Electronic Journal of Mathematical Analysis and Applications
Vol. 8(2) July 2020, pp. 70-74.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

CATALAN TRANSFORM OF THE $k-$JACOBSTHAL SEQUENCE

MERVE TAŞTAN AND ENGIN ÖZKAN

ABSTRACT. In this study, we present Catalan transform of the $k-$Jacobsthal sequence and examine the properties of the sequence. Then we put in for the Hankel transform to the Catalan transform of the $k-$Jacobsthal sequence. Furthermore, we acquire an interesting characteristic related to determinant of Hankel transform of the sequence.

1. Introduction

For any integer $n \in Z$, it is called a generalized Fibonacci-type sequence for any recurrence sequence of the following form $G(n+1) = aG(n) + bG(n-1), G(0) = m, G(1) = t$ where m, t, a and b are any complex numbers [3].

The known Jacobsthal numbers have some applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra, etc [9, 10]. Bruhn, et al. [5] introduced that generalized Petersen graph is equal to kth Jacobsthal number

There is an extensive work in the literature concerning Fibonacci-type sequences and their applications in modern science (for more detail, see [3, 6, 9, 11, 12, 13, 14] and the references therein).

There exist generalizations of the Jacobsthal numbers. This paper is an extension of the work of Falcon [14]. Falcon [14] gave an application of the Catalan transform to the $k-$Fibonacci sequences. In this paper, we put in for Catalan transform to the $k-$Jacobsthal sequence and present application of the Hankel transform to the Catalan transform of the $k-$Jacobsthal sequence.

The other section of the paper is prepared as follows. The following, we introduce some fundamental definitions of $k-$Jacobsthal numbers. In section 3, Catalan transform of $k-$Jacobsthal sequence is given. Finally, we give Henkel transform of the new sequence obtained $k-$Jacaobsthal sequence.

2. $k-$Jacobsthal number

For any positive number k, the $k-$Jacobsthal sequence, say $\{J_{k,n}\}_{(n\in N)}$ is defined by the recurrence relation

---

2010 *Mathematics Subject Classification.* 11B39, 11B83, 11C20.
Key words and phrases. Jacobstal sequence, Catalan numbers, Catalan Fibonacci sequence.
Submitted Aug. 5, 2019. Revised Oct. 30, 2019.

$$J_{k,n+1} = J_{k,n} + kJ_{k,n-1}, n \geq 1$$

with initial conditions $J_{k,0} = 0$ and $J_{k,1} = 1$ [2].

The $k-$Jacobsthal numbers is expressed function of the roots of σ_1 and σ_2 of characteristic equation $r^2 = kr+2$ via the well-known Binet's formula of Jacobsthal numbers. Hence, The $k-$Jacobsthal numbers is given as follow

$$J_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$$

where $\sigma_1 = \frac{k+\sqrt[2]{k^2+8}}{2}$ and $\sigma_2 = \frac{k-\sqrt[2]{k^2+8}}{2}$.

Note that, since $k > 0$, then $\sigma_2 < 0 < \sigma_1$ and $|\sigma_1| < |\sigma_1|$, $\sigma_1+\sigma_2 = k$, $\sigma_1.\sigma_2 = -2$ and $\sigma_1 - \sigma_2 = \sqrt[2]{k^2+8}$. Therefore, the general term of the $k-$Jacobsthal sequence may be expressed in the form: $J_{k,n} = c_1\sigma_1^n + c_2\sigma_2^n$ for some coefficients c_1 and c_2. If $n = 0$ and $n = 1$, then it is acquired $c_1 = \frac{1}{\sigma_1-\sigma_2} = -c_2$, and $J_{k,n} = \frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}$.

Proposition 2.1.

$$J_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i}(k^2+8)^i$$

where $\lfloor a \rfloor$ is the floor function of a.

Proof. By using the values of σ_1 and σ_2 obtained in equation , we get

$$J_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$$

$$= \frac{1}{\sqrt[2]{k^2+8}} \left[\left(\frac{k+\sqrt[2]{k^2+8}}{2} \right)^n - \left(\frac{k-\sqrt[2]{k^2+8}}{2} \right)^n \right]$$

from where, by developing the nth powers, it follows:

$$= \frac{1}{\sqrt[2]{k^2+8}} \left\{ \frac{k^n}{2^{n-1}} \left[\binom{n}{1} \frac{\sqrt[2]{k^2+8}}{k} + \binom{n}{2} \frac{\left(\sqrt[2]{k^2+8}\right)^3}{k^3} + ... \right] \right\}$$

$$= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i}(k^2+8)^i$$

The limit of the quotient of $J_{k,n}$ and $J_{k,n-1}$ as $n \to \infty$ is equal to σ_1. That is $\lim_{n\to\infty} \frac{J_{k,n}}{J_{k,n-1}} = \sigma_1$. □

2.2. **The Catalan transformation.** The Catalan transform is a sequence transform introduced by Barry [11] The Catalan numbers are defined by

$$c_n = \frac{1}{n+1} \binom{2n}{n}$$

in [11]. The latter can be written as

$$c_n = \frac{(2n)!}{(n+1)!n!}$$

The first few Catalan numbers are $1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$.

Also, one can be obtained the recurrence relation for $C(n)$ from

$$\frac{c_{n+1}}{c_n} = \frac{2(2n+1)}{n+2}$$

in [13].

It is given that the ordinary generating function of the Catalan sequence as follow

$$\begin{aligned} c(x) &= \frac{1-\sqrt[2]{1-4x}}{2x} \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + ... \end{aligned}$$

Definition 2.3. $(a_n)_{n\geq 0}$ *be a sequence with the generating function*

$$A(x) = a_0 + a_1 x + a_2 x^2 + ...$$

The Catalan transform of the sequence (a_n) *is defined to be the sequence whose o.g.f. is* $A(xc(x))$.

3. Catalan transform of the $k-$Jacobsthal sequence

Following [11], we define the Catalan transform of the $k-$Jacobsthal sequence $\{J_{k,n}\}$ as

$$CJ_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} J_{k,i}$$

for $n \geq 1$ with $CJ_{k,0} = 0$.

We can give some of them as follow:

$$\begin{aligned} CJ_{k,1} &= \sum_{1}^{1} \frac{i}{2-i} \binom{2-i}{1-i} J_{k,i} = 1, \\ CJ_{k,2} &= \sum_{1}^{2} \frac{i}{4-i} \binom{4-i}{2-i} J_{k,i} = 2, \\ CJ_{k,3} &= \sum_{1}^{3} \frac{i}{6-i} \binom{6-i}{3-i} J_{k,i} = 5 + k, \\ CJ_{k,4} &= 14 + 5k, \\ CJ_{k,5} &= 42 + 20k + k^2, \\ CJ_{k,6} &= 132 + 75k + 8k^2, \\ CJ_{k,7} &= 429 + 275k + 44k^2 + k^3. \end{aligned}$$

It can be written the following equation as the product of matrix C and $n \times 1$ matrix J_k

$$\begin{bmatrix} CJ_{k,1} \\ CJ_{k,2} \\ CJ_{k,3} \\ CJ_{k,4} \\ CJ_{k,5} \\ CJ_{k,6} \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} . \begin{bmatrix} J_{k,1} \\ J_{k,2} \\ J_{k,3} \\ J_{k,4} \\ J_{k,5} \\ J_{k,6} \\ . \\ . \\ . \end{bmatrix}$$

The entries of the matrix C verify the recurrence relation $C_{i,j} = \sum_{r=j-1}^{i-1} C_{i-1,r}$.The first column equals the second for $i > 1$ which are the Catalan numbers.

The lower triangular matrix $C_{n,n-i}$ is called Catalan triangle. Also, for $0 \leq i \leq n$,

$$C_{n,n-i} = \frac{(2n-i)!(i+1)}{(n-i)!(n+1)!}$$

We obtain first few Catalan transform of the $k-$Jaccobsthal sequence as follow:

$CJ_1 = \{0, 1, 2, 6, 19, 63, 215, 749, \ldots\}$, indexed in OEIS as A109262.
$CJ_2 = \{0, 1, 2, 7, 24, 86, 314, 1163, \ldots\}$,
$CJ_3 = \{0, 1, 2, 8, 29, 111, 429, 1677, \ldots\}$,
$CJ_4 = \{0, 1, 2, 9, 34, 138, 560, 2297, \ldots\}$,
$CJ_5 = \{0, 1, 2, 10, 39, 167, 707, 3029, \ldots\}$.

4. Hankel Transform

Let $R = \{r_0, r_1, r_2, ...\}$ be a sequence of real numbers. The Hankel transform of R is the sequence of determinants $H_n = Det\,[r_{i+j-2}]$ [10].That is

$$H_n = \begin{bmatrix} r_0 & r_1 & r_2 & r_3 & r_4 & \cdots \\ r_1 & r_2 & r_3 & r_4 & r_5 & \cdots \\ r_2 & r_3 & r_4 & r_5 & r_6 & \cdots \\ r_3 & r_4 & r_5 & r_6 & r_7 & \cdots \\ r_4 & r_5 & r_6 & r_7 & r_8 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

The Hankel determinant of order n of R is the upper-left $n \times n$ subdeterminant of H_n [6].

The sequence {1,1,1,...} is the Hankel transform of the Catalan sequence [1]. The Hankel transform of the sum of consecutive generalized Catalan numbers is the bisection of Fibonacci numbers [12].

$$HCJ_1 = Det[1] = 1,$$

$$HCJ_2 = \begin{vmatrix} 1 & 2 \\ 2 & 5+k \end{vmatrix} = 1 + k,$$

$$HCJ_3 = \begin{vmatrix} 1 & 2 & 5+k \\ 2 & 5+k & 14+5k \\ 5+k & 14+5k & 42+20k+k^2 \end{vmatrix} = k^2 + 3k + 1,$$

$$HCJ_4 = \begin{vmatrix} 1 & 2 & 5+k & 14+5k \\ 2 & 5+k & 14+5k & 42+20k+k^2 \\ 5+k & 14+5k & 42+20k+k^2 & 132+75k+k^2 \\ 14+5k & 42+20k+k^2 & 132+75k+k^2 & 429+275k+44k^2+k^3 \end{vmatrix} = k^3 + 5k^2 + 6k + 1.$$

We can continue in this form and then we will find that the Hankel transform of the Catalan transform of the k- Jacobsthal sequence $\{J_{k,n}\}$:

$$HCJ_1 = J_1,$$
$$HCJ_2 = J_3,$$

$$HCJ_3 = J_5,$$
$$HCJ_4 = J_7,$$

thus

$$HCJ_n = J_{k,2n-1}.$$

Conclusion 4.1. *In the present paper, we define Catalan $k-$Jacobsthal sequence and give some identites between the $k-$Jacobsthal and Catalan numbers. Also, we present some properties of the Catalan $k-$Jacobsthal sequence. This enables us to give in a straight forward way several formulas for the sums of such sequences. We put in for the Hankel transform to the Catalan transform of the $k-$Jacobsthal sequence and get an unknown property. These identities can be used to develop new identities of polynomials.*

Acknowledgement 4.2. *The authors would like to thank the anonymous reviewers for their useful comments and suggestions.*

References

[1] N. Anuradha, Number of points on certain hyperelliptic curves defined over finite fields. Finite Fields and Their Applications, 14, 314-328, 2008.
[2] P., Barry, A Catalan transform and related transformations on integer sequences. J.Integer Seq., 8(4), 1-24, 2005.
[3] H. Bruhn, L. Gellert, J. Günther, Jacobsthal numbers in generalized Petersen graphs. Electronic Notes in Discrete Mathematics, 49 465-472, 2015.
[4] A., Cvetkovi´c, Rajkovi´c, R., and M., Ivkovi´c, Catalan numbers, and Hankel transform, and Fibonacci numbers. J. Integer Seq. 5(1), 1-8, 2002.
[5] A. Daşdemir, On The Jacobsthal Numbers By Matrix Method 7(1), 69-76, 2012.
[6] S. Falcon, Catalan Transform of the $k-$Fibonacci sequence. Commun. Korean Math. Soc. 28(4), 827–832, 2013.
[7] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, Fibonacci Quarterly 3 (3), 161-176, 1965.
[8] J. W. Layman, The Hankel transform and some of its properties. J. Integer Seq., 4(1), 1- 11, 2001.,
[9] M. Marin, Contributions on uniqueness in thermoelastody-namics on bodies with voids, Ciencias Matematicas (Havana), 16 (2), 101-109, 1998.
[10] M. Marin, Generalized solutions in elasticity of micropolar bodies with voids, Revista de la Academia Canaria de Ciencias, vol. 8(1), 101-106, 1996.
[11] E. Özkan, A. Geçer, and İ. Altun, A new sequence realizing Lucas numbers, and the Lucas bound, Electronic Journal of Mathematical Analysis and Applications, vol.5, pp.148-154, 2017.
[12] P. M., Rajkovi´c, M. D. Petkovi´c, and P. Barry, The Hankel transform of the sum of consecutive generalized Catalan numbers. Integral Transforms Spec. Funct. 18(4), 285–296, 2007.
[13] N. J. A. Sloane,. The on-line encyclopedia of integer sequences. Annales Mathematicae et Informaticae, 41, 219-234, 2013.
[14] R., Stanley and E. W. Weisstein, Catalan Number, http://mathworld.wolfram.com/CatalanNumber.html. (Erişim Tarihi: 15.05.2019) 2019.

Merve TAŞTAN, Graduate School of Natural and Applied Sciences, Erzincan Binali Yildirim University, Erzincan, Turkey
E-mail address: mervetastan24@gmail.com

Engin ÖZKAN, Department of Mathematics, Erzincan Binali Yildirim University, Faculty of Arts and Sciences, Erzincan, Turkey
E-mail address: eozkanmath@gmail.com or enginozkan@erzincan.edu.tr