Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 305-325.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

M-POLYNOMIAL OF GENERALIZED TRANSFORMATION GRAPHS

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Abstract. In this paper, we show geomertric-arithmetic index (GA) can be obtained from the M-polynomial of a graph by giving a suitable operator and obtain M-polynomial of generalized transformation graph G and their complements. Furthermore, we derive some degree-based topological indices from the obtained polynomials. The topological indices play an important role in determining physico-chemical properties of chemical graphs and topological indices such as the Randic index, geomertric-arithmetic index are used to predict the bioactivity of chemical compounds, among them the degree-based topological indices can be easily driven from an algebraic expression corresponding to the chemical graphs called M-polynomial.

1. Introduction

Throughout this paper, by a graph $G = (V, E)$ we mean a simple, undirected graph of order n, size m. let $V(G)$ be the vertex set and $E(G)$ be the edge set of the graph G. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of G and we denote $d_v = d_G(v)$.

The *generalized transformation graph* G^{xy}, introduced by Basavanagoud et al.[5] is a graph whose vertex set is $V(G) \cup E(G)$ and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:

(a) $\alpha, \beta \in V(G)$, $\alpha,\ \beta$ are adjacent in G if $x = +$ and $\alpha,\ \beta$ are not adjacent in G if $x = -$
(b) $\alpha \in V(G)$ and $\beta \in E(G)$, $\alpha,\ \beta$ are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

One can obtain the four graphical trasformation of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. An example of generalized transformation graphs and their complements are depicted in Fig 1. Note that, G^{xy} is just the *semitotal-point graph* of G, which was introduced by Sampathkumar et al.[19]. The vertex v of G^{xy} corresponding to a vertex v of G is referred to as a *point vertex.* The vertex e of G^{xy} corresponding to an edge e of G is referred to as a *line vertex.*

2010 *Mathematics Subject Classification.* 05C07, 05C09, 05C31.
Key words and phrases. M-polynomial, generalized semitotal point transformation graphs and their complements, edge partions, degree-based topological indices.
Submitted Jan. 15, 2020.

Proposition 1.1. [5] *Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then the degree of point and line vertices in G^{xy} are:*

(i) $d_{G^{++}}(u) = 2d_G(u)$ *and* $d_{G^{++}}(e) = 2$
(ii) $d_{G^{+-}}(u) = m$ *and* $d_{G^{+-}}(e) = n - 2$
(iii) $d_{G^{-+}}(u) = n - 1$ *and* $d_{G^{-+}}(e) = 2$
(iv) $d_{G^{--}}(u) = n + m - 1 - 2d_G(u)$ *and* $d_{G^{--}}(e) = n - 2$.

The complement of G will be denoted by $\overline{G}$. If G has n vertices and m edges, then the number of vertices of G^{xy} is $n + m$. By Proposition 1.1 and taking into account that $d_{\overline{G}}(u) = n - 1 - d_G(u)$, we have the following proposition:

Proposition 1.2. [5] *Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then the degrees of point and line vertices in $\overline{G^{xy}}$ are:*

(i) $d_{\overline{G^{++}}}(u) = n + m - 1 - 2d_G(u)$ *and* $d_{\overline{G^{++}}}(e) = n + m - 3$
(ii) $d_{\overline{G^{+-}}}(u) = n - 1$ *and* $d_{\overline{G^{+-}}}(e) = m + 1$
(iii) $d_{\overline{G^{-+}}}(u) = m$ *and* $d_{\overline{G^{-+}}}(e) = n + m - 3$
(iv) $d_{\overline{G^{--}}}(u) = 2d_G(u)$ *and* $d_{\overline{G^{--}}}(e) = m + 1$.

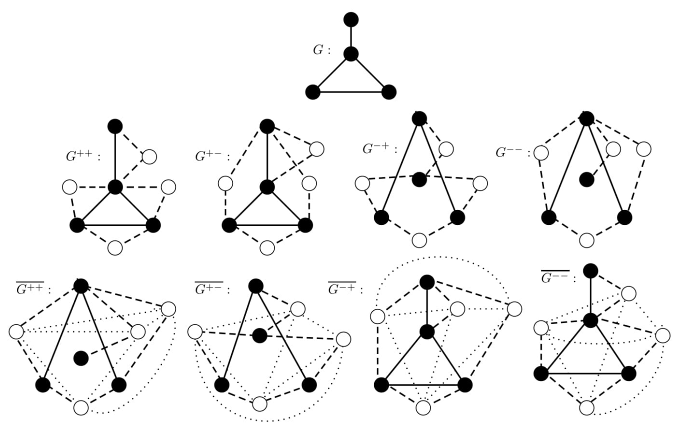

FIGURE 1. The graph G, its generalized transformations G^{xy} and their complements $\overline{G^{xy}}$.

In this paper, we obtain the M-polynomial for some degree-based topological indices of generalized transformation graphs G^{xy} and their complements $\overline{G^{xy}}$. The general form of degree-based topological index of a graph is given by

$$TI = \sum_{uv \in E(G)} f(d_G(u), d_G(v)),$$

where $f = f(x, y)$ is a function appropriately chosen for the computation. Table 1, 2, gives the standard topological indices defined by $f(x, y)$.
The M-polynomial was introduced in 2015 by Deutch and Klavžar [7] and is found useful in determining many degree-based topological indices (listed in Table 1, 2).

Definition 1. [7] *Let G be a graph. Then M-polynomial of G is defined as*

$$M(G; x, y) = \sum_{i \le j} m_{ij}(G) x^i y^j,$$

Where $m_{ij}, i, j \geq 1$, *is the number of edges* uv *of* G *such that* $\{d_G(u)), d_G(v)\} = \{i, j\}$ [11]. *Recently, the study of* $M-polynomial$ *are reported in* [9, 16, 17, 18, 6, 4, 3].

Table 1 was given by Deutch and Klavžar to derive degree-based topological indices from the *M-polynomial.*

Table 1. [7] Derivation of some degree-based topological indices from M-polynomial.

Notaion	Topological index	f(x,y)	derivation from $M(G;x,y)$
$M_1(G)$	First Zagreb	$x+y$	$(D_x+D_y)(M(G;x,y))\|_{x=y=1}$
$M_2(G)$	Second Zagreb	xy	$(D_xD_y)(M(G;x,y))\|_{x=y=1}$
$M_2^m(G)$	Second modified Zagreb	$\frac{1}{xy}$	$(S_xS_y)(M(G;x,y))\|_{x=y=1}$
$S_D(G)$	Symmetric division index	$\frac{x^2+y^2}{xy}$	$(D_xS_y+D_yS_x)(M(G;x,y))\|_{x=y=1}$
$H(G)$	Harmonic	$\frac{2}{x+y}$	$2S_xJ(M(G;x,y))\|_{x=1}$
$I_n(G)$	Invesre sum index	$\frac{xy}{x+y}$	$S_xJD_xD_y(M(G;x,y))\|_{x=1}$
$R_\alpha(G)$	General Randic index	$(xy)^\alpha$	$D_x^\alpha D_y^\alpha(M(G;x,y))\|_{x=y=1}$

Where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t}dt$, $S_y = \int_0^y \frac{f(x,t)}{t}dt$, $D_x^\alpha = D_x(D_x^{\alpha-1})(f(x,y))$ and $J(f(x,y)) = f(x,x)$ are the operators.

The Table 2 which gives operators to derive general sum connectivity index, the first general Zagreb index, and general Zagreb index from $M-polynomial$.

Table 2. [3, 4] New operators to derive degree-based topological indices from M-polynomial.

Notation	Topological index	$f(x,y)$	derivation from$M(G;x,y)$
$\chi_\alpha(G)$	General sum connectivity [12]	$(x+y)^\alpha$	$D_x^\alpha(J(M(G;x,y)))\|_{x=1}$
$M_1^\alpha G$	First general Zagreb [15]	$x^{\alpha-1}+y^{\alpha-1}$	$(D_x^{\alpha-1}+D_y^{\alpha-1})(M(G;x,y))\|_{x=y=1}$
$M_{(a,b)}(G)$	General Zagreb index [2, 8]	$x^ay^b+x^by^a$	$(D_x^aD_y^b+D_x^bD_y^a)(M(G;x,y))\|_{x=y=1}$

Note 1. The hyper Zagreb index is obtained by taking $\alpha = 2$ in general sum connectivity index.

2. Taking $\alpha = 2,\ 3$ in first general Zagreb index. First Zagreb and forgotton (F-index) topological indices are obtained respectively.

Vukičević and Furtula [21] proposed index named the geometric-arithmetic index, let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The geometric-arithmetic index (GA index for short) of G is defined as

$$GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_ud_v}}{d_u+d_v}.$$

The Table 3 is given by us which gives operator to derive geometric-arithmetic index from M-polynomial.

TABLE 3. [20] Operator to derive geometric-arithmetic index from M-polynomial.

Notation	Topological index	$f(x,y)$	derivation from $M_(G;x,y)$
$GA(G)$	Geometric-Arithmetic index [10]	$\frac{2\sqrt{xy}}{x+y}$	$2S_xJD_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(M(G;x,y))\|_{x=1}$

2. M-polynomial of generalized semitotal point transformation graphs G^{xy} and their complements $\overline{G^{xy}}$

In this section , we obtain M-polynomial of generalized semitotal point graph[5]. Namely G^{++}, G^{+-}, G^{-+}, G^{--} and their complements $\overline{G^{++}}$, $\overline{G^{+-}}$, $\overline{G^{-+}}$, $\overline{G^{--}}$ respectively. Also, we derive some topological indices of these graphs (mentioned in Table 1, 2, and 3) from the respective M-polynomial.

Theorem 2.1. *If G is any graph of order n and size m with the M-polynomial $M(G;x,y)=\sum_{i\leq j} m_{ij}(G)x^iy^j$, then*

$$M(G^{++};x,y)=\sum_{i\leq j} m_{ij}(G)x^{2i}y^{2j}+\sum_{\delta\leq i\leq\Delta} d_G(v_i)x^{2i}y^2.$$

Proof. Let G^{++} be the transformation graph of G has $n+m$ vertices and $3m$ edges. Using the Proposition 1.1, we have $d_{G^{++}}(u)=2d_G(u)$, $d_{G^{++}}(e)=2$. Therefore, the edge partition of G^{++} is given by

$$\begin{aligned}|E_{\{2i,\ 2j\}}| &= |uv\in E(G^{++}):d_u=2i \text{ and } d_v=2j|=m_{ij}(G)\\ |E_{\{2,\ 2i\}}| &= |uv\in E(G^{++}):d_u=2 \text{ and } d_v=2i|=d_G(v_i).\end{aligned}$$

From the definition, The M-polynomial of G^{++} is obtained below.

$$\begin{aligned}M(G^{++};x,y) &= \sum_{i\leq j} m_{ij}(G^{++})x^iy^j\\ &= \sum_{i\leq j} m_{ij}(G)x^{2i}y^{2j}+\sum_{\delta\leq i\leq\Delta} d_G(v_i)x^{2i}y^2.\end{aligned}$$

□

Corollary 2.2. *If G is any r-regular graph of order n and size m, then*

$$M(G^{++};x,y)=\frac{nr}{2}x^{2r}y^{2r}+nrx^2y^{2r}.$$

Corollary 2.3. *If G^{++} be the trasformation graph of any r-regular graph G, then*

1. $M_1(G^{++})=2nr(2r+1)$,
2. $M_2(G^{++})=nr^2(r+4)$,
3. $M_2^m(G^{++})=\frac{n(1+2r)}{8r}$,
4. $S_D(G^{++})=n(2r+1)$,
5. $H(G^{++})=\frac{n(r+5)}{4(1+r)}$,
6. $I_n(G^{++})=\frac{nr(5r+r^2)}{2(1+r)}$,
7. $R_\alpha(G^{++})=n2^{2\alpha}r^{\alpha+1}\left(\frac{r^\alpha+2}{2}\right)$,
8. $\chi_\alpha(G^{++})=nr^{\alpha+1}(2)^{2\alpha-1}+nr(2+2r)^\alpha$,
9. $M_1^\alpha(G^{++})=r^\alpha n2^{\alpha-1}+2^{\alpha-1}nr$,
10. $M_{(a,b)}(G^{++})=n2^{a+b}r^{a+b+1}+nr^{b+1}2^{a+b}+n2^{a+b}r^{a+1}$,

11. $GA(G^{++}) = \frac{nr}{2} + \frac{(nr)(2)^{\frac{1}{2}}(2r)^{\frac{1}{2}}}{1+r}$.

Proof. Let $M(G^{++}; x, y) = f(x, y) = \frac{nr}{2}x^{2r}y^{2r} + nrx^2y^{2r}$. Then we have

$$
\begin{aligned}
D_x(f(x,y)) &= nr^2x^{2r}y^{2r} + 2nrx^2y^{2r},\\
D_y(f(x,y)) &= nr^2x^{2r}y^{2r} + 2nr^2x^2y^{2r},\\
D_xD_y(f(x,y)) &= 2nr^3x^{2r}y^{2r} + 4nr^2x^2y^{2r},\\
S_x(f(x,y)) &= \frac{n}{4}x^{2r}y^{2r} + \frac{nr}{2}x^2y^{2r},\\
S_y(f(x,y)) &= \frac{n}{4}x^{2r}y^{2r} + \frac{n}{2}x^2y^{2r},\\
S_xS_y(f(x,y)) &= \frac{n}{(8r)}x^{2r}y^{2r} + \frac{n}{4}x^2y^{2r},\\
D_xS_y(f(x,y)) &= \frac{nr}{2}x^{2r}y^{2r} + nx^2y^{2r},\\
D_yS_x(f(x,y)) &= \frac{nr}{2}x^{2r}y^{2r} + nrx^2y^{2r},\\
D_x^{\alpha}(f(x,y)) &= \frac{nr}{2}(2r)^{\alpha}x^{2r}y^{2r} + (nr)(2)^{\alpha}x^2y^{2r},\\
D_y^{\alpha}(f(x,y)) &= \frac{nr}{2}(2r)^{\alpha}x^{2r}y^{2r} + (nr)(2r)^{\alpha}x^2y^{2r},\\
D_x^{\alpha}D_y^{\alpha}(f(x,y)) &= \frac{nr}{2}(2r)^{\alpha}(2r)^{\alpha}x^{2r}y^{2r} + (nr)(2r)^{\alpha}(2)^{\alpha}x^2y^{2r},\\
D_x^{a}D_y^{b}(f(x,y)) &= \frac{nr}{2}(2r)^{a}(2r)^{b}x^{2r}y^{2r} + (nr)(2r)^{b}(2)^{a}x^2y^{2r},\\
D_x^{b}D_y^{a}(f(x,y)) &= \frac{nr}{2}(2r)^{b}(2r)^{a}x^{2r}y^{2r} + (nr)(2r)^{a}(2)^{b}x^2y^{2r},\\
D_x^{\frac{1}{2}}(f(x,y)) &= \frac{nr}{2}(2r)^{\frac{1}{2}}x^{2r}y^{2r} + (nr)2^{\frac{1}{2}}x^2y^{2r},\\
D_y^{\frac{1}{2}}(f(x,y)) &= \frac{nr}{2}(2r)^{\frac{1}{2}}x^{2r}y^{2r} + (nr)(2r)^{\frac{1}{2}}x^2y^{2r},\\
D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) &= \frac{nr}{2}(2r)x^{2r}y^{2r} + 2^{\frac{1}{2}}(nr)(2r)^{\frac{1}{2}}x^2y^{2r}.
\end{aligned}
$$

Using the Theorem 2.1, and column 4 of Tables 1, 2, and 3. We get the desired result. □

Theorem 2.4. *If G is any graph of order n and size m, then*

$$M(G^{+-}; x, y) = mx^my^m + m(n-2)x^{n-2}y^m.$$

Proof. The G^{+-} be a transformation graph of any graph G, has $m + n$ vertices, and $m + m(n-2)$ edges. Using the Proposition 1.1, we have $d_{G^{+-}}(v_i) = m$ and $d_{G^{+-}}(e_i) = n-2$. Therefore, the edge partition of G^{+-} is given by

$$
\begin{aligned}
|E_{\{m,m\}}| &= |uv \in E(G^{+-}) : d_u = m \text{ and } d_v = m| = m\\
|E_{\{n-2,m\}}| &= |uv \in E(G^{+-}) : d_u = n-2 \text{ and } d_v = m| = m(n-2).
\end{aligned}
$$

From the definition, The M-polynomial of G^{+-} is obtained below.

$$
\begin{aligned}
M(G^{+-}; x, y) &= \sum_{i \le j} m_{ij}(G^{+-})x^iy^j\\
&= |E_{\{m,m\}}|x^my^m + |E_{\{n-2,m\}}|x^{n-2}y^m\\
&= mx^my^m + m(n-2)x^{n-2}y^m.
\end{aligned}
$$

□

Corollary 2.5. *If G^{+-} be the trasformation graph of any graph G, then*

1. $M_1(G^{+-}) = m(mn + n^2 + 4 - 4n)$,
2. $M_2(G^{+-}) = m^2(m + n^2 + 4 - 4n)$,
3. $M_2^m(G^{+-}) = \frac{m+1}{m}$,
4. $S_D(G^{+-}) = (n-2)^2 + m(m+2)$,
5. $H(G^{+-}) = \frac{n+2mn-3m-2}{n+m-2}$,
6. $I_n(G^{+-}) = \frac{m^2(n+m-2)+2m^2(n-2)^2}{2(n+m-2)}$,
7. $R_\alpha(G^{+-}) = m^{2\alpha+1} + m^{\alpha+1}(n-2)^{\alpha+1}$,
8. $\chi_\alpha(G^{+-}) = 2^\alpha m^{\alpha+1} + m(n-2)(m+n-2)^\alpha$,
9. $M_1^\alpha(G^{+-}) = 2m^\alpha + m(n-2)^\alpha + m^\alpha(n-2)$,
10. $M_{(a,b)}(G^{+-}) = 2m^{a+b+1} + m^{b+1}(n-2)^{a+1} + m^{a+1}(n-2)^{b+1}$,
11. $GA(G^{+-}) = m + 2\left(\frac{m^{\frac{3}{2}}(n-2)^{\frac{3}{2}}}{m+n-2}\right)$.

Proof. Let $M(G^{+-}; x, y) = f(x, y) = mx^m y^m + m(n-2)x^{n-2}y^m$. Then we have

$$\begin{aligned}
D_x(f(x,y)) &= m^2x^my^m + m(n-2)^2x^{n-2}y^m,\\
D_y(f(x,y)) &= m^2x^my^m + m^2(n-2)x^{n-2}y^m,\\
D_xD_y(f(x,y)) &= m^3x^my^m + m^2(n-2)^2x^{n-2}y^m,\\
S_x(f(x,y)) &= x^my^m + mx^{n-2}y^m,\\
S_y(f(x,y)) &= x^my^m + (n-2)x^{n-2}y^m\\
S_xS_y(f(x,y)) &= \frac{1}{m}x^my^m + x^{n-2}y^m,\\
D_xS_y(f(x,y)) &= mx^my^m + (n-2)^2x^{n-2}y^m,\\
D_yS_x(f(x,y)) &= mx^my^m + m^2x^{n-2}y^m,\\
D_x^\alpha(f(x,y)) &= m^{\alpha+1}x^my^m + m(n-2)^{\alpha+1}x^{n-2}y^m,\\
D_y^\alpha(f(x,y)) &= m^{\alpha+1}x^my^m + m^{\alpha+1}(n-2)x^{n-2}y^m,\\
D_x^\alpha D_y^\alpha(f(x,y)) &= m^{2\alpha+1}x^my^m + m^{\alpha+1}(n-2)^{\alpha+1}x^{n-2}y^m,\\
D_x^aD_y^b(f(x,y)) &= m^{a+b+1}x^my^m + m^{b+1}(n-2)^{a+1}x^{n-2}y^m,\\
D_x^bD_y^a(f(x,y)) &= m^{a+b+1}x^my^m + m^{a+1}(n-2)^{b+1}x^{n-2}y^m,\\
D_x^{\frac{1}{2}}(f(x,y)) &= mm^{\frac{1}{2}}x^my^m + m(n-2)(n-2)^{\frac{1}{2}}x^{n-2}y^m,\\
D_y^{\frac{1}{2}}(f(x,y)) &= mm^{\frac{1}{2}}x^my^m + m(n-2)m^{\frac{1}{2}}x^{n-2}y^m,\\
D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) &= m^2x^my^m + m(n-2)m^{\frac{1}{2}}(n-2)^{\frac{1}{2}}x^{n-2}y^mx^{n-2}y^m.
\end{aligned}$$

Using the Theorem 2.4, and column 4 of Tables 1, 2, and 3. We get the desired result. □

Theorem 2.6. *If G is any graph of order n and size m, then*

$$M(G^{-+}; x, y) = \left(\binom{n}{2} - m\right)x^{n-1}y^{n-1} + 2mx^2y^{n-1}.$$

Proof. The G^{-+} be a transformation graph of any graph G, has $n+m$ vertices, and $\left(\binom{n}{2} + m\right)$ edges. Using the Proposition 1.1, we have $d_{G^{-+}}(v_i) = n-1$ and

$d_{G^{-+}}(e_i) = 2$. Therefore, the edge partition of G^{-+} is given by

$$\begin{aligned} |E_{\{n-1,n-1\}}| &= |uv \in E(G^{-+}) : d_u = n-1 \, and \, d_v = n-1| = \left(\binom{n}{2} - m\right) \\ |E_{\{2,n-1\}}| &= |uv \in E(G^{-+}) : d_u = 2 \, and \, d_v = n-1| = 2m. \end{aligned}$$

From the definition, The M-polynomial of G^{-+} is obtained below.
$M(G^{-+}; x, y) = \sum_{i \le j} m_{ij}(G^{-+}) x^i y^j = \left(\binom{n}{2} - m\right) x^{n-1} y^{n-1} + 2m x^2 y^{n-1}$. □

Corollary 2.7. *If G^{-+} be the trasformation graph of graph G, then*

1. $M_1(G^{-+}) = 2\left(\binom{n}{2} - m\right)(n-1) + 2m(n+1)$,
2. $M_2(G^{-+}) = \left(\binom{n}{2} - m\right)(n-1)^2 + 4m(n-1)$,
3. $M_2^m(G^{-+}) = \frac{\left(\binom{n}{2}-m\right)(n-1)+m(n-1)^2}{(n-1)^3}$,
4. $S_D(G^{-+}) = 2\left(\binom{n}{2} - m\right) + \frac{m(n^2-2n+5)}{(n-1)}$,
5. $H(G^{-+}) = \frac{(n+1)\left(\binom{n}{2}-m\right)+4m(n-1)}{n^2-1}$,
6. $I_n(G^{-+}) = \frac{\left(\binom{n}{2}-m\right)(n^2-1)+8m(n-1)}{2(n+1)}$,
7. $R_\alpha(G^{-+}) = \left(\binom{n}{2} - m\right)(n-1)^{2\alpha} + m2^{\alpha+1}(n-1)^\alpha$,
8. $\chi_\alpha(G^{-+}) = \left(\binom{n}{2} - m\right)(2n-2)^\alpha + 2m(n+1)^\alpha$,
9. $M_1^\alpha(G^{-+}) = 2\left(\binom{n}{2} - m\right)(n-1)^{\alpha-1} + 2m(2^{\alpha-1} + (n-1)^{\alpha-1})$,
10. $M_{(a,b)}(G^{-+}) = 2\left(\binom{n}{2} - m\right)(n-1)^{a+b} + 2m(2^a(n-1)^b + 2^b(n-1)^a)$,
11. $GA(G^{-+}) = \binom{n}{2} - m + \frac{2^{\frac{5}{2}} m(n-1)^{\frac{1}{2}}}{n+1}$.

Proof. Let $M(G^{-+}; x, y) = f(x,y) = \left(\binom{n}{2} - m\right) x^{n-1} y^{n-1} + 2m x^2 y^{n-1}$. Then we have

$$\begin{aligned} D_x(f(x,y)) &= \left(\binom{n}{2} - m\right)(n-1)x^{n-1}y^{n-1} + 4mx^2y^{n-1}, \\ D_y(f(x,y)) &= \left(\binom{n}{2} - m\right)(n-1)x^{n-1}y^{n-1} + 2m(n-1)x^2y^{n-1}, \\ D_y(f(x,y)) &= \left(\binom{n}{2} - m\right)(n-1)x^{n-1}y^{n-1} + 2m(n-1)x^2y^{n-1}, \\ D_xD_y(f(x,y)) &= \left(\binom{n}{2} - m\right)(n-1)^2x^{n-1}y^{n-1} + 4m(n-1)x^2y^{n-1}, \\ S_x(f(x,y)) &= \frac{\left(\binom{n}{2} - m\right)}{n-1}x^{n-1}y^{n-1} + mx^2y^{n-1}, \\ S_y(f(x,y)) &= \frac{\left(\binom{n}{2} - m\right)}{n-1}x^{n-1}y^{n-1} + \frac{2m}{n-1}x^2y^{n-1}, \\ S_xS_y(f(x,y)) &= \frac{\left(\binom{n}{2} - m\right)}{(n-1)^2}x^{n-1}y^{n-1} + \frac{m}{n-1}x^2y^{n-1}, \\ D_xS_y(f(x,y)) &= \left(\binom{n}{2} - m\right)x^{n-1}y^{n-1} + \frac{4m}{n-1}x^2y^{n-1}, \\ D_yS_x(f(x,y)) &= \left(\binom{n}{2} - m\right)x^{n-1}y^{n-1} + m(n-1)x^2y^{n-1}, \end{aligned}$$

$$\begin{aligned}
D_x^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\alpha}x^{n-1}y^{n-1}+m2^{\alpha+1}x^2y^{n-1},\\
D_y^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\alpha}x^{n-1}y^{n-1}+2m(n-1)^{\alpha}x^2y^{n-1},\\
D_x^{\alpha}D_y^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{2\alpha}x^{n-1}y^{n-1}+m2^{\alpha+1}(n-1)^{\alpha}x^2y^{n-1},\\
D_x^{a}D_y^{b}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{a+b}x^{n-1}y^{n-1}+m2^{a+1}(n-1)^{b}x^2y^{n-1},\\
D_x^{b}D_y^{a}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{a+b}x^{n-1}y^{n-1}+m2^{b+1}(n-1)^{a}x^2y^{n-1},\\
D_x^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\frac{1}{2}}x^{n-1}y^{n-1}+2m2^{\frac{1}{2}}x^2y^{n-1},\\
D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\frac{1}{2}}x^{n-1}y^{n-1}+2m(n-1)^{\frac{1}{2}}x^2y^{n-1},\\
D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\frac{1}{2}}(n-1)^{\frac{1}{2}}x^{n-1}y^{n-1}+2m2^{\frac{1}{2}}(n-1)^{\frac{1}{2}}x^2y^{n-1}.
\end{aligned}$$

Using the Theorem 2.6, and column 4 of Tables 1, 2, and 3. We get the desired result. □

Theorem 2.8. *If G is any r-regular graph of order n and size m, then*

$$M(G^{--};x,y)=\left(\binom{n}{2}-\frac{nr}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}}+\frac{nr}{2}(n-2)x^{n-2}y^{\frac{2n+nr-2-4r}{2}}.$$

Proof. The G^{--} be a transformation graph of any r-regular graph G, has $\frac{n(2+r)}{2}$ vertices, and $\binom{n}{2}+\frac{n^2r}{2}-\frac{3nr}{2}$ edges. Using the Proposition 1.1, we have $d_{G^{--}}(v_i)=\frac{2n+nr-2-4r}{2}$ and $d_{G^{--}}(e_i)=n-2$. Therefore, the edge partition of G^{--} is given by

$$\begin{aligned}
\left|E_{\{\frac{2n+nr-2-4r}{2},\frac{2n+nr-2-4r}{2}\}}\right| &= \left|uv\in E(G^{--}): d_u=\frac{2n+nr-2-4r}{2}\ \textit{and}\ d_v=\frac{2n+nr-2-4r}{2}\right|\\
&= \binom{n}{2}-\frac{nr}{2}\\
\left|E_{\{n-2,\frac{2n+nr-2-4r}{2}\}}\right| &= |uv\in E(G^{--}): d_u=n-2\ \textit{and}\ d_v=n+m-1-2r|=\frac{nr}{2}(n-2).
\end{aligned}$$

From the definition, The M-polynomial of G^{--} is obtained below.

$$\begin{aligned}
M(G^{--};x,y) &= \sum_{i\le j} m_{ij}(G^{--})x^iy^j\\
&= \left(\binom{n}{2}-\frac{nr}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}}+\frac{nr}{2}(n-2)x^{n-2}y^{\frac{2n+nr-2-4r}{2}}.
\end{aligned}$$

□

Corollary 2.9. *If G^{--} be the trasformation graph of any r-regular graph G, then*

1. $M_1(G^{--})=2\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)+\frac{nr}{2}(n-2)^2+\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)$,
2. $M_2(G^{--})=\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^2+\frac{nr}{2}(n-2)^2\left(\frac{2n+nr-2-4r}{2}\right)$,
3. $M_2^m(G^{--})=\frac{4\left(\binom{n}{2}-\frac{nr}{2}\right)}{(2n+nr-2-4r)^2}+\frac{\frac{nr}{2}}{2n+nr-2-4r}$,

4. $S_D(G^{--}) = 2\left(\binom{n}{2} - \frac{nr}{2}\right) + \frac{\frac{nr}{2}(n-2)^2}{2n+nr-2-4r} + \frac{nr}{4}(2n+nr-2-4r)$,
5. $H(G^{--}) = \frac{2\left(\binom{n}{2}-\frac{nr}{2}\right)}{2n+nr-2-4r} + \frac{2nr(n-2)}{4n+nr-6-4r}$,
6. $I_n(G^{--}) = \frac{\left(\binom{n}{2}-\frac{nr}{2}\right)(2n+nr-2-4r)}{4} + \frac{nr(n-2)^2(2n+nr-2-4r)}{4(4n+nr-6-4r)}$,
7. $R_\alpha(G^{--}) = \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{2\alpha} + \frac{nr}{2}(n-2)^{\alpha+1}\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha}$,
8. $\chi_\alpha(G^{--}) = \left(\binom{n}{2} - \frac{nr}{2}\right)(nr+2n-2-4r)^\alpha + \frac{nr}{2}(n-2)\left(\frac{4n+nr-2-4r}{2}\right)^\alpha$,
9. $M_1^\alpha(G^{--}) = 2\left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha-1} + \frac{nr}{2}(n-2)^\alpha$
, $+\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha-1}$,
10. $M_{(a,b)}(G^{--}) = 2\left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{a+b} + \frac{nr}{2}(n-2)^{a+1}\left(\frac{2n+nr-2-4r}{2}\right)^b$
, $+\frac{nr}{2}(n-2)^{b+1}\left(\frac{2n+nr-2-4r}{2}\right)^a$,
11. $GA(G^{--}) = \left(\binom{n}{2} - \frac{nr}{2}\right) + \frac{nr(n-2)^{\frac{3}{2}}2^{\frac{1}{2}}(2n+nr-2-4r)^{\frac{1}{2}}}{2n+nr-2-4r}$.

Proof. Let $M(G^{--};x,y) = f(x,y) = \left(\binom{n}{2} - \frac{nr}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} + \frac{nr}{2}(n-2)x^{n-2}y^{\frac{2n+nr-2-4r}{2}}$.

Then we have

$$\begin{aligned}
D_x(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} \\
&\quad + \frac{nr}{2}(n-2)^2x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
D_y(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} \\
&\quad + \frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
D_xD_y(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^2x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} \\
&\quad + \frac{nr}{2}(n-2)^2\left(\frac{2n+nr-2-4r}{2}\right)x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
S_x(f(x,y)) &= \frac{2\left(\binom{n}{2} - \frac{nr}{2}\right)}{2n+nr-2-4r}x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} + \frac{nr}{2}x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
S_y(f(x,y)) &= \frac{2\left(\binom{n}{2} - \frac{nr}{2}\right)}{2n+nr-2-4r}x^{n+m-1-2r}y^{n+m-1-2r} + \frac{nr(n-2)}{2n+nr-2-4r}x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
S_xS_y(f(x,y)) &= \frac{4\left(\binom{n}{2} - \frac{nr}{2}\right)}{(2n+nr-2-4r)^2}x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} + \frac{nr}{2n+nr-2-4r}x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
D_xS_y(f(x,y)) &= \left(\binom{n}{2} - m\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} + \frac{nr(n-2)^2}{(2n+nr-2-4r)}x^{n-2}y^{\frac{2n+nr-2-4r}{2}}, \\
D_yS_x(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)x^{n+m-1-2r}y^{n+m-1-2r} + m(n+m-1-2r)x^{n-2}y^{n+m-1-2r}, \\
D_x^\alpha(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^\alpha x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}} \\
&\quad + \frac{nr}{2}(n-2)^{\alpha+1}x^{n-2}y^{\frac{2n+nr-2-4r}{2}},
\end{aligned}$$

$$
\begin{aligned}
D_y^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}, \\
D_x^{\alpha}D_y^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{2\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\frac{nr}{2}(n-2)^{\alpha+1}\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}, \\
D_x^{a}D_y^{b}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{a+b} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\frac{nr}{2}(n-2)^{a+1}\left(\frac{2n+nr-2-4r}{2}\right)^{b} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}, \\
D_x^{b}D_y^{a}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{a+b} x^{\frac{2n+nr-2-4r}{2}} y^{n+m-1-2r} \\
&\quad +\frac{nr}{2}(n-2)^{b+1}\left(\frac{2n+nr-2-4r}{2}\right)^{a} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}, \\
D_x^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +m(n-2)^{\frac{3}{2}} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}, \\
D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right))^{\frac{1}{2}} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}, \\
D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\frac{nr}{2}(n-2)^{\frac{3}{2}}\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{n-2} y^{\frac{2n+nr-2-4r}{2}}.
\end{aligned}
$$

Using the Theorem 2.8, and column 4 of Tables 1, 2, and 3. We get the desired result. □

Theorem 2.10. *If G is any r-regular graph of order n and size m, then*

$$
\begin{aligned}
M(\overline{G^{++}};x,y) &= \left(\binom{n}{2}-\frac{nr}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} + \binom{\frac{nr}{2}}{2} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}.
\end{aligned}
$$

Proof. The $\overline{G^{++}}$ be a transformation graph of any r-regular graph G, has $\frac{n(2+r)}{2}$ vertices, and $\binom{n}{2}+\binom{\frac{nr}{2}}{2}+\frac{3nr}{2}$ edges. Using the Proposition 1.2, we have $d_{\overline{G^{++}}}(v_i) = \frac{2n+nr-2-4r}{2}$ and $d_{\overline{G^{++}}}(e_i) = \frac{2n+nr-6}{2}$. Therefore, the edge partition of $\overline{G^{++}}$ is given by

$$\begin{aligned}
\left|E_{\{\frac{2n+nr-2-4r}{2},\frac{2n+nr-2-4r}{2}\}}\right| &= \left|uv\in E(\overline{G^{++}}): d_u=\frac{2n+nr-2-4r}{2} \text{ and } d_v=\frac{2n+nr-2-4r}{2}\right| \\
&= \binom{n}{2}-\frac{nr}{2} \\
\left|E_{\{\frac{2n+nr-6}{2},\frac{2n+nr-6}{2}\}}\right| &= \left|uv\in E(\overline{G^{++}}): d_u=\frac{2n+nr-6}{2} \text{ and } d_v=\frac{2n+nr-6}{2}\right| = \binom{\frac{nr}{2}}{2} \\
\left|E_{\{\frac{2n+nr-2-4r}{2},\frac{2n+nr-6}{2}\}}\right| &= \left|uv\in E(\overline{G^{++}}): d_u=\frac{2n+nr-2-4r}{2} \text{ and } d_v=\frac{2n+nr-6}{2}\right| \\
&= \frac{nr}{2}(n-2).
\end{aligned}$$

From the definition, The M-polynomial of $\overline{G^{++}}$ is obtained below.

$$\begin{aligned}
M(\overline{G^{++}};x,y) &= \sum_{i\le j} m_{ij}(\overline{G^{++}})x^i y^j \\
&= \left(\binom{n}{2}-\frac{nr}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}}+\binom{\frac{nr}{2}}{2}x^{\frac{2n+nr-6}{2}}y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-6}{2}}.
\end{aligned}$$

□

Corollary 2.11. *If $\overline{G^{++}}$ be the trasformation graph of any r-regular graph G, then*

1. $M_1(\overline{G^{++}})=2\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)+2\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)+\frac{nr}{2}(n-2)[\left(\frac{2n+nr-2-4r}{2}\right)$ $+\left(\frac{2n+nr-6}{2}\right)]$,
2. $M_2(\overline{G^{++}})=\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^2+\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^2+\frac{nr}{2}(n-2)\left(\frac{2n+nr-6}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)$,
3. $M_2^m(\overline{G^{++}})=\frac{4\left(\binom{n}{2}-\frac{nr}{2}\right)}{(2n+nr-2-4r)^2}+\frac{4\binom{\frac{nr}{2}}{2}}{(2n+nr-6)^2}+\frac{2(n-2)}{(2n+nr-2-4r)(2n+nr-6)}$,
4. $S_D(\overline{G^{++}})=2\left(\binom{n}{2}-\frac{nr}{2}\right)+2\binom{\frac{nr}{2}}{2}+\frac{nr}{2}(n-2)[\left(\frac{2n+nr-2-4r}{2}\right)+\left(\frac{2n+nr-6}{2}\right)]$,
5. $H(\overline{G^{++}})=\frac{2\left(\binom{n}{2}-\frac{nr}{2}\right)}{(2n+nr-2-4r)}+\frac{2\binom{m}{2}}{(2n+nr-6)}+\frac{nr(n-2)}{2n+nr-4-2r}$,
6. $I_n(\overline{G^{++}})=\frac{\left(\binom{n}{2}-\frac{nr}{2}\right)(2n+nr-2-4r)}{4}+\frac{\binom{\frac{nr}{2}}{2}(2n+nr-6)}{4}+\frac{nr(n-2)(2n+nr-2-4r)(2n+nr-6)}{8(2n+nr-4-2r)}$,
7. $R_\alpha(\overline{G^{++}})=\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{2\alpha}+\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{2\alpha}$ $+\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha}\left(\frac{2n+nr-6}{2}\right)^{\alpha}$,
8. $\chi_\alpha(\overline{G^{++}})=\left(\binom{n}{2}-\frac{nr}{2}\right)(2n+nr-2-4r)^\alpha+\binom{\frac{nr}{2}}{2}(2n+nr-6)^\alpha$ $+\frac{nr}{2}(n-2)(2n+nr-4-2r)^\alpha$,
9. $M_1^\alpha(\overline{G^{++}})=2\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha-1}+2\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{\alpha-1}$ $+\frac{nr}{2}(n-2)[\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha-1}+\left(\frac{2n+nr-6}{2}\right)^{\alpha-1}]$,
10. $M_{(a,b)}(\overline{G^{++}})=2\left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{a+b}+2\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{a+b}$ $+\frac{nr}{2}(n-2)[\left(\frac{2n+nr-2-4r}{2}\right)^{a}\left(\frac{2n+nr-6}{2}\right)^{b}]$,
11. $GA(\overline{G^{++}})=\left(\binom{n}{2}-\frac{nr}{2}\right)+\binom{\frac{nr}{2}}{2}+\frac{nr(n-2)(2n+nr-2-4r)^{\frac{1}{2}}(2n+nr-6)^{\frac{1}{2}}}{4n+2nr-8-4r}$.

Proof. Let $M(\overline{G^{++}};x,y)=f(x,y)=\left(\binom{n}{2}-\frac{nr}{2}\right)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-2-4r}{2}}+\binom{\frac{nr}{2}}{2}x^{\frac{2n+nr-6}{2}}y^{\frac{2n+nr-6}{2}}$ $+\frac{nr}{2}(n-2)x^{\frac{2n+nr-2-4r}{2}}y^{\frac{2n+nr-6}{2}}$.

Then we have

$$
\begin{aligned}
D_x(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&+ \binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right) x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_y(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&+ \binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right) x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ m(n-2)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_xD_y(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)(\frac{2n+nr-2-4r}{2})^2 x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&+ \binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^2 x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
S_x(f(x,y)) &= \frac{2\left(\binom{n}{2} - \frac{nr}{2}\right)}{2n+nr-2-4r} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} + \frac{2\binom{\frac{nr}{2}}{2}}{2n+nr-6} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr(n-2)}{2n+nr-2-4r} x^{2n+nr-2-4r2} y^{\frac{2n+nr-6}{2}}, \\
S_y(f(x,y)) &= \frac{2\left(\binom{n}{2} - \frac{nr}{2}\right)}{(2n+nr-2-4r)} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} + \frac{2\binom{\frac{nr}{2}}{2}}{2n+nr-6} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr(n-2)}{2n+nr-6} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
S_xS_y(f(x,y)) &= \frac{4\left(\binom{n}{2} - \frac{nr}{2}\right)}{(2n+nr-2-4r)^2} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} + \frac{4\binom{\frac{nr}{2}}{2}}{(2n+nr-6)^2} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{2nr(n-2)}{(2n+nr-6)(2n+nr-2-4r)} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_xS_y(f(x,y)) &= \binom{n}{2} - \frac{nr}{2} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} + \binom{\frac{nr}{2}}{2} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr(n-2)(2n+nr-2-4r)}{2(2n+nr-6)} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_yS_x(f(x,y)) &= \binom{n}{2} - \frac{nr}{2} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} + \binom{\frac{nr}{2}}{2} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr(n-2)(2n+nr-6)}{2(2n+nr-2-4r)} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_x^{\alpha}(f(x,y)) &= \left(\binom{n}{2} - \frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&+ \binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{\alpha} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&+ \frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}},
\end{aligned}
$$

$$
\begin{aligned}
D_y^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{\alpha} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_x^{\alpha}D_y^{\alpha}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{2\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{2\alpha} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\alpha}\left(\frac{2n+nr-6}{2}\right)^{\alpha} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_x^{a}D_y^{b}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{a+b} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{a+b} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{a}\left(\frac{2n+nr-6}{2}\right)^{b} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_x^{b}D_y^{a}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{a+b} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{a+b} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{b}\left(\frac{2n+nr-6}{2}\right)^{a} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_x^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-6}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}, \\
D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-\frac{nr}{2}\right)\left(\frac{2n+nr-2-4r}{2}\right) x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-2-4r}{2}} \\
&\quad +\binom{\frac{nr}{2}}{2}\left(\frac{2n+nr-6}{2}\right) x^{\frac{2n+nr-6}{2}} y^{\frac{2n+nr-6}{2}} \\
&\quad +\frac{nr}{2}(n-2)\left(\frac{2n+nr-6}{2}\right)^{\frac{1}{2}}\left(\frac{2n+nr-2-4r}{2}\right)^{\frac{1}{2}} x^{\frac{2n+nr-2-4r}{2}} y^{\frac{2n+nr-6}{2}}.
\end{aligned}
$$

Using the Theorem 2.10, and column 4 of Tables 1, 2, and 3. We get the desired result. □

Theorem 2.12. *If G is any graph of order n and size m,then*

$$M(\overline{G^{+-}};x,y) = \left(\binom{n}{2} - m\right) x^{n-1}y^{n-1} + 2mx^{n-1}y^{m+1} + \binom{m}{2}x^{m+1}y^{m+1}.$$

Proof. The $\overline{G^{+-}}$ be a transformation graph of any graph G, has $n+m$ vertices, and $\binom{n}{2}+\binom{m}{2}+m$ edges. Using the Proposition 1.2, we have $d_{\overline{G^{+-}}}(v_i) = n-1$ and $d_{\overline{G^{+-}}}(e_i) = m+1$. Therefore, the edge partition of $\overline{G^{+-}}$ is given by

$$\begin{aligned}
|E_{\{n-1,n-1\}}| &= |uv \in E(\overline{G^{+-}}) : d_u = n-1 \text{ and } d_v = n-1| = \binom{n}{2} - m \\
|E_{\{n-1,m+1\}}| &= |uv \in E(\overline{G^{+-}}) : d_u = n-1 \text{ and } d_v = m+1| = 2m \\
|E_{\{m+1,m+1\}}| &= |uv \in E(\overline{G^{+-}}) : d_u = m+1 \text{ and } d_v = m+1| = \binom{m}{2}.
\end{aligned}$$

From the definition, The M-polynomial of $\overline{G^{+-}}$ is obtained below.

$$\begin{aligned}
M(\overline{G^{+-}};x,y) &= \sum_{i\leq j} m_{ij}(\overline{G^{+-}})x^iy^j \\
&= \left(\binom{n}{2} - m\right) x^{n-1}y^{n-1} + 2mx^{n-1}y^{m+1} + \binom{m}{2}x^{m+1}y^{m+1}.
\end{aligned}$$

□

Corollary 2.13. *If $\overline{G^{+-}}$ be the trasformation graph of any graph G, then*

1. $M_1(\overline{G^{+-}}) = 2(\left(\binom{n}{2} - m\right)(n-1)) + 2m(n+m) + 2\left(\binom{m}{2}(m+1)\right)$,
2. $M_2(\overline{G^{+-}}) = \left(\binom{n}{2} - m\right)(n-1)^2 + 2m(n-1)(m+1) + \binom{m}{2}(m+1)^2$,
3. $M_2^m(\overline{G^{+-}}) = \frac{\binom{n}{2}-m}{(n-1)^2} + \frac{2m}{(n-1)(m+1)} + \frac{\binom{m}{2}}{(m+1)^2}$,
4. $S_D(\overline{G^{+-}}) = 2\left(\binom{n}{2} - m\right) + 2m\left(\frac{(n-1)^2+(m+1)^2}{(m+1)(n+1)}\right) + 2\binom{m}{2}$,
5. $H(\overline{G^{+-}}) = \frac{\binom{n}{2}-m}{n-1} + \frac{4m}{m+n} + \frac{\binom{m}{2}}{(m+1)}$,
6. $I_n(\overline{G^{+-}}) = \frac{\left(\binom{n}{2}-m\right)(n-1)}{2} + \frac{2m(m+1)(n-1)}{m+n} + \frac{\binom{m}{2}(m+1)}{2}$,
7. $R_\alpha(\overline{G^{+-}}) = \left(\binom{n}{2} - m\right)(n-1)^{2\alpha} + 2m(m+1)^\alpha(n+1)^\alpha + \binom{m}{2}(m+1)^{2\alpha}$,
8. $\chi_\alpha(\overline{G^{+-}}) = \left(\binom{n}{2} - m\right)(2n-2)^\alpha + 2m(m+n)^\alpha + \binom{m}{2}(2m+2)^\alpha$,
9. $M_1^\alpha(\overline{G^{+-}}) = 2[\left(\binom{n}{2} - m\right)(n-1)^{\alpha-1}] + 2m[(n-1)^{\alpha-1} + (m+1)^{\alpha-1}]$ $+ 2\binom{m}{2}(m+1)^{\alpha-1}$,
10. $M_{(a,b)}(\overline{G^{+-}}) = 2[\left(\binom{n}{2} - m\right)(n-1)^{a+b}] + 2m[(n-1)^a(m+1)^b$ $+ (n-1)^b(m+1)^a] + 2\binom{m}{2}(m+1)^{a+b}$,
11. $GA(\overline{G^{+-}}) = \left(\binom{n}{2} - m\right) + \frac{4m(n-1)^{\frac{1}{2}}(m+1)^{\frac{1}{2}}}{m+n} + \frac{\binom{m}{2}(n-1)^{\frac{1}{2}}(m+1)^{\frac{1}{2}}}{(m+1)}$.

Proof. Let $M(\overline{G^{+-}};x,y) = f(x,y) = \left(\binom{n}{2}-m\right)x^{n-1}y^{n-1}+2mx^{n-1}y^{m+1}+\binom{m}{2}x^{m+1}y^{m+1}$.
Then we have

$$\begin{aligned}
D_x(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)x^{n-1}y^{n-1}+2m(n-1)x^{n-1}y^{m+1}+\binom{m}{2}(m+1)x^{m+1}y^{m+1},\\
D_y(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)x^{n-1}y^{n-1}+2m(m+1)x^{n-1}y^{m+1}+\binom{m}{2}(m+1)x^{m+1}y^{m+1},\\
D_xD_y(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^2x^{n-1}y^{n-1}+2m(m+1)(n-1)x^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^2x^{m+1}y^{m+1},\\
S_x(f(x,y)) &= \frac{\left(\binom{n}{2}-m\right)}{n-1}x^{n-1}y^{n-1}+\frac{2m}{(n-1)}x^{n-1}y^{m+1}+\frac{\binom{m}{2}}{m+1}x^{m+1}y^{m+1},\\
S_y(f(x,y)) &= \frac{\left(\binom{n}{2}-m\right)}{n-1}x^{n-1}y^{n-1}+\frac{2m}{m+1}x^{n-1}y^{m+1}+\frac{\binom{m}{2}}{(m+1)}x^{m+1}y^{m+1},\\
S_xS_y(f(x,y)) &= \frac{\left(\binom{n}{2}-m\right)}{(n-1)^2}x^{n-1}y^{n-1}+\frac{2m}{(n-1)(m+1)}x^{n-1}y^{m+1}+\frac{\binom{m}{2}}{(m+1)^2}x^{m+1}y^{m+1},\\
D_xS_y(f(x,y)) &= \left(\binom{n}{2}-m\right)x^{n-1}y^{n-1}+\frac{2m(n-1)}{m+1}x^{n-1}y^{m+1}+\binom{m}{2}x^{m+1}y^{m+1},\\
D_yS_x(f(x,y)) &= \left(\binom{n}{2}-m\right)x^{n-1}y^{n-1}+\frac{2m(m+1)}{n-1}x^{n-1}y^{m+1}+\binom{m}{2}x^{m+1}y^{m+1},\\
D_x^\alpha(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^\alpha x^{n-1}y^{n-1}+2m(n-1)^\alpha x^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^\alpha x^{m+1}y^{m+1},\\
D_y^\alpha(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^\alpha x^{n-1}y^{n-1}+2m(m+1)^\alpha x^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^\alpha x^{m+1}y^{m+1},\\
D_x^\alpha D_y^\alpha(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{2\alpha}x^{n-1}y^{n-1}+2m(m+1)^\alpha(n-1)^\alpha x^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^{2\alpha}x^{m+1}y^{m+1},\\
D_x^aD_y^b(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{a+b}x^{n-1}y^{n-1}+2m(n-1)^a(m+1)^bx^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^{a+b}x^{m+1}y^{m+1},\\
D_x^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\frac{1}{2}}x^{n-1}y^{n-1}+2m(n-1)^{\frac{1}{2}}x^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^{\frac{1}{2}}x^{m+1}y^{m+1},\\
D_y^{\frac{1}{2}}(f(x,y)) &= \left(\binom{n}{2}-m\right)(n-1)^{\frac{1}{2}}x^{n-1}y^{n-1}+2m(m+1)^{\frac{1}{2}}x^{n-1}y^{m+1}\\
&\quad+\binom{m}{2}(m+1)^{\frac{1}{2}}x^{m+1}y^{m+1},
\end{aligned}$$

$$D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) = \left(\binom{n}{2}-m\right)(n-1)^{\frac{1}{2}}(n-1)^{\frac{1}{2}}x^{n-1}y^{n-1}+2m(m+1)^{\frac{1}{2}}(n-1)^{\frac{1}{2}}x^{n-1}y^{m+1} + \binom{m}{2}(m+1)^{\frac{1}{2}}(m+1)^{\frac{1}{2}}x^{m+1}y^{m+1}.$$

Using the Theorem 2.12, and column 4 of Tables 1, 2, and 3. We get the desired result. □

Theorem 2.14. *If G is any graph of order n and size m, then*

$$M(\overline{G^{-+}};x,y) = mx^my^m + \binom{m}{2}x^{m+n-3}y^{m+n-3} + m(n-2)x^my^{m+n-3}.$$

Proof. The $\overline{G^{-+}}$ be a transformation graph of any graph G, has $n+m$ vertices, and $mn-m+\binom{m}{2}$ edges. Using the Proposition 1.2, we have $d_{\overline{G^{-+}}}(v_i)=m$ and $d_{\overline{G^{-+}}}(e_i)=m+n-3$. Therefore, the edge partition of $\overline{G^{-+}}$ is given by

$$|E_{\{m,m\}}| = |uv\in E(\overline{G^{-+}}) : d_u = m \textit{ and } d_v = m| = m$$

$$|E_{\{m+n-1,m+n-1\}}| = |uv\in E(\overline{G^{-+}}) : d_u = m+n-1 \textit{ and } d_v = m+n-1| = \binom{m}{2}$$

$$|E_{\{m,m+n-3\}}| = |uv\in E(\overline{G^{-+}}) : d_u = m \textit{ and } d_v = m+n-3| = m(n-2).$$

From the definition, The M-polynomial of $\overline{G^{-+}}$ is obtained below.

$$M(\overline{G^{-+}};x,y) = \sum_{i\le j} m_{ij}(\overline{G^{-+}})x^iy^j = mx^my^m + \binom{m}{2}x^{m+n-3}y^{m+n-3} + m(n-2)x^my^{m+n-3}.$$

□

Corollary 2.15. *If $\overline{G^{-+}}$ be the trasformation graph of any graph G, then*

1. $M_1(\overline{G^{-+}}) = 2m^2 + 2\binom{m}{2}(m+n-3) + m(n-2)(2m+n-3)$,
2. $M_2(\overline{G^{-+}}) = m^3 + \binom{m}{2}(m+n-3)^2 + m^2(n-2)(m+n-3)$,
3. $M_2^m(\overline{G^{-+}}) = \frac{1}{m} + \frac{\binom{m}{2}}{(m+n-3)^2} + \frac{(n-2)}{(m+n-3)}$,
4. $S_D(\overline{G^{-+}}) = 2m + 2\binom{m}{2} + \frac{m^2(n-2)}{(m+n-3)} + (m+n-3)(n-2)$,
5. $H(\overline{G^{-+}}) = \frac{(m+n-3)+\binom{m}{2}}{m+n-3} + \frac{2m(n-2)}{2m+n-3}$,
6. $I_n(\overline{G^{-+}}) = \frac{m^2}{2} + \frac{\binom{m}{2}(m+n-3)}{2} + \frac{m^2(n-2)(m+n-3)}{2m+n-3}$,
7. $R_\alpha(\overline{G^{-+}}) = m^{2\alpha+1} + \binom{m}{2}(m+n-3)^{2\alpha} + m^{\alpha+1}(n-2)(m+n-3)$,
8. $\chi_\alpha(\overline{G^{-+}}) = 2^\alpha m^{\alpha+1} + \binom{m}{2}(2m+2n-6)^\alpha + m(n-2)(2m+n-3)^\alpha$,
9. $M_1^\alpha(\overline{G^{-+}}) = 2m^\alpha + 2\binom{m}{2}(m+n-3)^\alpha + m(n-2)[m^\alpha + (m+n-3)^\alpha]$,
10. $M_{(a,b)}(\overline{G^{-+}}) = 2m^{a+b+1} + 2\binom{m}{2}(m+n-3)^{a+b} + m^{a+1}(n-2)(m+n-3)^b$, $+m^{b+1}(n-2)(m+n-3)^a$,
11. $GA(\overline{G^{-+}}) = m + \binom{m}{2} + \frac{2m^{\frac{3}{2}}(n-2)(m+n-3)^{\frac{1}{2}}}{2m+n-3}$.

Proof. Let $M(\overline{G^{-+}};x,y) = f(x,y) = mx^my^m + \binom{m}{2}x^{m+n-3}y^{m+n-3} + m(n-2)x^my^{m+n-3}$.

Then we have

$$
\begin{aligned}
D_x(f(x,y)) &= m^2x^my^m + \binom{m}{2}(m+n-3)x^{m+n-3}y^{m+n-3} + m^2(n-2)x^my^{m+n-3},\\
D_y(f(x,y)) &= m^2x^my^m + \binom{m}{2}(m+n-3)x^{m+n-3}y^{m+n-3} + m(n-2)(m+n-3)x^my^{m+n-3},\\
D_xD_y(f(x,y)) &= m^3x^my^m + \binom{m}{2}(m+n-3)^2x^{m+n-3}y^{m+n-3} + m^2(n-2)(m+n-3)x^my^{m+n-3},\\
S_x(f(x,y)) &= x^my^m + \frac{\binom{m}{2}}{(m+n-3)}x^{m+n-3}y^{m+n-3} + (n-2)x^my^{mn-3},\\
S_y(f(x,y)) &= x^my^m + \frac{\binom{m}{2}}{m+n-3}x^{m+n-3}y^{m+n-3} + \frac{m(n-2)}{(m+n-3)}x^my^{m+n-3},\\
S_xS_y(f(x,y)) &= \frac{1}{m}x^my^m + \frac{\binom{m}{2}}{(m+n-3)^2}x^{m+n-3}y^{m+n-3} + \frac{(n-2)}{(m+n-3)}x^my^{m+n-3},\\
D_xS_y(f(x,y)) &= mx^my^m + \binom{m}{2}x^{m+n-3}y^{m+n-3} + \frac{m^2(n-2)}{(m+n-3)}x^my^{m+n-3},\\
D_yS_x(f(x,y)) &= mx^my^m + \binom{m}{2}x^{m+n-3}y^{m+n-3} + (m+n-3)(n-2)x^my^{m+n-3},\\
D_x^\alpha(f(x,y)) &= m^{\alpha+1}x^my^m + \binom{m}{2}(m+n-3)^\alpha x^{m+n-3}y^{m+n-3} + m(n-2)m^\alpha x^my^{m+n-3},\\
D_y^\alpha(f(x,y)) &= m^{\alpha+1}x^my^m + \binom{m}{2}(m+n-3)^\alpha x^{m+n-3}y^{m+n-3}\\
&\quad +m(n-2)(m+n-3)^\alpha x^my^{m+n-3},\\
D_x^\alpha D_y^\alpha(f(x,y)) &= m^{2\alpha+1}x^my^m + \binom{m}{2}(m+n-3)^{2\alpha}x^{m+n-3}y^{m+n-3}\\
&\quad +m^{\alpha+1}(n-2)(m+n-3)^\alpha x^my^{m+n-3},\\
D_x^aD_y^b(f(x,y)) &= m^{a+b+1}x^my^m + \binom{m}{2}(m+n-3)^{a+b}x^{m+n-3}y^{m+n-3}\\
&\quad +m(n-2)m^a(m+n-3)^bx^my^{m+n-3},\\
D_x^bD_y^a(f(x,y)) &= m^{a+b+1}x^my^m + \binom{m}{2}(m+n-3)^{a+b}x^{m+n-3}y^{m+n-3}\\
&\quad +m(n-2)m^b(m+n-3)^ax^my^{m+n-3},\\
D_x^{\frac{1}{2}}(f(x,y)) &= mm^{\frac{1}{2}}x^my^m + \binom{m}{2}(m+n-3)^{\frac{1}{2}}x^{m+n-3}y^{m+n-3} + m(n-2)m^{\frac{1}{2}}x^my^{m+n-3},\\
D_y^{\frac{1}{2}}(f(x,y)) &= mm^{\frac{1}{2}}x^my^m + \binom{m}{2}(m+n-3)^{\frac{1}{2}}x^{m+n-3}y^{m+n-3}\\
&\quad +m(n-2)(m+n-3)^{\frac{1}{2}}x^my^{m+n-3},\\
D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) &= mm^{\frac{1}{2}}m^{\frac{1}{2}}x^my^m + \binom{m}{2}(m+n-3)\frac{1}{2}(m+n-3)^{\frac{1}{2}}x^{m+n-3}y^{m+n-3}\\
&\quad +m(n-2)(m+n-3)^{\frac{1}{2}}m^{\frac{1}{2}}x^my^{m+n-3}.
\end{aligned}
$$

Using the Theorem 2.14, and column 4 of Tables 1, 2 and 3. We get the desired result. □

Theorem 2.16. *If G is any r-regular graph of order n and size m, then*

$$M(\overline{G^{--}};x,y)=\tfrac{nr}{2}x^{2r}y^{2r}+\tbinom{\frac{nr}{2}}{2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}+nrx^{2r}y^{\frac{nr+2}{2}}.$$

Proof. The $\overline{G^{--}}$ be a transformation graph of any r-regular graph G, has $\frac{n(r+2)}{2}$ vertices, and $\binom{\frac{nr}{2}}{2}+\frac{3nr}{2}$ edges. Using the Proposition 1.2, we have $d_{\overline{G^{--}}}(v_i)=2r$ and $d_{\overline{G^{--}}}(e_i)=\frac{nr+2}{2}$. Therefore, the edge partition of $\overline{G^{--}}$ is given by

$$\begin{aligned}
|E_{\{2r,2r\}}| &= |uv\in E(\overline{G^{--}}):d_u=2r \textit{ and } d_v=2r|=\frac{nr}{2}\\
\left|E_{\{\frac{nr+2}{2},\frac{nr+2}{2}\}}\right| &= \left|uv\in E(\overline{G^{--}}):d_u=\frac{nr+2}{2} \textit{ and } d_v=\frac{nr+2}{2}\right|=\binom{\frac{nr}{2}}{2}\\
\left|E_{\{2r,\frac{nr+2}{2}\}}\right| &= \left|uv\in E(\overline{G^{--}}):d_u=2r \textit{ and } d_v=\frac{nr+2}{2}\right|=nr.
\end{aligned}$$

From the definition, The M-polynomial of $\overline{G^{--}}$ is obtained below.

$$\begin{aligned}
M(\overline{G^{--}};x,y) &= \sum_{i\le j} m_{ij}(\overline{G^{--}})x^iy^j\\
&= \frac{nr}{2}x^{2r}y^{2r}+\binom{\frac{nr}{2}}{2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}+nrx^{2r}y^{\frac{nr+2}{2}}.
\end{aligned}$$

□

Corollary 2.17. *If $\overline{G^{--}}$ be the trasformation graph of any r-regular graph G, then*

1. $M_1(\overline{G^{--}})=4nr^2+\binom{\frac{nr}{2}}{2}(nr+2)+nr(\frac{nr+2}{2})$,
2. $M_2(\overline{G^{--}})=2nr^3+\binom{\frac{nr}{2}}{2}(\frac{nr+2}{2})^2+nr(nr+2)$,
3. $M_2^m(\overline{G^{--}})=\frac{n}{8r}+\frac{4\binom{\frac{nr}{2}}{2}}{(nr+2)^2}+\frac{n}{(nr+2)}$,
4. $S_D(\overline{G^{--}})=2nr+2\binom{\frac{nr}{2}}{2}+\frac{2nr}{(nr+2)}$,
5. $H(\overline{G^{--}})=\frac{n}{4r}+\frac{2\binom{\frac{nr}{2}}{2}}{nr+2}+\frac{4nr}{nr+4r+2}$,
6. $I_n(\overline{G^{--}})=\frac{nr^2}{2}+\frac{\binom{\frac{nr}{2}}{2}(nr+2)}{4}+\frac{(nr+2)2nr^2}{nr+4r+2}$,
7. $R_\alpha(\overline{G^{--}})=\frac{nr}{2}(2r)^{2\alpha}+\binom{\frac{nr}{2}}{2}(\frac{nr+2}{2})^{2\alpha}+nr(2r)^\alpha(\frac{nr+2}{2})^\alpha$,
8. $\chi_\alpha(\overline{G^{--}})=\frac{nr}{2}(4r)^\alpha+\binom{\frac{nr}{2}}{2}(nr+2)^\alpha+nr(\frac{4r+nr+2}{2})^\alpha$,
9. $M_1^\alpha(\overline{G^{--}})=2m(2r)^{\alpha-1}+2\binom{m}{2}(m+1)^{\alpha-1}+2m[(2r)^{\alpha-1}+(m+1)^{\alpha-1}]$,
10. $M_{(a,b)}(\overline{G^{--}})=nr(2r)^{a+b}+2\binom{\frac{nr}{2}}{2}(\frac{nr+2}{2})^{a+b}+nr[(2r)^a(\frac{nr+2}{2})^b+(2r)^b(\frac{nr+2}{2})^a]$,
11. $GA(\overline{G^{--}})=\frac{nr}{2}+\binom{\frac{nr}{2}}{2}+\frac{2^{\frac{1}{2}}nr(2r)^{\frac{1}{2}}(nr+2)^{\frac{1}{2}}}{(m+2r+1)}$.

Proof. Let $M(\overline{G^{--}};x,y) = f(x,y) = \frac{nr}{2}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + nrx^{2r}y^{\frac{nr+2}{2}}$.
Then we have

$$\begin{aligned}
D_x(f(x,y)) &= nr^2x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + 2nr^2x^{2r}y^{\frac{nr+2}{2}},\\
D_y(f(x,y)) &= nr^2x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + nr\left(\frac{nr+2}{2}\right)x^{2r}y^{\frac{nr+2}{2}},\\
D_xD_y(f(x,y)) &= 2nr^3x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^2x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + 2nr^2\left(\frac{nr+2}{2}\right)x^{2r}y^{\frac{nr+2}{2}},\\
S_x(f(x,y)) &= \frac{n}{4r}x^{2r}y^{2r} + \frac{2\binom{\frac{nr}{2}}{2}}{nr+2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + \frac{n}{2}x^{2r}y^{\frac{nr+2}{2}},\\
S_y(f(x,y)) &= \frac{n}{4r}x^{2r}y^{2r} + \frac{2\binom{\frac{nr}{2}}{2}}{nr+2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + \frac{2nr}{nr+2}x^{2r}y^{\frac{nr+2}{2}},\\
S_xS_y(f(x,y)) &= \frac{n}{8r}x^{2r}y^{2r} + \frac{4\binom{\frac{nr}{2}}{2}}{(nr+2)^2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + \frac{nr}{2r(nr+2)}x^{2r}y^{\frac{nr+2}{2}},\\
D_xS_y(f(x,y)) &= \frac{nr}{2}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + \frac{4nr}{(nr+2)}x^{2r}y^{\frac{nr+2}{2}},\\
D_yS_x(f(x,y)) &= \frac{nr}{2}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + nrx^{2r}y^{\frac{nr+2}{2}},\\
D_x^{\alpha}(f(x,y)) &= \frac{nr}{2}(2r)^{\alpha}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{\alpha}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + nr(2r)^{\alpha}x^{2r}y^{\frac{nr+2}{2}},\\
D_y^{\alpha}(f(x,y)) &= \frac{nr}{2}(2r)^{\alpha}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{\alpha}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}\\
&\quad + nr\left(\frac{nr+2}{2}\right)^{\alpha}x^{2r}y^{\frac{nr+2}{2}},\\
D_x^{\alpha}D_y^{\alpha}(f(x,y)) &= \frac{nr}{2}(2r)^{2\alpha}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{2\alpha}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}\\
&\quad + nr\left(\frac{nr+2}{2}\right)^{\alpha}(2r)^{\alpha}x^{2r}y^{\frac{nr+2}{2}},\\
D_x^{a}D_y^{b}(f(x,y)) &= \frac{nr}{2}(2r)^{a+b}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{a+b}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}\\
&\quad + nr\left(\frac{nr+2}{2}\right)^{b}(2r)^{a}x^{2r}y^{\frac{nr+2}{2}},\\
D_x^{b}D_y^{a}(f(x,y)) &= \frac{nr}{2}(2r)^{a+b}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{a+b}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}\\
&\quad + nr\left(\frac{nr+2}{2}\right)^{a}(2r)^{b}x^{2r}y^{\frac{nr+2}{2}},\\
D_x^{\frac{1}{2}}(f(x,y)) &= \frac{nr}{2}(2r)^{\frac{1}{2}}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{\frac{1}{2}}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}}\\
&\quad + nr(2r)^{\frac{1}{2}}x^{2r}y^{\frac{nr+2}{2}},\\
D_y^{\frac{1}{2}}(f(x,y)) &= \frac{nr}{2}(2r)^{\frac{1}{2}}x^{2r}y^{2r} + \binom{\frac{nr}{2}}{2}\left(\frac{nr+2}{2}\right)^{\frac{1}{2}}x^{\frac{nr+2}{2}}y^{\frac{nr+2}{2}} + nr\left(\frac{nr+2}{2}\right)^{\frac{1}{2}}x^{2r}y^{\frac{nr+2}{2}},
\end{aligned}$$

$$D_x^{\frac{1}{2}} D_y^{\frac{1}{2}}(f(x,y)) = nr^2 x^{2r} y^{2r} + \binom{\frac{nr}{2}}{2} \left(\frac{nr+2}{2}\right) x^{\frac{nr+2}{2}} y^{\frac{nr+2}{2}} + nr \left(\frac{nr+2}{2}\right)^{\frac{1}{2}} (2r)^{\frac{1}{2}} x^{2r} y^{\frac{nr+2}{2}}.$$

Using the Theorem 2.16, and column 4 of Tables 1, 2 and 3. We get the desired result. □

3. Conclusion

In the present paper, we obtained M-polynomial of generalized semitotal point transformation graphs G^{xy} and their complements and its degree-based topological indices using the obtained polynomials.

4. Acknowledgement

B. Basavanagoud is partially supported by the University Grants Commission (UGC), Government of India, New Delhi, through UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016(SAP-I).

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