

GEOMETRIC PROPERTIES OF AN INTEGRAL OPERATOR ASSOCIATED WITH BESSEL FUNCTIONS

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ABSTRACT. The purpose of the present paper is to obtain some sufficient conditions for starlikeness and convexity of an integral operator involving Bessel functions of the first kind. Relevant connections of the results presented here with various known results are briefly indicated.

1. INTRODUCTION

Let \mathcal{A} denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by S the subclass of \mathcal{A} consisting of functions of the form (1.1) which are also univalent in \mathbb{U} . A function $f(z) \in \mathcal{A}$ is said to starlike of order α ($0 \leq \alpha < 1$), if it satisfies the following condition

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad (z \in \mathbb{U}).$$

A function $f(z) \in \mathcal{A}$ is said to be convex of order α ($0 \leq \alpha < 1$), if it satisfies the condition

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}).$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $K(\alpha)$. Further we denote by $S^*(0) = S^*$ and $K(0) = K$. The classes $S^*(\alpha)$, $K(\alpha)$, S^* and K were studied by Robertson [18] and Silverman [21].

In 1983, Salagean [19] generalized these subclasses into the class $S(p, \alpha)$ as

$$\Re \left(\frac{D^{p+1}f(z)}{D^p f(z)} \right) > \alpha, \quad (1.2)$$

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where $0 \leq \alpha < 1$, $p \in N_0 = \{0, 1, 2, \dots\}$ and D^p stands for the Salagean operator introduced by Salagean. It is easy to verify that $S(0, \alpha) \equiv S^*(\alpha)$ and $S(1, \alpha) \equiv K(\alpha)$.

The study of the integral operator is one of the main interesting research problems in Geometric Function Theory. Noteworthy contribution in this direction are given in ([2], [4], [6]-[12], [15] and [16]). Recently several authors introduced integral operators involving Bessel functions and obtained some beautiful results. This opens up a new and interesting direction of research in Geometric Function Theory.

First we recall the definition of the Bessel function of the first kind of order ν is defined by the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n! \Gamma(n+\nu+1)},$$

where Γ stands for the Euler Gamma function, $z \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Recently, Szász and Kupán [22] investigated the univalence of the normalized Bessel function of the first kind $g_\nu : \mathbb{U} \rightarrow \mathbb{C}$, defined by

$$\begin{aligned} g_\nu(z) &= 2^\nu \Gamma(\nu+1) z^{1-\nu/2} J_\nu(z^{1/2}) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{n+1}}{4^n n! (\nu+1)(\nu+2) \dots (\nu+n)}. \end{aligned} \quad (1.3)$$

After the appearance of this paper several researchers e.g. Baricz and Frasin [1], Frasin [3], Magesh *et al.* [5], Porwal and Breaz [13], Porwal and Kumar [14], Porwal *et al.* [17] studied some interesting properties of integral operator involving Bessel functions. Recently, Porwal and Kumar [14] introduced a general integral operator associated with Bessel functions of first order in the following way

$$F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z) = \int_0^z \prod_{i=1}^n \left(\frac{g_{\nu_i}(t)}{t} \right)^{\alpha_i} \prod_{j=1}^m \left(\frac{D^p f_j(t)}{t} \right)^{\beta_j} dt, \quad (1.4)$$

and studied the mapping properties of this integral operator on a subclass of analytic univalent functions. Motivating with the above mentioned work we obtain some sufficient conditions for starlikeness and convexity of this integral operator.

2. MAIN RESULTS

To prove our main results we shall require the following lemmas.

Lemma 2.1. ([22]) *Let $\nu > \frac{(-5+\sqrt{5})}{4}$ and consider the normalized Bessel function of the first kind $g_\nu : \mathbb{U} \rightarrow \mathbb{C}$, defined by $g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu/2} J_\nu(z^{1/2})$, where J_ν stands for the Bessel function of the first kind. Then the following inequality hold for all $z \in \mathbb{U}$*

$$\left| \frac{z g'_\nu(z)}{g_\nu(z)} - 1 \right| \leq \frac{\nu+2}{4\nu^2 + 10\nu + 5}. \quad (2.1)$$

Lemma 2.2. ([20]) *If $f \in \mathcal{A}$ satisfies*

$$\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > -\frac{\delta+1}{2\delta(\delta-1)}, \quad (z \in \mathbb{U}),$$

for some $\delta \leq -1$,
or

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{3\delta + 1}{2\delta(\delta + 1)}, \quad (z \in \mathbb{U}),$$

for some $\delta > 1$, then $f \in S^*(\frac{\delta+1}{2\delta})$.

Theorem 2.1. Let n, m be natural numbers and let $\nu_1, \nu_2, \dots, \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the functions $g_{\nu_i} : \mathbb{U} \rightarrow \mathbb{C}$, defined by

$$g_{\nu_i}(z) = 2^{\nu_i} \Gamma(\nu_i + 1) z^{1-\nu_i/2} J_{\nu_i}(z^{1/2}). \tag{2.2}$$

Let $\nu = \min\{\nu_1, \nu_2, \dots, \nu_n\}$ and $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \dots, \beta_m$ be positive real numbers and let $f_j(z), (j = 1, 2, \dots, m)$ be of the form (1.1) is in the class $S(p, \gamma_j)$ and suppose that the following inequality

$$0 \leq 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j) \leq 1,$$

is satisfied. Then the function $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)$ defined by (1.4) is in $K(\mu)$, where

$$\mu = 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j).$$

Proof. First we observe that, since for all $i \in \{1, 2, \dots, n\}$ we have $g_{\nu_i} \in \mathcal{A}$, i.e. $g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0$, and for all $j = 1, 2, \dots, m$, $f_j(z)$ be of the form (1.1), this shows that $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m} \in \mathcal{A}$, i.e. $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(0) = F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(0) - 1 = 0$. On the other hand, it is easy to see that

$$F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z) = \prod_{i=1}^n \left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i} \prod_{j=1}^m \left(\frac{D^p f_j(z)}{z}\right)^{\beta_j}$$

and

$$z \frac{F''_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)}{F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right) + \sum_{j=1}^m \beta_j \left(\frac{D^{p+1} f_j(z)}{D^p f_j(z)} - 1\right).$$

or, equivalently,

$$1 + z \frac{F''_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)}{F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right) + \sum_{j=1}^m \beta_j \frac{D^{p+1} f_j(z)}{D^p f_j(z)} + 1 - \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j. \tag{2.3}$$

Taking the real part of both sides of (2.3), we have

$$\Re \left\{ 1 + z \frac{F''_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)}{F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)} \right\} = \sum_{i=1}^n \alpha_i \Re \left\{ \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} + \sum_{j=1}^m \beta_j \Re \frac{D^{p+1} f_j(z)}{D^p f_j(z)} + \left(1 - \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j \right). \tag{2.4}$$

From Lemma 2.1, we have

$$\left| \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right| \leq \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5}.$$

Since $\Re(z) \leq |z|$

$$\begin{aligned} \Re \left\{ 1 - \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} &\leq \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \\ \Re \left\{ \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} &\geq 1 - \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \end{aligned} \quad (2.5)$$

Now using the fact that $f_j(z) \in S(p, \gamma_j)$ for each $j = 1, 2, \dots, m$ and the inequality (2.5) for each ν_i , where $i \in \{1, 2, \dots, n\}$, we obtain

$$\begin{aligned} &\Re \left\{ 1 + z \frac{F''_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)}{F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)} \right\} \\ &\geq \sum_{i=1}^n \alpha_i \left(1 - \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \right) + \sum_{j=1}^m \beta_j \gamma_j + \left(1 - \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j \right) \\ &= 1 - \sum_{i=1}^n \alpha_i \left(\frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \right) - \sum_{j=1}^m \beta_j (1 - \gamma_j) \\ &\geq 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j), \end{aligned}$$

for all $z \in \mathbb{U}$ and $\nu, \nu_1, \dots, \nu_n > \frac{-5 + \sqrt{5}}{4}$. Here we used that the function $\phi : \left(\frac{-5 + \sqrt{5}}{4}, \infty \right) \rightarrow \mathbb{R}$, defined by

$$\phi(x) = \frac{x + 2}{4x^2 + 10x + 5},$$

is decreasing and consequently for all $i \in \{1, 2, \dots, n\}$ we have

$$\frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \leq \frac{\nu + 2}{4\nu^2 + 10\nu + 5}.$$

Because $0 \leq 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j) \leq 1$, we have $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z) \in K(\mu)$, where

$$\mu = 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j).$$

Thus the proof of Theorem 2.1 is established. \square

Theorem 2.2. Let n, m be natural numbers and let $\nu_1, \nu_2, \dots, \nu_n > \left(\frac{-5 + \sqrt{5}}{4} \right)$. Consider the functions g_{ν_i} , defined by (2.2). Let $\nu = \min\{\nu_1, \nu_2, \dots, \nu_n\}$ and $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \dots, \beta_m$ be positive real numbers and let $f_j(z)$, ($j = 1, 2, \dots, m$) be of the form (1.1) in the class $S(p, \gamma_j)$. More over suppose that these numbers satisfy the following inequality

$$\frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j (1 - \gamma_j) < \frac{2\delta^2 - \delta + 1}{2\delta(\delta - 1)}.$$

Then the function $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)$ defined by (1.4) is in the class $S^*\left(\frac{\delta + 1}{2\delta}\right)$ for some $\delta \leq -1$.

Proof. Applying the same reasoning as in the proof of Theorem 2.1, we have

$$\begin{aligned} & \Re \left\{ 1 + z \frac{F''_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)}{F'_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)} \right\} \\ & \geq 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j). \end{aligned}$$

Since

$$\begin{aligned} 1 - \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j (1 - \gamma_j) & > -\frac{\delta + 1}{2\delta(\delta - 1)}. \\ \frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j (1 - \gamma_j) & < \frac{\delta + 1}{2(\delta - 1)} - 1 \\ & = \frac{2\delta^2 - \delta + 1}{2\delta(\delta - 1)}. \end{aligned}$$

Hence from Lemma 2.2, we have $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z) \in S^*(\frac{\delta+1}{2\delta})$ for some $\delta \leq -1$.

Thus the proof of Theorem 2.2 is established. \square

Theorem 2.3. Let n, m be natural numbers and let $\nu_1, \nu_2, \dots, \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the functions $g_{\nu_i}(z)$ defined by (2.2). Let $\nu = \min\{\nu_1, \nu_2, \dots, \nu_n\}$ and $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \dots, \beta_m$ be positive real numbers and let $f_j(z), (j = 1, 2, \dots, m)$ be of the form (1.1) in the class $S(p, \gamma_j)$ and suppose that the following inequality

$$\frac{2 + \nu}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j (1 - \gamma_j) < \frac{2\delta^2 - \delta - 1}{2\delta(\delta + 1)},$$

is satisfied. Then the function $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m}(z)$ defined by (1.4) is in the class $S^*(\frac{\delta+1}{2\delta})$ for some $\delta > 1$.

Proof. The proof of above theorem is much akin to that of Theorem 2.2, therefore we omit the details involved. \square

Remark 1. If we take $\beta_j = 0, j = 1, 2, 3, \dots, m$ in Theorem 2.1, then we obtain the corresponding results of Frasin [3].

Remark 2. If we take $\beta_j = 0, j = 1, 2, 3, \dots, m$ in Theorems 2.2 and 2.3, then we obtain the corresponding results of Porwal et al. [15].

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