\leq 3\bar{N}(r, \infty; f) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; L^n) + S(r, f) \end{aligned}$$

i.e., for any $\epsilon > 0$

$$(k+2)\Theta(\infty, f) + \delta_{k+2}(0, f^n) + \delta_{k+2}(0, L^n) \leq k+4-n,$$

which is a contradicts (4) of Theorem 1.

Subcase-1.1.2. Next we assume $l = 1$. Now inequality (7) and in view of Lemmas 3, 5 we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) + \bar{N}_E^{(2)}(r, 1; F) + 2\bar{N}_L(r, 1; F) \\ &\quad + 2\bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) + S(r, F) \\ &\leq 2\bar{N}(r, \infty; F) + \frac{3}{2}\bar{N}(r, \infty; G) + \frac{1}{2}\bar{N}(r, 0; G) + N_2(r, 0; F) + N_2(r, 0; G) + \bar{N}_E^{(2)}(r, 1; F) \\ &\quad + 2\bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) + S(r, F) \\ &\leq 2\bar{N}(r, \infty; f) + \frac{1}{2}N_1(r, 0; G) + N_2(r, 0; F) + N_2(r, 0; G) + N(r, 1; F) + S(r, f) \\ T(r, L^n) &\leq 2\bar{N}(r, \infty; f) + \frac{1}{2}N_1(r, 0; (L^n)^{(k)}) + N_2(r, 0; (f^n)^{(k)}) + N_2(r, 0; (L^n)^{(k)}) + S(r, f) \\ &\leq 2\bar{N}(r, \infty; f) + \frac{1}{2}N_{k+1}(r, 0; L^n) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; L^n) \\ &\quad + \frac{3}{2}k\bar{N}(r, L) + S(r, f) \\ &\leq 2\bar{N}(r, \infty; f) + \frac{1}{2}N_{k+1}(r, 0; L^n) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; L^n) + S(r, f) \end{aligned}$$

i.e., for any $\epsilon > 0$

$$(k+2)\Theta(\infty, f) + \delta_{k+2}(0, f^n) + \frac{3}{2}\delta_{k+2}(0, L^n) \leq k + \frac{9}{2} - n,$$

which is a contradicts (5) of Theorem 1.

Subcase-1.2. Next we assume $l = 0$. Then by using the second fundamental Theorem and Lemmas 3, 4, 5 and 6 we get

$$\begin{aligned}
T(r, F) + T(r, G) &\leq \bar{N}(r, \infty; F) + \bar{N}(r, 0; F) + \bar{N}(r, 1; F) + \bar{N}(r, \infty; G) + \bar{N}(r, 0; G) + \bar{N}(r, 1; G) \\
&\quad - N_0(r, 0; F') - N_0(r, 0; G') + S(r, F) + S(r, G) \\
&\leq \bar{N}(r, \infty; F) + \bar{N}(r, 0; F) + \bar{N}(r, \infty; G) + \bar{N}(r, 0; G) + N(r, \infty; H) + \bar{N}_E^{(2)}(r, 1; F) \\
&\quad + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) - N_0(r, 0; F') - N_0(r, 0; G') + S(r, f) \\
&\leq 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + \frac{1}{2}\bar{N}(r, 0; G) + N_2(r, 0; F) + N_2(r, 0; G) + \bar{N}_E^{(2)}(r, 1; F) \\
&\quad + 2\bar{N}_L(r, 1; F) + 2\bar{N}_L(r, 1; G) + \bar{N}(r, 1; G) + S(r, f) \\
&\leq 3\bar{N}(r, \infty; f) + N_2(r, 0; F) + N_2(r, 0; G) + 2\bar{N}(r, 0; G) + \bar{N}(r, 0; F) + N(r, 1; F) + S(r, f) \\
T(r, L^n) &\leq 3\bar{N}(r, \infty; f) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_2(r, 0; (L^n)^{(k)}) + \bar{N}(r, 0; (f^n)^{(k)}) \\
&\quad + 2\bar{N}(r, 0; G) + S(r, f) \\
&\leq 3\bar{N}(r, \infty; f) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; L^n) + N_{k+1}(r, 0; f^n) \\
&\quad + k\bar{N}(r, f) + 2N_{k+1}(r, 0; L^n) + 3k\bar{N}(r, L) + S(r, f) \\
&\leq 3\bar{N}(r, \infty; f) + N_{k+2}(r, 0; f^n) + k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; L^n) + N_{k+1}(r, 0; f^n) \\
&\quad + k\bar{N}(r, \infty; f) + 2N_{k+1}(r, 0; L^n) + S(r, f)
\end{aligned}$$

i.e., for any $\epsilon > 0$

$$(2k + 3)\Theta(\infty, f) + 2\delta_{k+2}(0, f^n) + 3\delta_{k+2}(0, L^n) \leq 2k + 8 - n,$$

which is a contradicts (6) of Theorem 1.

Case 2. If $H \equiv 0$, then on integration, we get

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \quad (8)$$

where C, D are constants and $C \neq 0$. From (3.2) it is clear that F and G share 1 CM. We first assume that $D \neq 0$. Then by (8) we get

$$\bar{N}(r, \infty; f) = S(r, f). \quad (9)$$

Now we can write (8) as

$$\frac{1}{F-1} = \frac{D \left(G-1 + \frac{C}{D} \right)}{G-1}. \quad (10)$$

Consequently,

$$\bar{N} \left(r, 1 - \frac{C}{D}; G \right) = \bar{N}(r, \infty; F) = \bar{N}(r, \infty; G) = S(r, F). \quad (11)$$

Subcase-2.1. If $\frac{C}{D} \neq 1$, by the second fundamental theorem, Lemma 3, we have

$$\begin{aligned} T(r, G) &\leq \bar{N}(r, \infty; G) + N_1(r, 0; G) + \bar{N}\left(r, 1 - \frac{C}{D}; G\right) + S(r, G) \\ &\leq \bar{N}(r, 0; G) + S(r, f) \\ &\leq N_2(r, 0; G) + S(r, f) \\ T(r, G) &\leq N_{k+2}(r, 0; L) + S(r, f) \\ T(r, L^n) &\leq (k + 2)\bar{N}(r, 0; L) + S(r, f) \end{aligned}$$

i.e., $n \leq k + 2$, which is a contradiction since $n > k + 2$.

Subcase-2.2. If $\frac{C}{D} = 1$, we get from (8)

$$\left(F - 1 - \frac{1}{C}\right)G \equiv -\frac{1}{C} \tag{12}$$

i.e.,

$$\frac{1}{L^n[(f^n)^{(k)} - (1 + \frac{1}{C})a]} \equiv -\frac{C}{a^2} \frac{[L^n]^{(k)}}{L^n} \tag{13}$$

From (12) it follows that

$$\begin{aligned} N\left(r, 0; [f^n]^{(k)} \geq k + 1\right) &\leq N\left(r, 0; [L^n]^{(k)}\right) \\ &\leq N(r, 0; G) \\ &\leq N(r, 0; a) = S(r, f). \end{aligned} \tag{14}$$

Applying the first fundamental theorem, (9), (13) and (14)

$$\begin{aligned} (n - k)T(r, f) + nT(r, L) &= T\left(r, \frac{1}{L^n[(f^n)^{(k)} - (1 + \frac{1}{C})a]}\right) \\ &= m\left(r, \frac{(L^n)^{(k)}}{L^n}\right) + N\left(r, \frac{(L^n)^{(k)}}{L^n}\right) + S(r, f). \end{aligned} \tag{15}$$

From (15) it follows that

$$(n - k)T(r, f) \leq S(r, f), \text{ which is impossible.}$$

Hence $D = 0$ and So $\frac{G - 1}{F - 1} = C$ or $\frac{(L^n)^{(k)} - 1}{(f^n)^{(k)} - 1} = C$.

Theorem 2. Let f be a non-constant entire function in C , let L be a nonconstant L -function and let n and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(a(z), l)$, where $a(z) (\neq 0, \infty)$ be a meromorphic function. If $l \geq 2$ and

$$\delta_{k+2}(0, f^n) + \delta_{k+2}(0, L^n) \leq 2 - n, \tag{16}$$

or $l = 1$ and

$$\delta_{k+2}(0, f^n) + \frac{3}{2}\delta_{k+2}(0, L^n) \leq \frac{5}{2} - n, \tag{17}$$

or $l = 0$ and

$$2\delta_{k+2}(0, f^n) + 3\delta_{k+2}(0, L^n) \leq 5 - n, \tag{18}$$

then $\frac{[L^n]^{(k)} - a}{[f^n]^{(k)} - a} = c$.

Proof. Noting that $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; L) = 0$, and proceeding in the like manner as the proof of Theorem 1 we obtain the proof of the Theorem 2.

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