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GENERAL TRIPLE SERIES IDENTITY, LAURENT TYPE GENERATING RELATIONS AND APPLICATIONS

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Abstract. The main aim of this article is obtain certain Laurent type hypergeometric generating relations. A general triple series identity is established. By using triple series identity, a Laurent type hypergeometric generating relation is derived. Explicit expressions of some hybrid special functions related to the Bessel functions are also established as applications.

1. Introduction and preliminaries

The generalized hypergeometric functions and their extensions in the form of basic (or $q-$) hypergeometric functions, elliptic hypergeometric functions and multiple hypergeometric functions are ubiquitous. One characteristic of these functions is a tendency to appear in a variety of mathematical and physical circumstances. A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$ {}_pF_q\left[\begin{array}{ccc}(\alpha_p); & & \\ & z & \\ (\beta_q); & & \end{array}\right] = {}_pF_q\left[\begin{array}{ccc}\alpha_1,\alpha_2,\ldots,\alpha_p; & & \\ & z & \\ \beta_1,\beta_2,\ldots,\beta_q; & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!} \tag{1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\ ;\ j=1,2,\ldots,q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ is denoted by (α_p) with similar interpretation for others throughout this paper.

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Supposing that none of numerator parameters is zero or a negative integer and for $\beta_j \neq 0, -1, -2, \ldots;\ j = 1, 2, \ldots, q$, we note that the ${}_pF_q$ series defined by equation (1):

$$(i)\quad \text{converges for } |z| < \infty,\ \text{ if } p \leq q \tag{2}$$

$$(ii)\quad \text{converges for } |z| < 1,\ \text{ if } p = q + 1 \text{ and} \tag{3}$$

$$(ii)\quad \text{diverges for all } z,\ z \neq 0,\ \text{ if } p > q + 1. \tag{4}$$

Fox-Wright generalized hypergeometric function of one variable

$$
{}_p\Psi_q \left[\begin{array}{lc} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) \quad ; & \\ & z \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) \quad ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \cdots \Gamma(\alpha_p + nA_p)}{\Gamma(\beta_1 + nB_1) \cdots \Gamma(\beta_q + nB_q)} \frac{z^n}{n!}, \tag{5}
$$

$$
{}_p\Psi_q^{\star} \left[\begin{array}{lc} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) \quad ; & \\ & z \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) \quad ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{nA_1} \cdots (\alpha_p)_{nA_p}}{(\beta_1)_{nB_1} \cdots (\beta_q)_{nB_q}} \frac{z^n}{n!}, \tag{6}
$$

where $A_1, \cdots, A_p,\ B_1, \cdots, B_q$ are positive real numbers; subject to the convergence conditions:

$$(i)\ 1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i > 0 \text{ and } 0 < |z| < \infty; \quad z \neq 0$$

$$(ii)\ 1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0 \text{ and } 0 < |z| < A_1^{-A_1} \cdots A_p^{-A_p} B_1^{B_1} \cdots B_q^{B_q}.$$

Lauricella [5, p. 114] introduced fourteen complete hypergeometric functions of three variables and of the second order. He denoted his triple hypergeometric functions by the symbols $F_1,\ F_2,\ F_3,\ \cdots, F_{14}$ of which $F_1,\ F_2,\ F_5$ and F_9 correspond, respectively, to the three variable Lauricella functions $F_A^{(3)},\ F_B^{(3)},\ F_C^{(3)}$ and $F_D^{(3)}$. The remaining ten functions $F_3,\ F_4,\ F_6,\ F_7,\ F_8,\ F_{10}, \cdots,\ F_{14}$ of Lauricella's set apparently fell into oblivion. Saran [6] initiated a systematic study of these ten hypergeometric functions of Lauricella's set by giving them new notations as: $F_3 \equiv F_K$, $F_4 \equiv F_E$, $F_6 \equiv F_N$, $F_7 \equiv F_S$, $F_8 \equiv F_G$, $F_{10} \equiv F_R$, $F_{11} \equiv F_M$, $F_{12} \equiv F_P$, $F_{13} \equiv F_T$ and $F_{14} \equiv F_F$. In the course of a further investigation of the Lauricella's fourteen hypergeometric functions of three variables, Srivastava [7, 8] noticed the existence of three additional complete triple hypergeometric functions of second order; these functions $H_A,\ H_B$ and H_C had not been included in Lauricella's conjecture, nor they previously mentioned in the literature.

A general triple hypergeometric series was introduced by Srivastava [9] which is a unification of Lauricella's fourteen hypergeometric functions $F_1, \cdots,\ F_{14}$ and three additional functions $H_A,\ H_B$ and H_C of Srivastava and is defined as [9, p.

428]:

$$F^{(3)}\left[\begin{array}{ccccccccccccc}(a_A) & :: & (b_B) & ; & (b'_{B'}) & ; & (b''_{B''}) & : & (c_C) & ; & (c'_{C'}) & ; & (c''_{C''}) & ; \\ (e_E) & :: & (g_G) & ; & (g'_{G'}) & ; & (g''_{G''}) & : & (h_H) & ; & (h'_{H'}) & ; & (h''_{H''}) & ;\end{array} x,\ y,\ z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p)\, \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{7}$$

where

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p} \prod_{j=1}^{B}(b_j)_{m+n} \prod_{j=1}^{B'}(b'_j)_{n+p} \prod_{j=1}^{B''}(b''_j)_{p+m} \prod_{j=1}^{C}(c_j)_m \prod_{j=1}^{C'}(c'_j)_n \prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{E}(e_j)_{m+n+p} \prod_{j=1}^{G}(g_j)_{m+n} \prod_{j=1}^{G'}(g'_j)_{n+p} \prod_{j=1}^{G''}(g''_j)_{p+m} \prod_{j=1}^{H}(h_j)_m \prod_{j=1}^{H'}(h'_j)_n \prod_{j=1}^{H''}(h''_j)_p} \tag{8}$$

and (a_A) abbreviates the array of A parameters $a_1, \cdots, a_A$ with similar interpretations for (b_B), $(b'_{B'})$, $(b''_{B''})$, *et. cetera.*. The triple hypergeometric series (7) converges absolutely when

$$1 + E + G + G'' + H - A - B - B'' - C \geq 0, \tag{9a}$$

$$1 + E + G + G' + H' - A - B - B' - C' \geq 0, \tag{9b}$$

$$1 + E + G' + G'' + H'' - A - B' - B'' - C'' \geq 0, \tag{9c}$$

where the equalities hold true for suitably constrained values of $|x|$, $|y|$ and $|z|$.

About five decades ago, Srivastava and Daoust [10] first considered the two-variable case of their multiple hypergeometric function [11, p.454]; see also [12]). For the sake of ready reference, we choose to recall here their definition only in the two-variable case as follows [10, p.199, Eqn (2.1)]:

$$F^{A:B;B'}_{C:D;D'}\left(\begin{array}{c}[(a_A):\theta,\phi] : [(b_B):\psi] ; [(b'_{B'}):\psi'] ; \\ [(c_C):\xi,\eta] : [(d_D):\zeta] ; [(d'_{D'}):\zeta'] ;\end{array} x,\ y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m\theta_j+n\phi_j}}{\prod_{j=1}^{C}(c_j)_{m\xi_j+n\eta_j}} \frac{\prod_{j=1}^{B}(b_j)_{m\psi_j}}{\prod_{j=1}^{D}(d_j)_{m\zeta_j}} \frac{\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{D'}(d'_j)_{n\zeta'_j}} \frac{x^m\, y^n}{m!\, n!}, \tag{10}$$

where, for convergence of the double hypergeometric series,

$$1 + \sum_{j=1}^{C}\xi_j + \sum_{j=1}^{D}\zeta_j - \sum_{j=1}^{A}\theta_j - \sum_{j=1}^{B}\psi_j \geq 0 \tag{11}$$

and

$$1 + \sum_{j=1}^{C}\eta_j + \sum_{j=1}^{D'}\zeta'_j - \sum_{j=1}^{A}\phi_j - \sum_{j=1}^{B'}\psi'_j \geq 0, \tag{12}$$

with equality only when $|x|$ and $|y|$ are constrained appropriately (see, for details, [12]). Here, for the sake of convenience, $[(a_A):\theta,\phi]$ represents the set of "A" number of parameters $[a_1:\theta_1,\phi_1]$, $[a_2:\theta_2,\phi_2]$, $\ldots$, $[a_A:\theta_A,\phi_A]$. The values of positive real coefficients θ_1, θ_2, $\ldots$, θ_A may be equal or different with similar interpretation for coefficients ϕ_1, ϕ_2, $\ldots$, ϕ_A and others.

Remark 1.1. *The positivity of these coefficients was assumed by Srivastava-Daoust [12, pp. 153-158] in order merely to facilitate their investigations of the region of convergence of the multiple hypergeometric series* (10).

Remark 1.2. *For notational purposes the coefficients ξ_j, ζ_j, θ_j, ψ_j, η_j, ζ'_j, ψ'_j are allowed to take on all real values including, for example, zero and negative integers, see [13, pp.270-272].*

Many useful cases of reducibility of double hypergeometric functions are known to exist when the coefficients θ, ϕ, ψ, ψ', ξ, η, ζ and ζ' in equation (10) are chosen to be unity.

Lemma 1.1. *[14, p.100, Eqn(2)]*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \tag{13}$$

provided that concerned double series are absolutely convergent.

Lemma 1.2. *[14, p.102, Eqn(16)] For positive integers $m_1, \cdots, m_r \quad (r \geq 1)$,*

$$\sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty} \Theta(k_1,\cdots,k_r;n) = \sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{m_1k_1+\cdots+m_rk_r\leq n} \Theta(k_1,\cdots,k_r;n-m_1k_1-\cdots-m_rk_r), \tag{14}$$

provided that concerned multiple series are absolutely convergent.

Definition 1.1. Gauss's Multiplication Theorem [14, p.23]
For every positive integer m, we have

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^{m} \left(\frac{\lambda+j-1}{m}\right)_n, \quad n = 0,1,2,\cdots. \tag{15}$$

Bessel functions appear in a wide variety of physical problems. The hypergeometric forms of the ordinary Bessel functions $J_\nu(x)$ and modified Bessel functions $I_\mu(x)$ are defined as follows:

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left[\begin{array}{ccc} - & ; & \\ & & -\frac{x^2}{4} \\ \nu+1 & ; & \end{array}\right] \tag{16}$$

and

$$I_\mu(x) = \frac{\left(\frac{x}{2}\right)^\mu}{\Gamma(\mu+1)}\, {}_0F_1\left[\begin{array}{ccc} - & ; & \\ & & \frac{x^2}{4} \\ \mu+1 & ; & \end{array}\right], \tag{17}$$

respectively.

Tricomi functions $C_\nu(x)$ are Bessel like functions which are defined as:

$$C_\nu(x) = \frac{1}{\Gamma(\nu+1)}\, {}_0F_1\left[\begin{array}{ccc} - & ; & \\ & & -x \\ \nu+1 & ; & \end{array}\right] \tag{18}$$

and have the following relationship with Bessel functions:

$$C_\nu(x) = x^{-\frac{\nu}{2}} J_\nu(2\sqrt{x}). \tag{19}$$

The Hermite polynomials $H_n(x)$ are defined as:

$$H_n(x) = (2x)^n \ {}_2F_0 \left[\begin{array}{ccc} -\frac{n}{2}, \ \frac{-n+1}{2} & ; & \\ & & -\frac{1}{x^2} \\ \text{---} & ; & \end{array} \right]; \quad n = 0, 1, 2, \cdots . \tag{20}$$

The importance of the generalized Bessel functions stems from their wide use in applications and from their implications in different fields of applied mathematics and physics.

In [3], monomiality principle is used to develop a systematic treatment of Bessel functions of Hermite and Laguerre type. The Hermite-Bessel functions ${}_HJ_n(x,y)$ and Laguerre-Bessel functions ${}_LJ_n(x,y)$ are defined by the following generating functions:

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{4}\left(t-\frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_HJ_n(x,y)t^n \tag{21}$$

and

$$C_0\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} {}_LJ_n(x,y)t^n, \tag{22}$$

respectively.

In 2018, Khan *et. al.* [4] introduced the Laguerre-Gould-Hopper-Bessel functions ${}_{L}H^{(m,r)}J_n(x,y,z)$ by using the operational techniques.

The Laguerre-Gould-Hopper-Bessel functions ${}_{L}H^{(m,r)}J_n(x,y,z)$ are defined by the following generating function [4, p. 388]:

$$C_0\left(-\frac{x}{2^m}\left(t-\frac{1}{t}\right)^m\right)\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)+\frac{z}{2^r}\left(t-\frac{1}{t}\right)^r\right) = \sum_{n=-\infty}^{\infty} {}_{L}H^{(m,r)}J_n(x,y,z)t^n. \tag{23}$$

For the Laguerre-Gould-Hopper-Bessel functions ${}_{L}H^{(m,r)}J_n(x,y,z)$, the following explicit representation holds [4, p. 389]:

$${}_{L}H^{(m,r)}J_n(x,y,z) = \sum_{k=0}^{\infty}\sum_{s,l=0}^{rs+ml\leq n+2k} \frac{(-1)^k(n+2k)!z^s x^l y^{n+2k-rs-ml}}{2^{n+2k}k!\,\Gamma(1+n+k)s!(l!)^2(n+2k-rs-ml)!}. \tag{24}$$

Recently, Ali and Qureshi [1] introduced the extended form of Laguerre-Gould-Hopper-Bessel functions $G^{(m,r)}_{n,\lambda}(x,y,z;\xi)$ by combined use of integral transform and operational rules, which are defined by the following generating function:

$$\frac{C_0\left(-\frac{x}{2^m}\left(t-\frac{1}{t}\right)^m\right)\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)\right)}{\left(\xi-\frac{z}{2^r}\left(t-\frac{1}{t}\right)^r\right)^\lambda} = \sum_{n=-\infty}^{\infty} G^{(m,r)}_{n,\lambda}(x,y,z;\xi)t^n. \tag{25}$$

The generalized hybrid Bessel functions $G^{(m,r)}_{n,\lambda}(x, y, z; \xi)$ have the following explicit representation:

$$G^{(m,r)}_{n,\lambda}(x, y, z; \xi) = \frac{1}{\xi^{\lambda}} \sum_{k=0}^{\infty} \sum_{s,\ell=0}^{rs+m\ell\leq n+2k} \frac{(-1)^k (n+2k)!(\lambda)_s z^s x^\ell y^{n+2k-rs-m\ell}}{2^{n+2k}\xi^s k!\,\Gamma(1+n+k)s!(\ell!)^2(n+2k-rs-m\ell)!} \tag{26}$$

or

$$G^{(m,r)}_{n,\lambda}(x, y, z; \xi) = \frac{1}{\xi^{\lambda}} \sum_{k=0}^{\infty} \frac{(-1)^k y^{n+2k}}{2^{n+2k} k!(n+k)!}\, F^{1:1;0}_{0:0;1}\left(\begin{matrix} [-n-2k : r, m] & : & [\lambda : 1] & ; & \text{——} & ; \\ \text{———} & : & \text{——} & ; & [1:1] & ; \end{matrix}\ \frac{z}{\xi(-y)^r}, \frac{x}{(-y)^m}\right), \tag{27}$$

where $F^{1:1;0}_{0:0;1}$ is the Srivastava-Daoust double hypergeometric function defined by equation (10).

In Section 2, a general triple series identity is derived. Section 3 is dedicated to obtain the Laurent type hypergeometric generating relations. In Section 4, some special cases of the obtained results are presented.

2. General triple series identity

In this section, we derive a triple series identity in the form of the following theorem:

Theorem 2.1. *Let* $\{\Delta(\ell)\}$, $\{\Theta(\ell)\}$, $\{\Xi(\ell)\}$, $\{\Upsilon(\ell)\}$, $\{\Phi(\ell)\}$, $\{\Psi(\ell)\}$ *and* $\{\Omega(\ell)\}$; $\ell \in \{1, 2, 3, \cdots\}$ *are seven bounded sequences of arbitrary complex numbers, where* m *and* r *are positive integers and* $\Delta(0) \neq 0$, $\Theta(0) \neq 0$, $\Xi(0) \neq 0$, $\Upsilon(0) \neq 0$, $\Phi(0) \neq 0$, $\Psi(0) \neq 0$, $\Omega(0) \neq 0$. *Then*

$$\begin{aligned} &\sum_{\ell,k,s=0}^{\infty} \Delta(\ell+k+s)\ \Theta(\ell+k)\ \Xi(k+s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(k)\ \Omega(s) \times \\ &\times \frac{\left(ax\left(t-\frac{1}{t}\right)^m\right)^\ell \left(by\left(t-\frac{1}{t}\right)\right)^k \left(cz\left(t-\frac{1}{t}\right)^r\right)^s}{\ell!\ k!\ s!} \\ = \quad &\sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \sum_{\ell,s=0}^{m\ell+rs\leq 2j+2n^\star+n} \frac{(-1)^{j+n^\star}(2j+2n^\star+n)!}{(j+n^\star)!\ (j+n^\star+n)!} \times \\ &\times \Delta(2j+2n^\star+n-(m-1)\ell-(r-1)s)\ \Theta(2j+2n^\star+n-(m-1)\ell-rs) \times \\ &\times \Xi(2j+2n^\star+n-m\ell-(r-1)s)\ \Upsilon(s+\ell)\ \Phi(\ell) \times \\ &\times\ \Psi(2j+2n^\star+n-m\ell-rs)\ \Omega(s)\ \frac{(ax)^\ell\ (by)^{2j+2n^\star+n-m\ell-rs}(cz)^s}{\ell!\ (2j+2n^\star+n-m\ell-rs)!\ s!}\ t^n, \end{aligned} \tag{28}$$

where $n^\star$ *is defined as:*

$$n^\star = \max\{0, -n\} = \begin{cases} -n, & \text{when } n = \cdots, -3, -2, -1 \\ 0, & \text{when } n = 0, 1, 2, \cdots, \end{cases} \tag{29}$$

provided that each of the multiple series involved is absolutely convergent.

Proof. Suppose the l.h.s. of equation (28) is denoted by Λ. Then, we have

$$\begin{aligned}\Lambda &= \sum_{\ell,k,s=0}^{\infty} \Delta(\ell+k+s)\ \Theta(\ell+k)\ \Xi(k+s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(k)\ \Omega(s)\times\\ &\quad\times\frac{(ax)^{\ell}(by)^{k}(cz)^{s}}{\ell!\ k!\ s!}\ \left(t-\frac{1}{t}\right)^{k+m\ell+rs}. \end{aligned}\tag{30}$$

Now, replacing k by $k-m\ell-rs$ and using Lemma 1.2, we get

$$\begin{aligned}\Lambda &= \sum_{k=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\le k} \Delta(k-(m-1)\ell-(r-1)s)\ \Theta(k-(m-1)\ell-rs)\times\\ &\quad\times\Xi(k-m\ell-(r-1)s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(k-m\ell-rs)\ \Omega(s)\times\\ &\quad\times\frac{(ax)^{\ell}(by)^{k-m\ell-rs}(cz)^{s}}{\ell!\ (k-m\ell-rs)!\ s!}\ \left(t-\frac{1}{t}\right)^{k}\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j}k!}{j!\ (k-j)!}\times\\ &\quad\times\sum_{\ell,s=0}^{m\ell+rs\le k} \Delta(k-(m-1)\ell-(r-1)s)\ \Theta(k-(m-1)\ell-rs)\times\\ &\quad\times\Xi(k-m\ell-(r-1)s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(k-m\ell-rs)\ \Omega(s)\times\\ &\quad\times\frac{(ax)^{\ell}(by)^{k-m\ell-rs}(cz)^{s}}{\ell!\ (k-m\ell-rs)!\ s!}\ t^{k-2j}. \end{aligned}\tag{31}$$

On replacing k by $k+j$ in equation (31) and using Lemma 1.1, we obtain

$$\begin{aligned}\Lambda &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j}(k+j)!}{j!\ k!}\times\\ &\quad\times\sum_{\ell,s=0}^{m\ell+rs\le k+j} \Delta(k+j-(m-1)\ell-(r-1)s)\ \Theta(k+j-(m-1)\ell-rs)\times\\ &\quad\times\Xi(k+j-m\ell-(r-1)s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(k+j-m\ell-rs)\ \Omega(s)\times\\ &\quad\times\frac{(ax)^{\ell}(by)^{k+j-m\ell-rs}(cz)^{s}}{\ell!\ (k+j-m\ell-rs)!\ s!}\ t^{k-j}. \end{aligned}\tag{32}$$

Further, putting $k=j+n$, in equation (32), we find

$$\begin{aligned}\Lambda &= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j}(2j+n)!}{j!\ (j+n)!}\times\\ &\quad\times\sum_{\ell,s=0}^{m\ell+rs\le 2j+n} \Delta(2j+n-(m-1)\ell-(r-1)s)\ \Theta(2j+n-(m-1)\ell-rs)\times\\ &\quad\times\Xi(2j+n-m\ell-(r-1)s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(2j+n-m\ell-rs)\ \Omega(s)\times\\ &\quad\times\frac{(ax)^{\ell}(by)^{2j+n-m\ell-rs}(cz)^{s}}{\ell!\ (2j+n-m\ell-rs)!\ s!}\ t^{n}. \end{aligned}\tag{33}$$

Since n varies from $-\infty$ to ∞ and j varies from 0 to ∞, therefore due to the presence of $(j+n)!$ in denominator of above equation, equation (33) can be modified in the following form:

$$\Lambda = \sum_{n=-\infty}^{\infty}\sum_{j=n^\star}^{\infty}\frac{(-1)^j(2j+n)!}{j!\,(j+n)!}\times$$
$$\times\sum_{\ell,s=0}^{m\ell+rs\le 2j+n}\Delta(2j+n-(m-1)\ell-(r-1)s)\ \Theta(2j+n-(m-1)\ell-rs)\times$$
$$\times\Xi(2j+n-m\ell-(r-1)s)\ \Upsilon(s+\ell)\ \Phi(\ell)\ \Psi(2j+n-m\ell-rs)\ \Omega(s)\times$$
$$\times\frac{(ax)^\ell(by)^{2j+n-m\ell-rs}(cz)^s}{\ell!\,(2j+n-m\ell-rs)!\,s!}\ t^n, \tag{34}$$

where $n^\star$ is defined by equation (29).

On replacing j by $j+n^\star$ in equation (34), we get equation (28). □

In the next section, we derive certain Laurent type hypergeometric generating relations by using the triple series identity (28). It should be noted that in single, double and triple hypergeometric functions, we are assuming that numerator and denominator parameters are neither zero nor negative integers.

3. Laurent type hypergeometric generating relations

Theorem 3.1. *The following generating function (in terms of general triple hypergeometric series $F^{(3)}$ of Srivastava) for the Srivastava-Daoust double hypergeometric function $F^{1+E+U+R+Q+C:G;V}_{D+A+B+P+M\ :H;W}$ holds true:*

$$F^{(3)}\left[\begin{array}{ccccccccccccccc}(d_D) & :: & (a_A) & ; & (b_B) & ; & (c_C) & : & (g_G) & ; & (p_P) & ; & (v_V) & ; \\ (e_E) & :: & (u_U) & ; & (r_R) & ; & (m_M) & : & (h_H) & ; & (q_Q) & ; & (w_W) & ;\end{array}\right.$$
$$\left. ax\left(t-\frac{1}{t}\right)^m,\ by\left(t-\frac{1}{t}\right),\ cz\left(t-\frac{1}{t}\right)^r\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+n^\star}\frac{(by)^{2j+2n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}\times$$
$$\times\frac{\prod_{i=1}^{D}(d_i)_{2j+2n^\star+n}\prod_{i=1}^{A}(a_i)_{2j+2n^\star+n}\prod_{i=1}^{B}(b_i)_{2j+2n^\star+n}\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n}}{\prod_{i=1}^{E}(e_i)_{2j+2n^\star+n}\prod_{i=1}^{U}(u_i)_{2j+2n^\star+n}\prod_{i=1}^{R}(r_i)_{2j+2n^\star+n}\prod_{i=1}^{Q}(q_i)_{2j+2n^\star+n}}\times$$

$$\times F^{1+E+U+R+Q+C:G;V}_{D+A+B+P+M\ :H;W}\left(\begin{array}{c}[-2j-2n^{\star}-n:m,r],\ [1-(e_E)-2j-2n^{\star}-n:m-1,r-1],\\ [1-(d_D)-2j-2n^{\star}-n:m-1,r-1],\end{array}\right.$$

$$[1-(u_U)-2j-2n^{\star}-n:m-1,r],\ [1-(r_R)-2j-2n^{\star}-n:m,r-1],$$
$$[1-(a_A)-2j-2n^{\star}-n:m-1,r],\ [1-(b_B)-2j-2n^{\star}-n:m,r-1],$$

$$\begin{array}{lcll}[1-(q_Q)-2j-2n^{\star}-n:m,r],\ [(c_C):1,1] & : & [(g_G):1]; & [(v_V):1];\\ [1-(p_P)-2j-2n^{\star}-n:m,r],\ [(m_M):1,1] & : & [(h_H):1]; & [(w_W):1];\end{array}$$

$$\left.(-1)^{\theta}\tfrac{ax}{(by)^m},\ (-1)^{\phi}\tfrac{cz}{(by)^r}\right)t^n,\quad t\neq 0, \tag{35}$$

where

$$\theta=(D-E+A-U)(m-1)+(B-R+P-Q+1)m$$

and

$$\phi=(D-E+B-R)(r-1)+(A-U+P-Q+1)r,$$

for suitably constrained values of $\left|ax\left(t-\frac{1}{t}\right)^m\right|$, $\left|by\left(t-\frac{1}{t}\right)\right|$ *and* $\left|cz\left(t-\frac{1}{t}\right)^r\right|$, m *and* r *are positive integers and* $n^{\star}$ *is defined by equation* (29) *and* $[1-(e_E)-2j-2n^{\star}-n:m,r]$ *represent* E *parameters given by* $[1-e_1-2j-2n^{\star}-n:m,r]$, $[1-e_2-2j-2n^{\star}-n:m,r]$ $,\cdots,$ $[1-e_E-2j-2n^{\star}-n:m,r]$, *subject to the following convergence conditions:*

$$1+E+U+M+H-D-A-C-G\geq 0, \tag{36a}$$

$$1+E+U+R+Q-D-A-B-P\geq 0, \tag{36b}$$

$$1+E+R+M+W-D-B-C-V\geq 0. \tag{36c}$$

Proof. Taking

$$\Delta(\ell+k+s)=\frac{\prod_{i=1}^{D}(d_i)_{\ell+k+s}}{\prod_{i=1}^{E}(e_i)_{\ell+k+s}},\quad \Theta(\ell+k)=\frac{\prod_{i=1}^{A}(a_i)_{\ell+k}}{\prod_{i=1}^{U}(u_i)_{\ell+k}},\quad \Xi(k+s)=\frac{\prod_{i=1}^{B}(b_i)_{k+s}}{\prod_{i=1}^{R}(r_i)_{k+s}}$$

$$\Upsilon(s+\ell)=\frac{\prod_{i=1}^{C}(c_i)_{s+\ell}}{\prod_{i=1}^{M}(m_i)_{s+\ell}},\quad \Phi(\ell)=\frac{\prod_{i=1}^{G}(g_i)_{\ell}}{\prod_{i=1}^{H}(h_i)_{\ell}},\quad \Psi(k)=\frac{\prod_{i=1}^{P}(p_i)_{k}}{\prod_{i=1}^{Q}(q_i)_{k}},\quad \Omega(s)=\frac{\prod_{i=1}^{V}(v_i)_{s}}{\prod_{i=1}^{W}(w_i)_{s}}$$

in general triple series identity (28), applying some algebraic properties of Pochhammer symbols and after simplification, we obtain:

$$F^{(3)}\left[\begin{array}{ccccccccccccccc}(d_D) & :: & (a_A) & ; & (b_B) & ; & (c_C) & : & (g_G) & ; & (p_P) & ; & (v_V) & ; \\ (e_E) & :: & (u_U) & ; & (r_R) & ; & (m_M) & : & (h_H) & ; & (q_Q) & ; & (w_W) & ;\end{array}\right.$$

$$\left. ax\left(t-\frac{1}{t}\right)^m,\ by\left(t-\frac{1}{t}\right),\ cz\left(t-\frac{1}{t}\right)^r\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell,s=0}^{m\ell+rs\le 2j+2n^\star+n}(-1)^{j+n^\star}\frac{(2j+2n^\star+n)!}{(j+n^\star)!\ (j+n^\star+n)!}\times$$

$$\times\frac{\prod_{i=1}^{D}(d_i)_{2j+2n^\star+n-(m-1)\ell-(r-1)s}\prod_{i=1}^{A}(a_i)_{2j+2n^\star+n-(m-1)\ell-rs}\prod_{i=1}^{B}(b_i)_{2j+2n^\star+n-m\ell-(r-1)s}}{\prod_{i=1}^{E}(e_i)_{2j+2n^\star+n-(m-1)\ell-(r-1)s}\prod_{i=1}^{U}(u_i)_{2j+2n^\star+n-(m-1)\ell-rs}\prod_{i=1}^{R}(r_i)_{2j+2n^\star+n-m\ell-(r-1)s}}\times$$

$$\times\frac{\prod_{i=1}^{C}(c_i)_{s+\ell}\prod_{i=1}^{G}(g_i)_\ell\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n-m\ell-rs}\prod_{i=1}^{V}(v_i)_s}{\prod_{i=1}^{M}(m_i)_{s+\ell}\prod_{i=1}^{H}(h_i)_\ell\prod_{i=1}^{Q}(q_i)_{2j+2n^\star+n-m\ell-rs}\prod_{i=1}^{W}(w_i)_s}\times$$

$$\times\frac{(ax)^\ell(by)^{2j+2n^\star+n-m\ell-rs}(cz)^s}{\ell!\ (2j+2n^\star+n-m\ell-rs)!\ s!}\ t^n$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+n^\star}\frac{(by)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!}\times$$

$$\times\frac{\prod_{i=1}^{D}(d_i)_{2j+2n^\star+n}\prod_{i=1}^{A}(a_i)_{2j+2n^\star+n}\prod_{i=1}^{B}(b_i)_{2j+2n^\star+n}\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n}}{\prod_{i=1}^{E}(e_i)_{2j+2n^\star+n}\prod_{i=1}^{U}(u_i)_{2j+2n^\star+n}\prod_{i=1}^{R}(r_i)_{2j+2n^\star+n}\prod_{i=1}^{Q}(q_i)_{2j+2n^\star+n}}\times$$

$$\times\sum_{\ell,s=0}^{m\ell+rs\le 2j+2n^\star+n}\frac{(-2j-2n^\star-n)_{m\ell+rs}\prod_{i=1}^{E}(1-e_i-2j-2n^\star-n)_{(m-1)\ell+(r-1)s}}{\prod_{i=1}^{D}(1-d_i-2j-2n^\star-n)_{(m-1)\ell+(r-1)s}}\times$$

$$\times\frac{\prod_{i=1}^{U}(1-u_i-2j-2n^\star-n)_{(m-1)\ell+rs}\prod_{i=1}^{R}(1-r_i-2j-2n^\star-n)_{m\ell+(r-1)s}}{\prod_{i=1}^{A}(1-a_i-2j-2n^\star-n)_{(m-1)\ell+rs}\prod_{i=1}^{B}(1-b_i-2j-2n^\star-n)_{m\ell+(r-1)s}}\times$$

$$\times \frac{\prod_{i=1}^{Q}(1-q_i-2j-2n^\star-n)_{m\ell+rs}\ \prod_{i=1}^{C}(c_i)_{\ell+s}\ \prod_{i=1}^{G}(g_i)_\ell\ \prod_{i=1}^{V}(v_i)_s}{\prod_{i=1}^{P}(1-p_i-2j-2n^\star-n)_{m\ell+rs}\ \prod_{i=1}^{M}(m_i)_{\ell+s}\ \prod_{i=1}^{H}(h_i)_\ell\ \prod_{i=1}^{W}(w_i)_s} \times$$

$$\times(-1)^{[(D-E)(m-1)+(A-U)(m-1)+(B-R)m+(P-Q)m+m]\ell}\frac{\left(\frac{ax}{(by)^m}\right)^\ell}{\ell!}\times$$

$$\times(-1)^{[(D-E)(r-1)+(A-U)r+(B-R)(r-1)+(P-Q)r+r]s}\frac{\left(\frac{cz}{(by)^r}\right)^s}{s!}. \tag{37}$$

On using definition of the Srivastava and Daoust hypergeometric function (10) in the r.h.s. of equation (37), we obtain assertion (35).

□

Remark 3.1. *Taking $D = A = B = C = E = U = R = M = 0$ in equation (35), we deduce the following consequence of Theorem 3.1:*

Corollary 3.1. *For $G \leq H+1$, $P \leq Q+1$, $V \leq W+1$ and $t \neq 0$, the following generating function (in terms of the product of three generalized hypergeometric functions of one variable) for the Srivastava-Daoust double hypergeometric function holds true:*

$${}_GF_H\left[\begin{array}{ll}(g_G)\ ; & \\ & ax\left(t-\frac{1}{t}\right)^m\\ (h_H)\ ; & \end{array}\right]{}_PF_Q\left[\begin{array}{ll}(p_P)\ ; & \\ & by\left(t-\frac{1}{t}\right)\\ (q_Q)\ ; & \end{array}\right]{}_VF_W\left[\begin{array}{ll}(v_V)\ ; & \\ & cz\left(t-\frac{1}{t}\right)^r\\ (w_W)\ ; & \end{array}\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+n^\star}(by)^{2j+2n^\star+n}\prod_{i=1}^{P}(p_i)_{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!\ \prod_{i=1}^{Q}(q_i)_{2j+2n^\star+n}}\times$$

$$\times F^{Q+1:G;\,V}_{P\ \ :H;W}\left(\begin{array}{l}[-2j-2n^\star-n:m,r],\ [1-(q_Q)-2j-2n^\star-n:m,r]\ :\ [(g_G):1]\ ;\ [(v_V):1]\ ;\\ [1-(p_P)-2j-2n^\star-n:m,r]\ :\ [(h_H):1]\ ;\ [(w_W):1]\ ;\end{array}\ (-1)^{m(P-Q+1)}\frac{(ax)}{(by)^m},\ (-1)^{r(P-Q+1)}\frac{(cz)}{(by)^r}\right)t^n, \tag{38}$$

where m and r are positive integers and $n^\star$ is defined equation (29).

4. Applications

In this section, we consider certain special cases of our main result.

I. Taking $x = 0$; $P = Q = 0$, $b = \frac{1}{2}$; $V = W = 0$, $r = 2$, $c = \frac{1}{2^r}$ in equation (38), we get

$$\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)+\frac{z}{4}\left(t-\frac{1}{t}\right)^2\right)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{j+n^\star} \left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!} \left(\frac{\sqrt{z}}{iy}\right)^{2j+2n^\star+n} H_{2j+2n^\star+n}\left(\frac{iy}{2\sqrt{z}}\right) t^n. \quad (39)$$

Comparing equation (39) with the equation (21), we find that the Hermite-Bessel functions ${}_HJ_n(y,z)$ have the following series definition in terms of the Hermite polynomials $H_n(x)$:

$$ {}_HJ_n(y,z) = \sum_{j=0}^{\infty} \frac{(-1)^{j+n^\star} \left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!} \left(\frac{\sqrt{z}}{iy}\right)^{2j+2n^\star+n} H_{2j+2n^\star+n}\left(\frac{iy}{2\sqrt{z}}\right). \quad (40)$$

II. Taking $G=0$, $H=1$, $h_1=1$, $m=1$, $a=-\frac{1}{2}$; $P=Q=0$, $b=\frac{1}{2}$; $z=0$ in equation (38), we get

$$C_0\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{j+n^\star} \left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!} \left(1-\frac{x}{y}\right)^{2j+2n^\star+n} t^n. \quad (41)$$

Comparing equation (41) with the equation (22), we find that the Laguerre-Bessel functions ${}_LJ_n(x,y)$ have the following series definition:

$$ {}_LJ_n(x,y) = \sum_{j=0}^{\infty} \frac{(-1)^{j+n^\star} \left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!} \left(1-\frac{x}{y}\right)^{2j+2n^\star+n} t^n. \quad (42)$$

III. Taking $G=0$, $H=1$, $h_1=1$, $a=\frac{1}{2^m}$; $P=Q=0$, $b=\frac{1}{2}$; $V=W=0$, $c=\frac{1}{2^r}$ in equation (38), we get

$$C_0\left(-\frac{x}{2^m}\left(t-\frac{1}{t}\right)^m\right)\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)+\frac{z}{2^r}\left(t-\frac{1}{t}\right)^r\right)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{j+n^\star} \left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!} \times$$

$$\times F^{1:0;0}_{0:1;0}\left(\begin{array}{ccccc} [-2j-2n^\star-n:m,r] & : & — & ; & — ; \\ & & & & \\ — & : & [1:1] & ; & — ; \end{array} (-1)^m \frac{x}{y^m},\ (-1)^r \frac{z}{y^r}\right) t^n. \quad (43)$$

Comparing equation (43) with the equation (23), we find that the Laguerre-Gould-Hopper-Bessel functions ${}_{LH^{(m,r)}}J_n(x,y,z)$ have the following series definition in terms of the Srivastava-Daoust hypergeometric function:

$$ {}_{LH^{(m,r)}}J_n(x,y,z) = \sum_{j=0}^{\infty} \frac{(-1)^{j+n^\star} \left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\ (j+n^\star+n)!} \times$$

$$\times F^{1:0;0}_{0:1;0}\left(\begin{array}{ccccc} [-2j-2n^\star-n:m,r] & : & — & ; & — ; \\ & & & & \\ — & : & [1:1] & ; & — ; \end{array} (-1)^m \frac{x}{y^m},\ (-1)^r \frac{z}{y^r}\right). \quad (44)$$

IV. Taking $G = 0$, $H = 1$, $h_1 = 1$, $a = \frac{1}{2^m}$; $P = Q = 0$, $b = \frac{1}{2}$; $V = 1$, $W = 0$, $v_1 = \lambda$, $c = \frac{1}{2^r\xi}$ in equation (38), we get

$$\xi^{\lambda}\frac{C_0\left(-\frac{x}{2^m}\left(t-\frac{1}{t}\right)^m\right)\exp\left(\frac{y}{2}\left(t-\frac{1}{t}\right)\right)}{\left(\xi-\frac{z}{2^r}\left(t-\frac{1}{t}\right)^r\right)^{\lambda}} = \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+n^\star}\left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}\times$$

$$\times F^{1:0;1}_{0:1;0}\left(\begin{array}{ccccccc}[-2j-2n^\star-n:m,r] & : & \text{—} & ; & [\lambda:1] & ; & \\ & & & & & & (-1)^m\frac{x}{y^m},\ (-1)^r\frac{z}{\xi y^r} \\ \text{—} & : & [1:1] & ; & \text{—} & ; & \end{array}\right)t^n. \tag{45}$$

Comparing equation (45) with the equation (25), we find that the extended Laguerre-Gould-Hopper-Bessel functions $G^{(m,r)}_{n,\lambda}(x,y,z;\xi)$ have the following series definition in terms of the Srivastava-Daoust hypergeometric function:

$$G^{(m,r)}_{n,\lambda}(x,y,z;\xi) = \frac{1}{\xi^\lambda}\sum_{j=0}^{\infty}\frac{(-1)^{j+n^\star}\left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}\times$$

$$\times F^{1:0;1}_{0:1;0}\left(\begin{array}{ccccccc}[-2j-2n^\star-n:m,r] & : & \text{—} & ; & [\lambda:1] & ; & \\ & & & & & & (-1)^m\frac{x}{y^m},\ (-1)^r\frac{z}{\xi y^r} \\ \text{—} & : & [1:1] & ; & \text{—} & ; & \end{array}\right) \tag{46}$$

or

$$G^{(m,r)}_{n,\lambda}(x,y,z;\xi) = \frac{1}{\xi^\lambda}\sum_{j=0}^{\infty}\frac{(-1)^{j+n^\star}\left(\frac{y}{2}\right)^{2j+2n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}\times$$

$$\times F^{1:1;0}_{0:0;1}\left(\begin{array}{ccccccc}[-2j-2n^\star-n:m,r] & : & [\lambda:1] & ; & \text{—} & ; & \\ & & & & & & (-1)^r\frac{z}{\xi y^r},\ (-1)^m\frac{x}{y^m} \\ \text{—} & : & \text{—} & ; & [1:1] & ; & \end{array}\right), \tag{47}$$

which is the modified form of the equation (27).

V. Taking $G = 0$, $H = 1$, $h_1 = \nu + 1$, $a = -\frac{1}{4}$; $P = 0$, $Q = 1$, $q_1 = \mu + 1$, $b = \frac{1}{4}$, $z = 0$ in equation (38), we get

$$\frac{2^{(\nu+\mu)}\Gamma(\nu+1)\,\Gamma(\mu+1)}{x^{\frac{\nu}{2}}\,y^{\frac{\mu}{2}}\left(t-\frac{1}{t}\right)^{\frac{m\nu+\mu}{2}}}\,J_\nu\left(\sqrt{x\left(t-\frac{1}{t}\right)^m}\right)\,I_\mu\left(\sqrt{y\left(t-\frac{1}{t}\right)}\right)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+n^\star}\left(\frac{y}{4}\right)^{2j+2n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!\,(\mu+1)_{2j+2n^\star+n}}\times$$

$$\times\ {}_2\Psi^\star_1\left[\begin{array}{ccc}(-2j-2n^\star-n,m),\ (-\mu-2j-2n^\star-n,m) & ; & \\ & & \frac{4^{m-1}(-x)}{y} \\ (\nu+1,1) & ; & \end{array}\right]t^n. \tag{48}$$

In this article, we have derived certain Laurent type hypergeometric generating relations using a general triple series identity. This approach can be extended to obtain multiple hypergeometric generating relations for other complex special functions by considering the general multiple series identity. This will be taken in forthcoming investigation.

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