

AROUND A CONJECTURE OF K. TRAN

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ABSTRACT. We study the root distribution of a sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ with the rational generating function

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^\ell + A(z)t^k}$$

for $(k, \ell) = (3, 2)$ and $(4, 3)$ where $A(z)$ and $B(z)$ are arbitrary polynomials in z with complex coefficients. We show that the zeros of $P_n(z)$ which satisfy $A(z)B(z) \neq 0$ lie on a real algebraic curve which we describe explicitly.

1. INTRODUCTION

The study of zeros of sequences of polynomials plays an important role in many areas of mathematics such as analysis, probability theory, combinatorics and geometry. In this article, we study the distribution of zeros of polynomials in a polynomial sequence $\{P_n\}_{n=0}^{\infty}$ generated by a certain type of three-term recurrence relation. Such polynomials are of interest due to several remarkable properties they possess. For example, three-term recurrence relations are useful in numerical mathematics for producing sequences of orthogonal polynomials. Let $Q_1(z), Q_2(z), \dots, Q_k(z)$ be fixed complex polynomials and $\{P_n(z)\}_{n=0}^{\infty}$ be the sequence of polynomials generated by recurrence relation of the form:

$$P_n(z) + Q_1(z)P_{n-1}(z) + Q_2(z)P_{n-2}(z) + \dots + Q_k(z)P_{n-k}(z) = 0, \quad n = 1, 2, \dots \quad (1)$$

subject to certain initial conditions.

The problem of describing the location of the zeros of polynomials in the sequence $\{P_n(z)\}_{n=0}^{\infty}$ might have two different versions. The first is asymptotic and it aims at finding the limiting curve for the zeros of $P_n(z)$ as $n \rightarrow \infty$. This is the approach taken in the papers ([3], [4], [6] and [12]). The second is of exact type; it aims at finding the curve where all the zeros of $P_n(z)$ lie for all n (or at least for all large n), for example ([1] and [2]). For general recurrence relations, such curves do not exist. However, for three-term recurrences with $2 \leq k \leq 5$ and appropriate initial conditions, such a curve containing all the zeros of $P_n(z)$ exists and is given in [1]. More generally in [1, Conjecture 6], K. Tran conjectured existence of such a curve

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for all $k \geq 6$. Below we reformulate Tran's Conjecture.

Conjecture A For an arbitrary pair of complex polynomials $A(z)$ and $B(z)$, every zero of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length k

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0 \quad \text{for } n = 1, 2, \dots$$

with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \dots = P_{1-k}(z) = 0$ which is not a zero of $A(z)$ lies on the portion of the real algebraic curve $\mathcal{C} \subset \mathbb{C}$ given by

$$\Im\left(\frac{B^k(z)}{A(z)}\right) = 0 \quad \text{and} \quad 0 \leq (-1)^k \Re\left(\frac{B^k(z)}{A(z)}\right) \leq \frac{k^k}{(k-1)^{k-1}}.$$

The first part of the above conjecture explicitly defines the real algebraic curve on which all the zeros of the polynomials $P_n(z)$ (except the zeros shared with $A(z)$) are located. The second part describes the exact portion on this curve where these zeros lie. In [1], Conjecture A was settled for $k \leq 5$. In a subsequent paper [2], K. Tran proved existence of such a curve containing all the zeros of $P_n(z)$ for sufficiently large n and arbitrary k . More supporting results can be found in [7]. Additionally, a criterion for the reality of all the zeros of every polynomial in the sequence $\{P_n(z)\}_{n=0}^{\infty}$ for $k = 2$ can be found in [10].

Based on numerical experiments, B. Shapiro has formulated the following generalization of Conjecture A (private communication).

Conjecture 1 For an arbitrary pair of complex polynomials $A(z)$ and $B(z)$, every zero of every polynomial in the sequence $\{P_n(z)\}_{n=0}^{\infty}$ satisfying the three-term recurrence relation of length k

$$P_n(z) + B(z)P_{n-\ell}(z) + A(z)P_{n-k}(z) = 0 \quad \text{for } n = 1, 2, \dots \quad (2)$$

with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \dots = P_{1-k}(z) = 0$ and coprime $1 \leq \ell < k$ which is not a zero of $A(z)$ or $B(z)$ lies on the real algebraic curve $\mathcal{C} \subset \mathbb{C}$ given by

$$\Im\left(\frac{B^k(z)}{A^\ell(z)}\right) = 0.$$

The purpose of this paper is to prove some specific cases of Conjecture 1 and make them more precise. It is important to note that the rational function $B^k(z)/A^\ell(z)$ can be written in terms of the discriminant of the denominator of the generating function of the polynomial sequence generated by (2). Our approach to the proof of the specific cases of Conjecture 1 uses the q -analogue of the discriminant of a polynomial, a concept introduced by Ismail in [9] and used in [1]. In particular, the ratios of zeros of the denominator of the generating function with equal modulus will play a fundamental role.

The paper is organized as follows. In Section 2, we review the most important notions and basic results. In Section 3, we prove some relevant general results. In Section 4, we prove the main results of the paper. In Section 5, we give some numerical examples.

2. DISCRIMINANTS

This paper uses discriminants, and therefore we present below a few key results about them from the literature. Our initial step is to recall the concept of discriminants of polynomials, both ordinary and q -discriminants.

Definition 1 (see [11]) Let $P(x)$ be a univariate polynomial of degree n with zeros x_1, \dots, x_n and leading coefficient a_n . The ordinary discriminant of $P(x)$ is defined as

$$Disc_x(P(x)) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

This ordinary discriminant of $P(x)$ can also be expressed in terms of the resultant of $P(x)$ and its derivative. Let $P(x), Q(x)$ be polynomials of degrees n, m and leading coefficients a_n, b_m respectively. The resultant of $P(x)$ and $Q(x)$ is defined by

$$Res(P(x), Q(x)) = a_n^m \prod_{P(x_i)=0} Q(x_i) = b_m^n \prod_{Q(x_i)=0} P(x_i). \quad (3)$$

The discriminant of $P(x)$ is then computed as follows

$$Disc_x(P(x)) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} Res(P(x), P'(x)). \quad (4)$$

For more details on the ordinary discriminants and resultants, (see [5] and [11]).

Example 1 If $f(x)$ and $g(x)$ are fixed complex polynomials in x with $f \neq 0$, then the discriminant of $P(x, t) = f(x)t^3 + g(x)t^2 + 1$ as a polynomial in t is given by

$$Disc_t(P(x, t)) = -27(f(x))^2 - 4(g(x))^3.$$

The proof/ verification of the above example follows directly from equation (4).

Generally, the discriminant of a polynomial connects with the ratio of its zeros in the sense that the discriminant is zero if and only if the polynomial has a multiple zero. In particular, the discriminant of a polynomial vanishes whenever there exist at least two zeros that are equal, i.e., zeros whose ratio is 1.

Definition 2 (see [9]) Let $P(t)$ be a polynomial of degree n with zeros x_1, \dots, x_n and leading coefficient a_n . The q -discriminant of $P(t)$ is

$$Disc_t(P(t); q) = q^{n(n-1)/2} a_n^{2n-2} \prod_{1 \leq i < j \leq n} (q^{-1/2}x_i - q^{1/2}x_j)(q^{1/2}x_i - q^{-1/2}x_j).$$

In other words,

$$Disc_t(P(t); q) = q^{n(n-1)/2} a_n^{2n-2} \prod_{1 \leq i < j \leq n} (x_i^2 + x_j^2 - (q^{-1} + q)x_i x_j). \quad (5)$$

It is clear from Definition 2 that $Disc_t(P(t); 1) = Disc_t(P(t))$, the ordinary discriminant of the polynomial. Additionally, the q -discriminant vanishes whenever there is a pair of zeros with ratio q . Below we give some examples of q -discriminants.

Example 2 Let $P(t) = at^2 + bt + c$ be a quadratic polynomial with complex coefficients. The q -discriminant of $P(t)$ is

$$Disc_t(P(t); q) = b^2q - (q + 1)^2ac.$$

This case can easily be computed from Equation (5). A similar but tedious calculation give the q -discriminant of cubic polynomial $P(t) = at^3 + bt^2 + ct + d$ as $Disc_t(P(t); q) = -a^2d^2(1+q+q^2)^3 - b^2q^2(-c^2q+bd(1+q)^2) + acq(-c^2q(1+q)^2 + bd\Phi)$ where $\Phi = 1 + 5q + 6q^2 + 5q^3 + q^4$.

In general computing q -discriminants using Definition 2 can be a tedious exercise. However, the following proposition proved in [9] is often used.

Proposition 1 Let $P(t)$ be a polynomial of degree n with zeros x_1, \dots, x_n and leading coefficient a_n . The q -discriminant of $P(t)$ is given by:

$$Disc_t(P(t); q) = (-1)^{n(n-1)/2} a_n^{n-2} \prod_{i=1}^n (D_q P)(x_i),$$

where

$$(D_q P)(t) = \frac{P(t) - P(qt)}{t - qt}.$$

Proposition 1 is used in ([2] and [14]) to derive the expression of the q -discriminant of the polynomial $D(t, x) = A(x)t^n + B(x)t + 1$. This is stated in Proposition 2, (we suppress the parameter x).

Proposition 2 (K. Tran [2]) Let $D(t) = At^n + Bt + 1$ be a polynomial in t of degree n where A and B are arbitrary fixed complex functions. The q -discriminant of $D(t) = At^n + Bt + 1$ is given by

$$Disc_t(D(t); q) = \pm A^{n-2} \left(B^n \frac{q^{n-1}(1-q^{n-1})^{n-1}}{(1-q)^{n-1}} + (-1)^{n-1} \frac{(1-q^n)^n}{(1-q)^n} A \right).$$

For completeness let us review some definitions (also obtained in [8, 12]) about the root distribution of a sequence of functions

$$f_m = \sum_{i=1}^n \alpha_i(z) \beta_i(z)^m,$$

where $\alpha_i(z)$ and $\beta_i(z)$ are analytic in a domain D . Let us call an index i dominant at z if $|\beta_i(z)| \geq |\beta_j(z)|$ for all $j(1 \leq j \leq n)$. Let

$$D_i = \{z \in D : i \text{ is dominant at } z\}.$$

Denote the set of zeros of $f_m(z)$ by $Z(f_m)$.

Definition 3 (see [8]) The set of all $z \in D$ such that every neighborhood U of z has a non-empty intersection with infinitely many of the sets $Z(f_m)$ is called limit superior of $Z(f_m)$ and is denoted by $\limsup Z(f_m)$. On the other hand, the set of all $z \in D$ such that every neighborhood U of z has a non-empty intersection with all but finitely many of the sets $Z(f_m)$ is called limit inferior of $Z(f_m)$ and is denoted by $\liminf Z(f_m)$.

We note that both $\liminf Z(f_m)$ and $\limsup Z(f_m)$ always exist. In addition, it always holds that $\liminf Z(f_m) \subset \limsup Z(f_m)$. Moreover, in the event that both the $\liminf Z(f_m)$ and $\limsup Z(f_m)$ coincide, then we call this set, the limit of $Z(f_m)$, namely, $\lim Z(f_m) = \liminf Z(f_m) = \limsup Z(f_m)$.

The following theorem, see ([8, Theorem 1.5]) will be useful in this work.

Theorem 1 Let D be a domain in \mathbb{C} , and let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$, ($n \geq 2$) be analytic functions on D , none of which is identically zero. Let us further assume a “no-degenerate-dominance” condition, that is, there do not exist indices $i \neq i'$ such that $\beta_i \equiv \omega \beta_{i'}$ for some constant ω with $|\omega| = 1$ and such that $D_i (= D_{i'})$ has nonempty interior. For each integer $m \geq 0$, define f_m by

$$f_m = \sum_{i=1}^n \alpha_i(z) \beta_i(z)^m.$$

Then $\liminf Z(f_m) = \limsup Z(f_m)$, and a point z lies in this set if and only if either

- (a) there is a unique dominant index i at z , and $\alpha_i(z) = 0$, or
- (b) there are two or more dominant indices at z .

Remark 1

Case (a) of Theorem 1 consists of isolated points in D while case (b) consists of curves (and possibly isolated points where all the β_i vanish simultaneously). The Beraha-Kahane-Weiss Theorem for finding limits of roots of family of polynomials is a special case of Theorem 1 in which f_m are polynomials satisfying a linear finite-order recurrence relation. See e.g the papers ([3] and [4]) for details. In our case f_m are the polynomials P_n satisfying a three-term recurrence relation.

If $z^* \in \mathbb{C}$ is a fixed complex number such that the zeros in t of $D(t, z^*) = A(z^*)t^k + B(z^*)t^\ell + 1$ are distinct, then by use of partial fraction decomposition and Theorem 1, z^* belongs to $\liminf Z(f_m)$ when the two smallest zeros (in modulus) of $D(t, z^*)$ have the same modulus, [15]. Since the solutions to $D(t, z) = 0$ never vanish (identically) in the required domain, it follows that case Theorem 1(a) is never satisfied (no unique dominant index) and we are left to be in case of Theorem 1(b) where we have some equimodularity condition.

Now with our setup, if we can obtain a point $z^* \in \mathbb{C}$ so that the zeros of $D(t, z^*)$ are distinct and the two smallest (in modulus) zeros of $D(t, z^*)$ have the same modulus, then for such a point z^* , we have that $z^* \in \liminf Z(f_m) = \limsup Z(f_m)$. This implies that on a small neighborhood of z^* , there is a zero of f_m for all large m by the Definition 3 of $\liminf Z(f_m)$. For details see [15].

It is important to note that Theorem 1 provides a description of the asymptotic behaviour of the zeros of $\{P_n(x)\}$ in the general case. On the other hand, Conjectures A and 1 describe specific cases where all the zeros z of $P_n(x)$ such that $A(z) \neq 0$ for Conjecture A or $(A(z)B(z)) \neq 0$ for Conjecture 1) lie on the curve \mathcal{C} for all n or for all sufficiently large n .

3. GENERAL RESULTS

In order to prove some specific cases of Conjecture 1, we first prove some general auxiliary lemmas. The results for cases when $(k, \ell) = (3, 2)$ and $(k, \ell) = (4, 3)$ then fit in as specific cases. In this section among other things, we compute the q -discriminant of a special polynomial $D(t, z) = A(z)t^k + B(z)t^\ell + 1$ for coprime $1 \leq \ell < k$. We begin as follows.

Lemma 1 Let $A(z), B(z)$ be fixed complex polynomials and $\{P_n(z)\}_{n=0}^\infty$ be a sequence of polynomials given by linear recurrence relation of length k

$$P_n(z) + B(z)P_{n-\ell}(z) + A(z)P_{n-k}(z) = 0 \tag{6}$$

with the standard initial conditions $P_0(z) = 1, P_{-1}(z) = \dots = P_{1-k}(z) = 0$ where $1 \leq \ell < k$ are coprime. The generating function of $\{P_n(z)\}_{n=0}^\infty$ is given by

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^\ell + A(z)t^k}. \tag{7}$$

Proof. Let $G(t) = \sum_{n \geq 0} P_n t^n$ be the generating function of $\{P_n\}_{n=0}^\infty$. Multiplying

(6) by t^n and summing over from $n = k$ to infinity, we obtain

$$\sum_{n \geq k} P_n t^n + B \sum_{n \geq k} P_{n-\ell} t^n + A \sum_{n \geq k} P_{n-k} t^n = 0.$$

Simplifying gives

$$\left(G(t) - \sum_{n=0}^{k-1} P_n t^n \right) + B t^\ell \left(G(t) - \sum_{n=0}^{k-\ell-1} P_n t^n \right) + A t^k G(t) = 0$$

which is equivalent to

$$\begin{aligned} G(t) &= \frac{1}{1 + B(z)t^\ell + A(z)t^k} \left(\sum_{n=0}^{k-1} P_n t^n + B t^\ell \sum_{n=0}^{k-\ell-1} P_n t^n \right) \\ &= \frac{1}{1 + B(z)t^\ell + A(z)t^k} \left(1 + \sum_{n=1}^{\ell-1} P_n t^n + \sum_{n=\ell}^{k-1} (P_n + B P_{n-\ell}) t^n \right). \end{aligned}$$

From Equation (6), we have $P_n = 0$ for $0 < n \leq \ell - 1$ and $P_n + B P_{n-\ell} = 0$ for $\ell \leq n \leq k - 1$. It follows that $1 + \sum_{n=1}^{\ell-1} P_n t^n + \sum_{n=\ell}^{k-1} (P_n + B P_{n-\ell}) t^n = 1$, which proves the required result.

Let us now consider the q -discriminant of $D(t, z)$. In the following theorem, we shall suppress the variable z .

Theorem 2 For coprime $1 \leq \ell < k$, the q -discriminant of $D(t) = At^k + Bt^\ell + 1$ is given by

$$Disc_t(D(t); q) = (-1)^{\frac{k(k+1)}{2}} \left((q^k - 1)^k A^\ell - B^k (1 - q^\ell)^\ell (q^\ell - q^k)^{k-\ell} \right) W \tag{8}$$

where $W = A^{k-\ell-1} B^{\ell-1} (1 - q)^{-k}$.

Proof. By Proposition 1, we have

$$Disc_t(D(t); q) = (-1)^{k(k-1)/2} A^{k-2} \prod_{\substack{D(s_i^*)=0 \\ 1 \leq i \leq k}} (D_q D)(s_i^*),$$

where

$$(D_q D)(t) = \frac{D(t) - D(qt)}{t - qt}.$$

Now

$$\begin{aligned} (D_q D)(t) &= \frac{At^k + Bt^\ell + 1 - (Aq^k t^k + Bq^\ell t^\ell + 1)}{t - qt} \\ &= At^{k-1} \frac{1 - q^k}{1 - q} + Bt^{\ell-1} \frac{1 - q^\ell}{1 - q}. \end{aligned} \tag{9}$$

Using equation (3) with $P = D$ and $Q = D_q D$ we obtain

$$\prod_{\substack{D(s_i^*)=0 \\ 1 \leq i \leq k}} (D_q D)(s_i^*) = A \left(\frac{1 - q^k}{1 - q} \right)^k \prod_{\substack{(D_q D)(t_i^*) \\ 1 \leq i \leq k-1}} D(t_i^*).$$

This implies that

$$\begin{aligned} \text{Disc}_t(D(t); q) &= (-1)^{\frac{k(k-1)}{2}} A^{k-1} \left(\frac{1-q^k}{1-q} \right)^k \prod_{\substack{(D_q D)(t_i^*) \\ 1 \leq i \leq k-1}} D(t_i^*) \\ &= (-1)^{\frac{k(k-1)}{2}} A^{k-1} \left(\frac{1-q^k}{1-q} \right) \prod_{\substack{(D_q D)(t_i^*) \\ 1 \leq i \leq k-1}} \frac{1-q^k}{1-q} D(t_i^*). \end{aligned} \quad (10)$$

Now, $D(t)$ and $D_q D(t)$ are related by the equation

$$(D_q D)(t) = \left(\frac{D(t)}{t} - Bt^{\ell-1} - \frac{1}{t} \right) \frac{1-q^k}{1-q} + Bt^{\ell-1} \left(\frac{1-q^\ell}{1-q} \right).$$

Therefore

$$\begin{aligned} D(t) \left(\frac{1-q^k}{1-q} \right) &= (Bt^\ell + 1) \left(\frac{1-q^k}{1-q} \right) - Bt^\ell \left(\frac{1-q^\ell}{1-q} \right) + t(D_q D)(t) \\ &= t(D_q D)(t) + Bt^\ell \left(\frac{q^\ell - q^k}{1-q} \right) + \frac{1-q^k}{1-q}. \end{aligned}$$

At the zeros t_i^* of $(D_q D)(t)$ for $1 \leq i \leq k-1$, we have $(D_q D)(t_i^*) = 0$, hence

$$D(t_i^*) \left(\frac{1-q^k}{1-q} \right) = (t_i^*)^\ell \left(\frac{q^\ell - q^k}{1-q} \right) B + \frac{1-q^k}{1-q}. \quad (11)$$

Now substituting equation (11) into equation (10) we obtain

$$\begin{aligned} \text{Disc}_t(D(t); q) &= (-1)^{\frac{k(k-1)}{2}} A^{k-1} \left(\frac{1-q^k}{1-q} \right) \prod_{\substack{(D_q D)(t_i^*) \\ 1 \leq i \leq k-1}} \left((t_i^*)^\ell \left(\frac{q^\ell - q^k}{1-q} \right) B + \frac{1-q^k}{1-q} \right) \\ &= (-1)^{\frac{k(k-1)}{2}} (AB)^{k-1} \left(\frac{1-q^k}{1-q} \right) \left(\frac{q^\ell - q^k}{1-q} \right)^{k-1} \\ &\quad \times \prod_{\substack{(D_q D)(t_i^*) \\ 1 \leq i \leq k-1}} \left((t_i^*)^\ell + \frac{1-q^k}{B(q^\ell - q^k)} \right). \end{aligned}$$

From Equation (9), the condition $(D_q D)(t_i^*) = 0$ for $1 \leq i \leq k-1$ implies that 0 is a root that occurs with multiplicity $\ell-1$. The other zeros satisfy the relation

$$(t_i^*)^{k-\ell} = -\frac{B}{A} \left(\frac{1-q^\ell}{1-q^k} \right). \quad (12)$$

Since the integers k and ℓ are coprime, which is equivalent $k-\ell$ and ℓ being coprime, we can make the change of variables $\xi = t^\ell$. We observe that t_i^* for $1 \leq i \leq k-\ell$ is a zero to the polynomial $r(t) = t^{k-\ell} + \frac{B}{A} \left(\frac{1-q^\ell}{1-q^k} \right)$. By the change of variable $\xi = t^\ell$, the polynomial $r(t)$ becomes

$$\Psi(\xi) = \xi^{k-\ell} - \left(\frac{-B}{A} \right)^\ell \left(\frac{1-q^\ell}{1-q^k} \right)^\ell. \quad (13)$$

In particular, the solutions ξ_i^* to $\Psi(\xi) = 0$ for $1 \leq i \leq k - \ell$ are of the form $(t_i^*)^\ell$ where t_i^* is a solution to $r(t) = 0$, (nonzero solution to $(D_q D)(t) = 0$). So

$$\prod_{\substack{(D_q D)(t_i^*)=0 \\ t_i^* \neq 0 \\ 1 \leq i \leq k-\ell}} \left((t_i^*)^\ell + \frac{1 - q^k}{B(q^\ell - q^k)} \right) = \prod_{\substack{\Psi(\xi_i^*)=0 \\ 1 \leq i \leq k-\ell}} \left(\xi_i^* + \frac{1 - q^k}{B(q^\ell - q^k)} \right).$$

The zero $t_i^* = 0$ for $1 \leq i \leq \ell - 1$ contributes the following factor to the discriminant

$$\left(\frac{1 - q^k}{B(q^\ell - q^k)} \right)^{\ell-1}$$

The expression for $Disc_t(D(t); q)$ now has the form

$$\begin{aligned} Disc_t(D(t); q) &= (-1)^{\frac{k(k-1)}{2}} (AB)^{k-1} \left(\frac{1 - q^k}{1 - q} \right) \left(\frac{q^\ell - q^k}{1 - q} \right)^{k-1} \left(\frac{1 - q^k}{B(q^\ell - q^k)} \right)^{\ell-1} \\ &\quad \times \prod_{\substack{\Psi(\xi_i^*)=0 \\ 1 \leq i \leq k-\ell}} \left(\xi_i^* + \frac{1 - q^k}{B(q^\ell - q^k)} \right). \end{aligned}$$

We now simplify the remaining product. Consider the following equation:

$$\xi^{k-\ell} = \left(\frac{-B}{A} \right)^\ell \left(\frac{1 - q^\ell}{1 - q^k} \right)^\ell. \quad (14)$$

Using the change of variable

$$\xi = Y - \frac{1 - q^k}{B(q^\ell - q^k)},$$

Equation (14) becomes

$$\left(Y - \frac{1 - q^k}{B(q^\ell - q^k)} \right)^{k-\ell} - \left(\frac{-B}{A} \right)^\ell \left(\frac{1 - q^\ell}{1 - q^k} \right)^\ell = 0.$$

Using Binomial Theorem, we get

$$g(Y) := Y^{k-\ell} + \dots + \left(-\frac{(1 - q^k)}{B(q^\ell - q^k)} \right)^{k-\ell} - \left(\frac{-B}{A} \right)^\ell \left(\frac{1 - q^\ell}{1 - q^k} \right)^\ell = 0.$$

The constant term in $g(Y)$ is

$$C = \left(-\frac{(1 - q^k)}{B(q^\ell - q^k)} \right)^{k-\ell} - \left(\frac{-B}{A} \right)^\ell \left(\frac{1 - q^\ell}{1 - q^k} \right)^\ell.$$

For $i = 1, 2, \dots, k - \ell$, let y_i are the solutions to $g(Y) = 0$. Then

$$\xi_j = y_i - \frac{1 - q^k}{B(q^\ell - q^k)}$$

for some $j \in \{1, 2, \dots, k - \ell\}$. So we can choose $i = j$ so that $\xi_i = y_i - \frac{1 - q^k}{B(q^\ell - q^k)}$.

Now the product of solutions to $g(Y) = 0$ is given by

$$C = (-1)^{k-\ell} \prod_{i=1}^{k-\ell} y_i = \left[\left(-\frac{(1 - q^k)}{B(q^\ell - q^k)} \right)^{k-\ell} - \left(\frac{-B}{A} \right)^\ell \left(\frac{1 - q^\ell}{1 - q^k} \right)^\ell \right].$$

or

$$\prod_{j=1}^{k-\ell} \left(\xi_j + \frac{1 - q^k}{B(q^\ell - q^k)} \right) = \prod_{i=1}^{k-\ell} y_i = (-1)^{k-\ell} C. \tag{15}$$

Plugging equation (15) into the expression for $Disc_t(D(t); q)$ gives

$$\begin{aligned} Disc_t(D(t); q) &= (-1)^{\frac{k(k-1)}{2}} \frac{(AB)^{k-1} (q^\ell - q^k)^{k-1} (1 - q^k)}{(1 - q)^k} \left(\frac{1 - q^k}{B(q^\ell - q^k)} \right)^{\ell-1} \\ &\quad \times (-1)^{k-\ell} \left[\frac{(q^k - 1)^{k-\ell}}{B^{k-\ell} (q^\ell - q^k)^{k-\ell}} - \frac{(-1)^\ell B^\ell (1 - q^\ell)^\ell}{A^\ell (1 - q^k)^\ell} \right]. \end{aligned} \tag{16}$$

Simplifying equation (16) gives

$$Disc_t(D(t); q) = (-1)^{\frac{k(k+1)}{2}} \left((q^k - 1)^k A^\ell - B^k (1 - q^\ell)^\ell (q^\ell - q^k)^{k-\ell} \right) W,$$

where $W = A^{k-\ell-1} B^{\ell-1} (1 - q)^{-k}$. The proof is complete.

Corollary 1 Let A and B be fixed complex polynomials. The q -discriminant of $D(t) = At^3 + Bt^2 + 1$ is

$$Disc_t(D(t); q) = \frac{B \left(A^2 (q^3 - 1)^3 + B^3 (q - 1) q^2 (q^2 - 1)^2 \right)}{(1 - q)^3}.$$

Proof. Set $k = 3$ and $\ell = 2$ in the Theorem 2 above and obtain the result.

The following lemma will be used in the proof of the main results.

Lemma 2 Let k, ℓ be integers and $q \in \mathbb{C}$ with $|q| = 1$. The function defined by

$$h(q) = \frac{(1 - q^k)^k}{(1 - q^\ell)^\ell (q^\ell - q^k)^{k-\ell}}$$

is real-valued.

Proof. Since $|q| = 1$, we have $q^s - q^{-s} = 2i\Im(q^s)$, for any integer s , (here $i^2 = -1$). It therefore follows that $(1 - q^s)^s = (q^{\frac{s}{2}}(q^{-\frac{s}{2}} - q^{\frac{s}{2}}))^s = q^{s(\frac{s}{2})} (-2i)^s (\Im(q^{\frac{s}{2}}))^s$. So,

$$\begin{aligned} h(q) &= \frac{(1 - q^k)^k}{(1 - q^\ell)^\ell (q^\ell - q^k)^{k-\ell}} \\ &= \frac{q^{k(\frac{k}{2})} (-2i)^k (\Im(q^{\frac{k}{2}}))^k}{q^{\ell(\frac{\ell}{2})} (-2i)^\ell (\Im(q^{\frac{\ell}{2}}))^\ell q^{\ell(k-\ell)} q^{(k-\ell)(\frac{k-\ell}{2})} (-2i)^{k-\ell} (\Im(q^{\frac{k-\ell}{2}}))^{k-\ell}} \\ &= \frac{q^{k(\frac{k}{2}) - (\ell(\frac{\ell}{2}) + \ell(k-\ell) + (k-\ell)(\frac{k-\ell}{2}))} (\Im(q^{\frac{k}{2}}))^k}{(\Im(q^{\frac{\ell}{2}}))^\ell \cdot (\Im(q^{\frac{k-\ell}{2}}))^{k-\ell}} \\ &= \frac{(\Im(q^{\frac{k}{2}}))^k}{(\Im(q^{\frac{\ell}{2}}))^\ell \cdot (\Im(q^{\frac{k-\ell}{2}}))^{k-\ell}}. \end{aligned}$$

Since the expressions $\Im(q^{\frac{k}{2}})$, $\Im(q^{\frac{\ell}{2}})$ and $\Im(q^{\frac{k-\ell}{2}})$ are all real, the result follows.

4. SPECIFIC CASES OF CONJECTURE 1 WHEN $(k, \ell) = (3, 2)$ AND $(k, \ell) = (4, 3)$

In this section, we settle the specific cases of $(k, \ell) = (3, 2)$ and $(k, \ell) = (4, 3)$ completely by showing that the zeros of $P_n(z)$ lie on a portion of real algebraic curve $\Im\left(\frac{B^k(z)}{A^\ell(z)}\right) = 0$. We also provide the relevant inequality constraint.

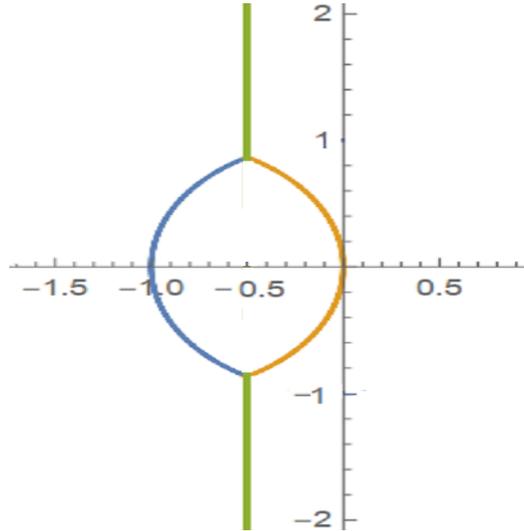


FIGURE 1. Distribution of the quotients of zeros of $D(t) = At^3 + Bt^2 + 1$. C_1 (blue), C_2 (orange) and C_3 (green).

For the case $(k, \ell) = (3, 2)$, we begin with the following lemma.

Lemma 3 Suppose $\xi_2, \xi_3 \neq 0$ are complex numbers such that $\xi_2 + \xi_3 + 1 = 0$ and

$$\frac{\xi_2^{n+1} - 1}{\xi_2 - 1} = \frac{\xi_3^{n+1} - 1}{\xi_3 - 1}$$

for some positive integer n . Then ξ_2 and ξ_3 lie on the curve $\Gamma = C_1 \cup C_2 \cup C_3$ and are dense there as $n \rightarrow \infty$. Here the equations of C_1, C_2 and C_3 (see Fig. 1) are given by

$$\Gamma = \begin{cases} C_1 : & x^2 + y^2 = 1, \quad x \leq -\frac{1}{2}, \\ C_2 : & (x + 1)^2 + y^2 = 1, \quad x \geq -\frac{1}{2}, \\ C_3 : & x = -\frac{1}{2} + iy, \quad |y| \geq \frac{\sqrt{3}}{2}. \end{cases}$$

The proof is similar to the one given in [2, Lemma 2] by letting $\xi \mapsto \frac{1}{\xi}$.

Remark 2 Note that t_1, t_2 and t_3 are zeros of $D(t, z) = A(z)t^3 + B(z)t^2 + 1$. By Vieta’s formulae we have $t_1t_2 + t_1t_3 + t_2t_3 = 0$. Dividing by t_1^2 and letting $q_i = t_i/t_1$ and $q_i = 1/\xi_i, i = 2, 3$ gives $q_2q_3 + q_2 + q_3 = 0$ if and only if $\xi_2 + \xi_3 + 1 = 0$, the first condition in Lemma 3.

Theorem 3 Let $\{P_n(z)\}$ be a polynomial sequence whose generating function is

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^2 + A(z)t^3},$$

where $B(z)$ and $A(z)$ are polynomials in z with complex coefficients. Given any $n \in \mathbb{N}$, those zeros z of $P_n(z)$ for which $A(z)B(z) \neq 0$ lie on the curve defined by

$$\Im \left(\frac{B^3(z)}{A^2(z)} \right) = 0 \text{ and } 0 \leq \Re \left(\frac{B^3(z)}{A^2(z)} \right) < \infty.$$

Proof. Let z_0 be a zero of $P_n(z)$ such that $A(z_0)B(z_0) \neq 0$ and let $t_1 = t_1(z_0), t_2 = t_2(z_0), t_3 = t_3(z_0)$ be the zeros of $D(t, z_0) = A(z_0)t^3 + B(z_0)t^2 + 1$. There are two cases we shall consider; namely, the case of repeated zeros and the case of distinct zeros of D .

Case 1: Suppose $D(t, z_0)$ has repeated zeros. The ordinary discriminant of $D(t, z_0)$ is

$$Disc_t(D(t, z_0)) = -27A^2(z_0) - 4B^3(z_0).$$

For repeated zeros, $Disc_t(D(t, z_0)) = 0$ which is equivalent to

$$\frac{B^3(z_0)}{A^2(z_0)} = -\frac{27}{4} \in \mathbb{R}.$$

Therefore, the point z_0 lies on the curve given by

$$\Im\left(\frac{B^3(z)}{A^2(z)}\right) = 0.$$

Case 2: Suppose all the zeros of $D(t, z_0)$ are distinct, i.e., $Disc_t(D(t, z_0)) \neq 0$. In this situation, let t_1, t_2 and t_3 be the three distinct zeros of $D(t, z)$. We first consider the distribution of quotients of zeros $q_i = t_i/t_1, 1 \leq i \leq 3$. Then we examine the root distribution of $P_n(z)$ using q -discriminants. We show that these quotients $q_i = t_i/t_1, 1 \leq i \leq 3$ lie on the curve in Fig. 1.

Let z_0 be a root of $P_n(z)$ which satisfies $A(z_0)B(z_0) \neq 0$ and let $t_1 = t_1(z_0), t_2 = t_2(z_0), t_3 = t_3(z_0)$ be the zeros of $D(t, z_0) = A(z_0)t^3 + B(z_0)t^2 + 1$.

By partial fraction decomposition, we have

$$\frac{1}{1 + B(z_0)t^2 + A(z_0)t^3} = \frac{1}{A(z_0)} \left(\frac{1}{(t - t_1)(t - t_2)(t - t_3)} \right) = \frac{1}{A(z_0)} \sum_{i=1}^3 \frac{1}{t - t_i} \prod_{i \neq j} \frac{1}{t_i - t_j}.$$

By using geometric progression, we get

$$\frac{1}{t - t_i} = -\frac{1}{t_i} \sum_{n=0}^{\infty} \frac{t^n}{t_i^n} = -\sum_{n=0}^{\infty} \frac{1}{t_i^{n+1}} t^n \quad (|t| < |t_i|).$$

Hence we obtain

$$\frac{1}{D(t, z_0)} = -\frac{1}{A(z_0)} \sum_{n=0}^{\infty} \left(\sum_{i=1}^3 \frac{1}{t_i^{n+1}} \prod_{i \neq j} \frac{1}{t_i - t_j} \right) t^n. \tag{17}$$

Comparing coefficients of t^n in equations (7) and (17) we obtain

$$P_n(z_0) = -\frac{1}{A(z_0)} \sum_{i=1}^3 \frac{1}{t_i^{n+1}} \prod_{i \neq j} \frac{1}{t_i - t_j}.$$

Let $q_i = t_i/t_1, i = 2, 3$. Now for any $n \geq 0$, the equation $P_n(z_0) = 0$ is equivalent to

$$q_2^n q_3^n (q_2 - q_3) - q_2^n (q_2 - 1) + q_3^n (q_3 - 1) = 0. \tag{18}$$

Since $q_2 - q_3 = q_2 - 1 - q_3 + 1$ it follows that

$$q_2^n q_3^n (q_2 - q_3) = q_2^n q_3^n (q_2 - 1) - q_2^n q_3^n (q_3 - 1).$$

Therefore equation (18) can be rewritten as

$$q_2^n q_3^n (q_2 - 1) - q_2^n q_3^n (q_3 - 1) = q_2^n (q_2 - 1) - q_3^n (q_3 - 1)$$

and consequently since $q_2, q_3 \neq 0, 1$ we obtain

$$\frac{q_2^n - 1}{q_2^n(q_2 - 1)} = \frac{q_3^n - 1}{q_3^n(q_3 - 1)}. \quad (19)$$

If we add 1 on both sides of the equation (19) and set $q_i = 1/\xi_i$, $i = 2, 3$ we obtain [one side]

$$\frac{q_2^n - 1}{q_2^n(q_2 - 1)} + 1 = \frac{q_2^{n+1} - 1}{q_2^n(q_2 - 1)} = \frac{q_2^{n+1}(1 - (1/q_2)^{n+1})}{q_2^{n+1}(1 - 1/q_2)} = \frac{\xi_2^{n+1} - 1}{\xi_2 - 1}.$$

Similarly on the other side of equation (19) we obtain

$$\frac{q_3^n - 1}{q_3^n(q_3 - 1)} + 1 = \frac{\xi_3^{n+1} - 1}{\xi_3 - 1}.$$

Thus equation (19) becomes

$$\frac{\xi_2^{n+1} - 1}{\xi_2 - 1} = \frac{\xi_3^{n+1} - 1}{\xi_3 - 1},$$

the second condition in Lemma 3.

Next, we need show that $\xi_i, i = 2, 3$ being on the curve Γ implies that $q_i \in \Gamma$ as well. In particular, if $\xi_i \in C_1$ then the corresponding $q_i \in C_1$. If $\xi_i \in C_2$ then the corresponding $q_i \in C_3$ and for $\xi_i \in C_3$, the corresponding $q_i \in C_2$. To do this, it is enough to show that Γ is invariant under the Möbius inversion $z \mapsto \frac{1}{z}$. This is shown as follows.

(i) For any $z = x + iy \in C_1$, we have $|z| = 1$ with $\Re(z) \leq -\frac{1}{2}$. In addition, $z \neq 0$ and so the image of z under inversion $w = \frac{1}{z}$ is $w = \frac{1}{z}$. Note that $|w| = |\frac{1}{z}| = \frac{1}{|z|} = 1$. Furthermore, $\Re(w) = \Re(\frac{1}{z}) = \frac{x}{x^2+y^2} \leq -\frac{1}{2}$. Hence $w \in C_1$.

(ii) For any $z = x + iy \in C_2$, we have $(x+1)^2 + y^2 = 1$ where $x \geq -\frac{1}{2}$. Under the inversion $w = \frac{1}{z}$, we have

$$w = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x - iy}{(x+1)^2 + y^2 - 2x - 1} = -\frac{1}{2} + \frac{y}{2x}i.$$

Note that on C_2 , $-\frac{1}{2} \leq x \leq 0$ and $-\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2}$. With these ranges on x and y , we obtain $|\Im(w)| = |\frac{y}{2x}| \geq \frac{\sqrt{3}}{2}$. Clearly $\Re(w) = -\frac{1}{2}$. Hence $w \in C_3$.

(iii) For any $z = x + iy \in C_3$, we have $z = -\frac{1}{2} + yi$ where $|y| \geq \frac{\sqrt{3}}{2}$. Clearly $z \neq 0$. So $|\bar{z}| = |z|$. Under the inversion $w = \frac{1}{z}$, we have

$$w = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{-1/2}{1/4 + y^2} + i\frac{-y}{1/4 + y^2}.$$

With the ranges of x and y for $z \in C_3$, we obtain

$$0 \geq \Re(w) = \frac{-1/2}{1/4 + y^2} \geq \frac{-1/2}{1/4 + 3/4} \geq -\frac{1}{2}. \quad (20)$$

In addition, since $\Im(w) = \frac{-y}{1/4+y^2}$ and $|y| \geq \frac{\sqrt{3}}{2}$, we have $0 \leq |\Im(w)| \leq \frac{\sqrt{3}}{2}$. Therefore $w \in C_2$.

The conclusion that Γ is invariant under the Möbius inversion $z \mapsto \frac{1}{z}$ thus follows. Therefore q_2 and q_3 lie on the curve Γ given in Lemma 3. The proof is complete.

Now we know that q_2 and q_3 are given by the ratio of zeros of D and are therefore the zeros of the discriminant. (Recall from Ismail [9] that the q -discriminant equals zero if there is a pair of zeros with the ratio q). From Corollary 1, we know that the q -discriminant of $D(t, z) = 1 + B(z)t^2 + A(z)t^3$ is given by

$$\text{Disc}_t(D(z, q)) = \frac{B \left(A^2 (q^3 - 1)^3 + B^3 (q - 1) q^2 (q^2 - 1)^2 \right)}{(1 - q)^3}.$$

Since q_2 and q_3 are the zeros of the discriminant then

$$\frac{B \left(A^2 (q_i^3 - 1)^3 + B^3 (q_i - 1) q_i^2 (q_i^2 - 1)^2 \right)}{(1 - q_i)^3} = 0$$

with $i = 2, 3$. Consequently since $q_i \neq 0, 1$ we have

$$\frac{B^3(z)}{A^2(z)} = \frac{-(1 + q_i + q_i^2)^3}{q_i^2(1 + q_i)^2}.$$

It remains to show that

$$f(q) = \frac{B^3(z)}{A^2(z)} = \frac{-(1 + q + q^2)^3}{q^2(1 + q)^2}$$

maps the curve Γ to a real interval so that we conclude $\Im(f) = 0$.

Now since q lies on the curve in Fig. 1, we have three possibilities below.

- (a) $|q| = 1$, and this corresponds to the points ξ on the curve C_1 .
- (b) $q + \bar{q} = -1$, and this corresponds to the points ξ on the curve C_2 .
- (c) $-1 - q = 1$, and this corresponds to the points ξ on the curve C_3 .

For part (a)

$$f(q) = -\frac{(1 + q + q^2)^3}{q^2(1 + q)^2} = -\frac{q^3(\frac{1}{q} + 1 + q)^3}{q^3(\frac{1}{q} + 2 + q)} \in \mathbb{R}$$

since $\frac{1}{q} + q \in \mathbb{R}$. Hence $\Im(f) = 0$.

For part (b), note that $\bar{q} = -1 - q$. Therefore

$$\begin{aligned} -\frac{(1 + \bar{q} + \bar{q}^2)^3}{\bar{q}^2(1 + \bar{q})^2} &= -\frac{(1 + (-1 - q) + (-1 - q)^2)^3}{(-1 - q)^2(1 + (-1 - q))^2} \\ &= -\frac{(-q + 2q + q^2 + 1)^3}{(-1 - q)^2 q^2} \\ &= -\frac{(1 + q + q^2)^3}{q^2(1 + q)^2} \end{aligned}$$

Consequently $f(\bar{q}) = \overline{f(q)} = f(q)$, and we conclude that $\Im(f) = 0$.

For Part (c), $-1 - q = 1 \Rightarrow \bar{q} = -\frac{q}{1+q}$. Therefore

$$f(\bar{q}) = -\frac{(1 + \bar{q} + \bar{q}^2)^3}{\bar{q}^2(1 + \bar{q})^2} = -\frac{(1 + \frac{-q}{1+q} + (\frac{-q}{1+q})^2)^3}{(\frac{-q}{1+q})^2(1 + \frac{-q}{1+q})^2} = -\frac{(1 + q + q^2)^3}{q^2(1 + q)^2} = f(q).$$

Hence $f(\bar{q}) = \overline{f(q)} = f(q)$, and we conclude that $\Im(f) = 0$.

The conclusion $\Im(f) = 0$ follows. Therefore, the point z_0 lies on the curve given by

$$\Im\left(\frac{B^3(z)}{A^2(z)}\right) = 0. \tag{21}$$

Next we prove the inequality constraint. From the recurrence relation with $\ell = 2, k = 3$ we have

$$P_n(z) + B(z)P_{n-2}(z) + A(z)P_{n-3}(z) = 0$$

whose characteristic equation is $t^3 + B(z)t + A(z) = 0$. The corresponding denominator of the generating function is $D(t, z) = A(z)t^3 + B(z)t^2 + 1$. Suppressing z , we can write $D(t) = At^3 + Bt^2 + 1$.

Let t_1, t_2, t_3 be the zeros of $D(t)$ labelled so that $0 < |t_1| \leq |t_2| \leq |t_3|$ and set $q_i := t_i/t_1$ for $i = 1, 2, 3$. Note these are functions of z . Vieta's formulae give $t_1t_2 + t_1t_3 + t_2t_3 = 0$. This implies that $q_3 = -\frac{u}{1+u}$ where $u = q_2$.

Equation (21) describes the algebraic closure of the Beraha-Kahane-Weiss limit (which is a real semialgebraic closure in \mathbb{R}^2). In other words the condition (21) describes the set of all z for which $D(t, z)$ has two roots with coinciding absolute values. In terms of the quantities q_1, q_2 and q_3 , the later condition is given by $q_1 = 1, |q_2| = |u| = 1$. Thus if z_0 is a zero of $P_n(z)$ and $A(z_0)B(z_0) \neq 0$ then $q_1 = 1, |u| = 1$. The Beraha-Kahane-Weiss theorem limit is given by the extra condition that $|q_3| \geq 1$.

So, we search for an u such that the above conditions are satisfied. Let $u = e^{i\theta}$ where $\theta \in [0, 2\pi]$. It follows from $|q_3| \geq 1$ that $1 = |u| \geq |1 + u| = 2(1 + \cos \theta)$. This implies that $\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$.

Let $\Omega = \{e^{i\theta} : \frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}\}$. Since $q \in \Omega \subset \Gamma$, it follows that

$$f(q) = -\frac{(1 + q + q^2)^3}{q^2(1 + q)^2}.$$

We now compute the range of f on Ω . To do this, we let $q = e^{i\theta}$ where $\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$. After this parametrization we obtain

$$f(q) = F(\theta) = -\frac{(2 \cos(\theta) + 1)^3}{2 \cos(\theta) + 2}.$$

It is clear that F is well-defined and continuous on the union $[\frac{2\pi}{3}, \pi) \cup (\pi, \frac{4\pi}{3}]$. Observe that for $\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$, we have $-1 \leq 2 \cos(\theta) + 1 \leq 0$ and $0 \leq 2 \cos(\theta) + 2 \leq 1$ hence $F(\theta) \geq 0$. Moreover, F attains its minimal values equal to 0 at $\theta = \frac{2\pi}{3}$ and at $\theta = \frac{4\pi}{3}$. Since $\lim_{\theta \rightarrow \pi} F(\theta) = +\infty$, it follows from continuity and non-negativity of F that the range of f is $\mathbb{R}_{\geq 0}$. This proves the inequality condition that

$$0 \leq \Re\left(\frac{B^3(z)}{A^2(z)}\right) < \infty.$$

Next we settle the specific case of Conjecture 1 where $(k, \ell) = (4, 3)$. We begin with the following lemma.

Lemma 4 Let z_0 be a zero of $P_n(z)$ and $q = q(z_0)$ be a quotient of two zeros in t of $D(t, z_0) = A(z_0)t^4 + B(z_0)t^3 + 1$. Then the set of all such quotients belongs to

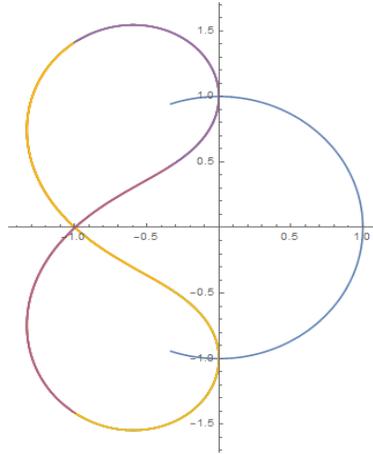


FIGURE 2. Distribution of the quotients of zeros of the $D(t) = At^4 + Bt^3 + 1$.

the curve depicted in Fig. 2 where the equation of the quartic curve C_5 (see the left part of Fig. 2) is

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0,$$

and the other curve C_4 (see the right part of Fig. 2) is the segment of the unit circle with real part at least $-1/3$. **Proof.** Let z_0 be a zero of $P_n(z)$ such that $A(z_0)B(z_0) \neq 0$ and let $t_1 = t_1(z_0), t_2 = t_2(z_0), t_3 = t_3(z_0)$ and $t_4 = t_4(z_0)$ be the zeros of $D(t, z_0) = A(z_0)t^4 + B(z_0)t^3 + 1$.

By partial fraction decomposition, we have

$$\frac{1}{1 + B(z_0)t^3 + A(z_0)t^4} = \frac{1}{A(z_0)} \left(\frac{1}{(t - t_1)(t - t_2)(t - t_3)(t - t_4)} \right) = \frac{1}{A(z_0)} \sum_{i=1}^4 \frac{1}{t - t_i} \prod_{i \neq j} \frac{1}{t_i - t_j}.$$

By using geometric progression, we get

$$\frac{1}{t - t_i} = -\frac{1}{t_i} \sum_{n=0}^{\infty} \frac{t^n}{t_i^n} = -\sum_{n=0}^{\infty} \frac{1}{t_i^{n+1}} t^n \quad (|t| < |t_i|).$$

Hence we obtain

$$\frac{1}{D(t, z_0)} = -\frac{1}{A(z_0)} \sum_{n=0}^{\infty} \left(\sum_{i=1}^4 \frac{1}{t_i^{n+1}} \prod_{i \neq j} \frac{1}{t_i - t_j} \right) t^n. \tag{22}$$

Comparing coefficients of t^n in Equation (7) and Equation (22) we obtain

$$P_n(z_0) = -\frac{1}{A(z_0)} \sum_{i=1}^4 \frac{1}{t_i^{n+1}} \prod_{i \neq j} \frac{1}{t_i - t_j}.$$

Let $q_i = t_i/t_1, i = 2, 3, 4$. Now for any $n \geq 0$, the equation $P_n(z_0) = 0$ is equivalent to

$$q_2^{n+1}q_3^{n+3}q_4^{n+2} - q_2^{n+1}q_3^{n+2}q_4^{n+3} - q_2^{n+2}q_3^{n+3}q_4^{n+1} + q_2^{n+2}q_3^{n+1}q_4^{n+3} + q_2^{n+3}q_3^{n+2}q_4^{n+1} - q_2^{n+3}q_3^{n+1}q_4^{n+2} + q_2^{n+1}q_3^{n+2} - q_2^{n+1}q_3^{n+3} + q_2^{n+2}q_3^{n+3} + q_2^{n+3}q_3^{n+1} - q_2^{n+3}q_3^{n+2} - q_2^{n+2}q_3^{n+1} - q_2^{n+1}q_4^{n+2} + q_2^{n+1}q_4^{n+3}$$

$$+q_2^{n+2}q_4^{n+1}-q_2^{n+2}q_4^{n+3}-q_2^{n+3}q_4^{n+1}+q_2^{n+3}q_4^{n+2}-q_3^{n+2}q_4^{n+1}+q_3^{n+3}q_4^{n+1}+q_3^{n+1}q_4^{n+2}-q_3^{n+3}q_4^{n+2}-q_3^{n+1}q_4^{n+3}+q_3^{n+2}q_4^{n+3}=0.$$

Observe that since $D(t, z_0) = A(z_0)t^4 + B(z_0)t^3 + 1$, Vieta's formulae give

$$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0, \tag{23}$$

and

$$t_1t_2t_3 + t_1t_3t_4 + t_1t_2t_4 + t_2t_3t_4 = 0. \tag{24}$$

Divide Equation (23) by t_1^2 and (24) by t_1^3 and solve the resulting equations simultaneously to get q_3 and q_4 in terms of $q_2 = u$ as follows. Either

$$q_3 = \frac{u(u+1) + iu\sqrt{3u^2+3+2u}}{2(u^2+u+1)} \quad \text{and} \quad q_4 = \frac{u(u+1) - iu\sqrt{3u^2+3+2u}}{2(u^2+u+1)}, \tag{25}$$

or

$$q_3 = \frac{u(u+1) - iu\sqrt{3u^2+3+2u}}{2(u^2+u+1)} \quad \text{and} \quad q_4 = \frac{u(u+1) + iu\sqrt{3u^2+3+2u}}{2(u^2+u+1)}. \tag{26}$$

Observe that the product and sum of the roots q_3 and q_4 are respectively

$$q_3q_4 = \frac{u^2}{u^2+u+1} \quad \text{and} \quad q_3+q_4 = \frac{-u(u+1)}{u^2+u+1}. \tag{27}$$

From (27), we have that q_3 and q_4 are two roots of the equation

$$G(q) := q^2 + \frac{u(u+1)}{u^2+u+1}q + \frac{u^2}{u^2+u+1} = 0. \tag{28}$$

Let $u = e^{i\theta}$ be a point on the unit circle such that $-1/3 \leq \cos \theta \leq 1$. The quadratic formula (28) thus gives

$$q = \frac{1}{2} \left(-1 + \frac{e^{-i\theta}}{2\cos\theta+1} \pm \frac{ie^{-i\theta/2}\sqrt{6\cos\theta+2}}{2\cos\theta+1} \right). \tag{29}$$

Splitting the real and imaginary parts of Equation (29), using Mathematica we obtain that G maps the interval $-\frac{1}{3} \leq \cos \theta \leq 1$ to the quartic curve

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0.$$

Theorem 4 Let $\{P_n(z)\}$ be a polynomial sequence whose generating function is

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^3 + A(z)t^4},$$

where $B(z)$ and $A(z)$ are polynomials in z with complex coefficients. Given any $n \in \mathbb{N}$, those zeros z of $P_n(z)$ which satisfy $A(z)B(z) \neq 0$ lie on the real algebraic curve defined by

$$\Im \left(\frac{B^4(z)}{A^3(z)} \right) = 0 \quad \text{and} \quad 0 \leq \Re \left(\frac{B^4(z)}{A^3(z)} \right) < \infty.$$

Proof. Let z_0 be a zero of $P_n(z)$ such that $A(z_0)B(z_0) \neq 0$ and let $t_1 = t_1(z_0), t_2 = t_2(z_0), t_3 = t_3(z_0), t_4 = t_4(z_0)$ be the zeros of $D(t, z_0) = A(z_0)t^4 + B(z_0)t^3 + 1$. As above, we consider two cases.

Case 1: Suppose $D(t, z_0)$ has repeated zeros. The ordinary discriminant of $D(t, z_0)$ is

$$Disc_t(D(t, z_0)) = 4^4 A^3(z_0) - 3^3 B^4(z_0).$$

For repeated zeros, $Disc_t(D(t, z_0)) = 0$ which is equivalent to

$$\frac{B^4(z_0)}{A^3(z_0)} = \frac{4^4}{3^3} = \frac{256}{27} \in \mathbb{R}.$$

Therefore, the point z_0 lies on the curve given by

$$\Im\left(\frac{B^4(z)}{A^3(z)}\right) = 0 \text{ and } \Re\left(\frac{B^4(z)}{A^3(z)}\right) > 0.$$

Case 2: Suppose all the zeros of $D(t, z_0)$ are distinct, i.e., $Disc_t(D(t, z_0)) \neq 0$. Let t_1, t_2, t_3 and t_4 be the distinct zeros of $D(t, z)$. As above, we consider the distribution of quotients of zeros $q_i = t_i/t_1, 1 \leq i \leq 4$. Then we examine the root distribution of $P_n(z)$ using q -discriminants and show that the quotient of zeros $q_i = t_i/t_1, 1 \leq i \leq 4$ lie on the curve given in Fig. 2.

By Theorem 2, the q -discriminant of $D(t, z) = 1 + B(z)t^3 + A(z)t^4$ is given by

$$Disc_t(D(t); q) = \frac{B^2\left(A^3(q^4 - 1)^4 - B^4(q - 1)q^3(q^3 - 1)^3\right)}{(q - 1)^4}.$$

If q is a quotient of two distinct zeros of $D(t, z_0) = 1 + B(z_0)t^3 + A(z_0)t^4$ then $Disc_t(D(t); q) = 0$ which implies

$$\frac{B^2\left(A^3(q^4 - 1)^4 - B^4(q - 1)q^3(q^3 - 1)^3\right)}{(q - 1)^4} = 0.$$

Since $q \neq 0, 1$, we have

$$\frac{B^4(z_0)}{A^3(z_0)} = \frac{(q^4 - 1)^4}{(q^3 - 1)^3(q^3 - q^4)} =: f(q) \quad (30)$$

It remains to show that for q on the curve depicted in Fig. 2, we have $\Im(f(q)) = 0$. To do this, let $u = e^{i\theta}$ be a point on a unit circle with $-\frac{1}{3} \leq \cos\theta \leq 1$. Then $u \in C_4$. To each u , there are two possible values of q . Moreover, $u \neq q$ since this would imply that $u = -1 \pm i\sqrt{2}$ contradicting that u is a point on the segment of the unit circle with real part at least $-1/3$. Now, let $u \in C_4$ with the property that $q(u) \in C_5$ (the quartic curve). Then q and u are related by the equation

$$q^2 + \frac{u(u+1)}{u^2+u+1}q + \frac{u^2}{u^2+u+1} = 0. \quad (31)$$

Multiplying Equation (31) by $u - q$, we obtain

$$q^3 + q^3u + q^3u^2 = u^3 + qu^3 + q^2u^3. \quad (32)$$

From (30) and the identity (32) we show that $f(q) = f(u)$ as follows.

$$\begin{aligned} f(q) &= \frac{(q^4 - 1)^4}{(q^3 - 1)^3(q^3 - q^4)} = -\frac{(q + 1)^4(q^2 + 1)^4}{(q^2 + q + 1)^3q^3} = -\frac{(q + 1)^4(q^2 + 1)^4u^{12}}{q^{12}(u^2 + u + 1)^3u^3} \\ &= -\frac{(1 + 1/q)^4(1 + 1/q^2)^4u^{12}}{(u^2 + u + 1)^3u^3} = \frac{(u^3(1 + 1/q)(1 + 1/q^2))^4(u - 1)^4}{(u^3 - 1)^3(u^3 - u^4)} \\ &= \frac{(u^3(1 + 1/q + 1/q^2 + 1/q^3))^4(u - 1)^4}{(u^3 - 1)^3(u^3 - u^4)} = \frac{(u^3(1 + 1/u + 1/u^2 + 1/u^3))^4(u - 1)^4}{(u^3 - 1)^3(u^3 - u^4)} \\ &= \frac{(1 + u + u^2 + u^3)^4(u - 1)^4}{(u^3 - 1)^3(u^3 - u^4)} = \frac{(u^4 - 1)^4}{(u^3 - 1)^3(u^3 - u^4)} = f(u). \end{aligned}$$

Since $u \in C_4$, it implies $|u| = 1$. Lemma 2 then gives $\Im(f(u)) = \Im(f(q)) = 0$. The conclusion $\Im(f) = 0$ thus follows. Therefore, the point z_0 lies on the curve given by

$$\Im\left(\frac{B^4(z)}{A^3(z)}\right) = 0.$$

Next we prove the inequality constraint. From the recurrence relation with $\ell = 3, k = 4$ we have

$$P_n(z) + B(z)P_{n-3}(z) + A(z)P_{n-4}(z) = 0$$

whose characteristic equation is $t^4 + B(z)t + A(z) = 0$. The corresponding denominator of the generating function is $D(t, z) = A(z)t^4 + B(z)t^3 + 1$. Suppressing z , we can write $D(t) = At^4 + Bt^3 + 1$.

Let t_1, t_2, t_3, t_4 be the zeros of $D(t)$ labelled so that $0 < |t_1| \leq |t_2| \leq |t_3| \leq |t_4|$ and set $q_i := t_i/t_1$ for $i = 1, 2, 3, 4$. Note these are functions of z .

From the Equations (25) and (26), it is enough to consider only one pair of solutions (q_3, q_4) since the other pair follows similarly and give the same results.

If z_0 is a zero of $P_n(z)$ and $A(z_0)B(z_0) \neq 0$, then as above $q_1 = 1, |u| = 1, |q_3| \geq 1$ and $|q_4| \geq 1$. So, we search for an u such that the above conditions are satisfied. Set $u = e^{i\theta}$ where $\theta \in [0, 2\pi]$. It follows from $|q_3| \geq 1$ that

$$\left| \frac{u}{u^2 + u + 1} \right| \left| u + 1 + i\sqrt{3u^2 + 3 + 2u} \right| \geq 2,$$

which becomes

$$\left| 1 + e^{i\theta} + i\sqrt{3e^{2i\theta} + 3 + 2e^{i\theta}} \right| \geq |2(e^{2i\theta} + e^{i\theta} + 1)|. \tag{33}$$

Simplifying (33) gives

$$(2 \cos \theta + 1) \cos \theta \leq 0. \tag{34}$$

Solving (34) gives $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{3\pi}{2}]$.

Similarly, for $|q_4| \geq 1$, we get $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{3\pi}{2}]$. We thus conclude that both $|q_3| \geq 1$ and $|q_4| \geq 1$ only when $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{3\pi}{2}]$.

Let $\Omega = \{e^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{3\pi}{2}]\}$. From

$$f(q) = \frac{B^4(z)}{A^3(z)} = \frac{(q^4 - 1)^4}{(q^3 - 1)^3(q^3 - q^4)}.$$

If $q \in \Omega$, it follows that

$$f(q) = \frac{(1 + q + q^2 + q^3)^4}{q^3(1 + q + q^2)^3}$$

We now compute the range of f on Ω . To do this, we set $q = e^{i\theta}$ where $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{3\pi}{2}]$. After this parametrization we obtain

$$f(q) = F(\theta) = \frac{(\cos(4\theta) - 1)^2}{(\cos(3\theta) - 1)(\cos(2\theta) - \cos\theta)}.$$

It is clear that F is well-defined and continuous on the union $[\frac{\pi}{2}, \frac{2\pi}{3}] \cup (\frac{4\pi}{3}, \frac{3\pi}{2}]$. For $\theta \in \Omega$, we obtain $0 \leq (\cos(4\theta) - 1)^2 \leq \frac{9}{4}$, $-1 \leq (\cos(3\theta) - 1) \leq 0$ and $-1 \leq (\cos(2\theta) - \cos\theta) \leq 0$. Hence $F(\theta) \geq 0$. Moreover, F attains its minimum value of 0 at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Since $\lim_{\theta \rightarrow \frac{2\pi}{3}^-} F(\theta) = +\infty = \lim_{\theta \rightarrow \frac{4\pi}{3}^+} F(\theta)$, it follows from continuity and non-negativity of F that the range of f is $\mathbb{R}_{\geq 0}$. This proves the inequality condition that

$$0 \leq \Re \left(\frac{B^4(z)}{A^3(z)} \right) < \infty.$$

5. EXAMPLES

In this section we present several concrete examples using numerical experiments. In these examples, we consider the sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ generated by the rational function

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^\ell + A(z)t^k},$$

where $A(z)$ and $B(z)$ are arbitrary complex polynomials. We plot a portion of the curve given by

$$\Im \left(\frac{B^k(z)}{A^\ell(z)} \right) = 0.$$

On each graph, we plot the zeros of one of the polynomials (of our choice) in the polynomial sequence described by the given parameters $k, \ell, A(z)$ and $B(z)$.

Example 3 For $k = 3, \ell = 2, A(z) = z + 5$ and $B(z) = -z^2 + 2z + 5$, we obtain Fig. 3 and Fig. 4. In Fig. 3, we plot the zeros of $P_{30}(z)$ while in Fig. 4, we plot the zeros of $P_{70}(z)$.

Example 4 For $k = 3, \ell = 2, A(z) = z^3 - z + 6$ and $B(z) = -z^2 + 7z - 5$, we obtain Fig. 5 and Fig. 6. In Fig. 5, we plot the zeros of $P_{120}(z)$ while in Fig. 6, we plot the zeros of $P_{200}(z)$.

Example 5 For $k = 4, \ell = 3, A(z) = z^2 + 1$ and $B(z) = z^3 - 1$, we obtain Fig. 7 and Fig. 8. In Fig. 7, we plot the zeros of $P_{40}(z)$ while in Fig. 8, we plot the zeros of $P_{70}(z)$.

Example 6 For $k = 4, \ell = 3, A(z) = 7z^5 - 2z + i$ and $B(z) = -z^2 - 2z + 5$, we obtain Fig. 9 and Fig. 10. In Fig. 9, we plot the zeros of $P_{50}(z)$ while in Fig. 10, we plot the zeros of $P_{150}(z)$.

We end this paper with the following conjecture which we obtain based on numerical experiments.

Conjecture 2 In the notation of Conjecture 1 with the condition $\ell > 1$, we have the following.

(a) If k is even, then

$$0 \leq \Re \left(\frac{B^k(z)}{A^\ell(z)} \right) < \infty.$$

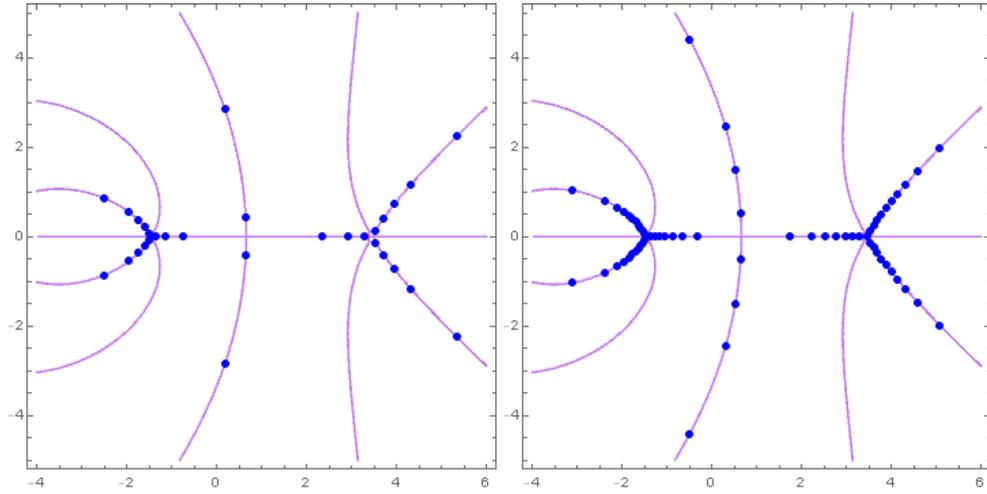


FIGURE 3. The graph of $\Im(B^3(z)/A^2(z)) = 0$ and the zeros of $P_{30}(z)$.

FIGURE 4. The graph of $\Im(B^3(z)/A^2(z)) = 0$ and the zeros of $P_{70}(z)$.

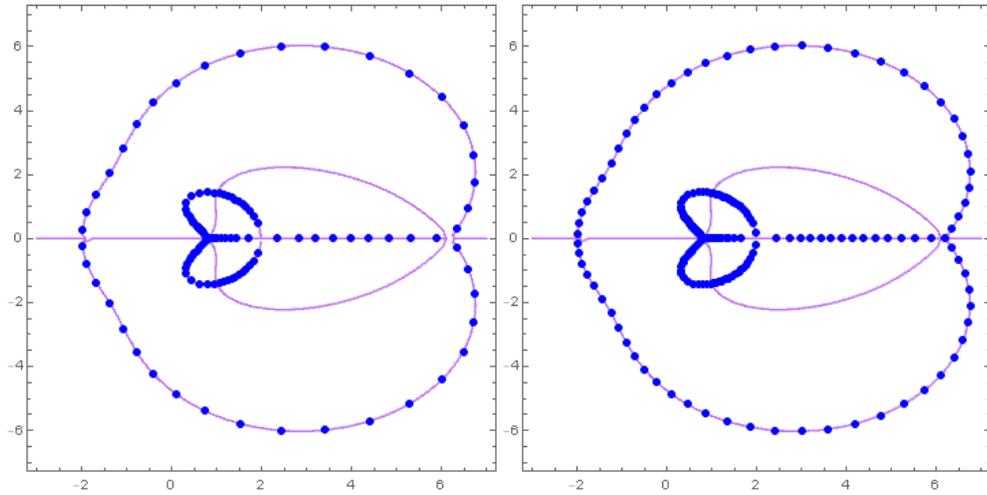


FIGURE 5. The graph of $\Im(B^3(z)/A^2(z)) = 0$ and the zeros of $P_{120}(z)$.

FIGURE 6. The graph of $\Im(B^3(z)/A^2(z)) = 0$ and the zeros of $P_{200}(z)$.

(b) If k is odd, then

$$0 \leq -\Re\left(\frac{B^k(z)}{A^\ell(z)}\right) < \infty \text{ for } \ell \text{ odd}$$

and

$$0 \leq \Re\left(\frac{B^k(z)}{A^\ell(z)}\right) < \infty \text{ for } \ell \text{ even.}$$

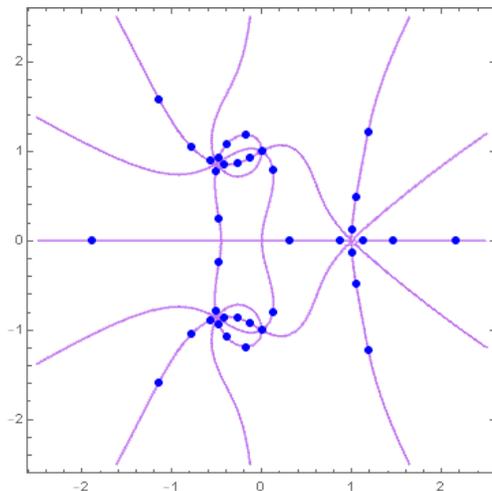


FIGURE 7. The graph of $\Im(B^4(z)/A^3(z)) = 0$ and the zeros of $P_{40}(z)$.

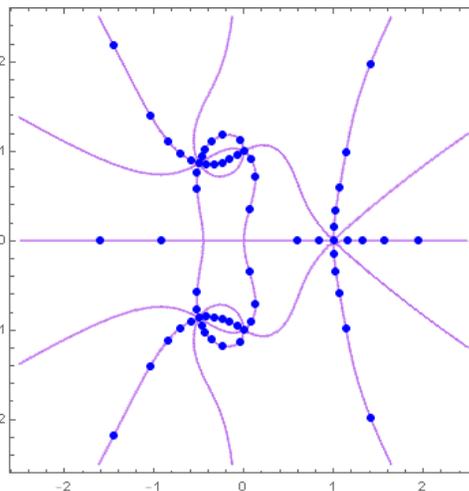


FIGURE 8. The graph of $\Im(B^4(z)/A^3(z)) = 0$ and the zeros of $P_{70}(z)$.

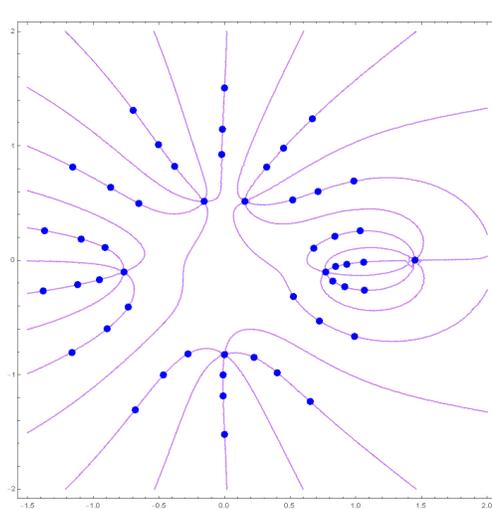


FIGURE 9. The graph of $\Im(B^4(z)/A^3(z)) = 0$ and the zeros of $P_{50}(z)$.

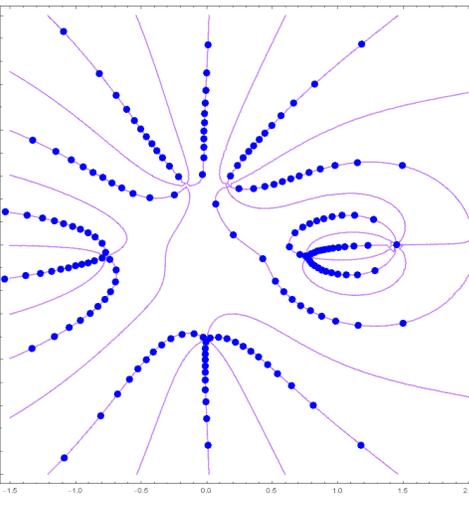


FIGURE 10. The graph of $\Im(B^4(z)/A^3(z)) = 0$ and the zeros of $P_{150}(z)$.

Remark 3

- (a) The complex number z referred to in the Conjecture 2 must satisfy the condition that $P_n(z) = 0$ and $A(z)B(z) \neq 0$.
- (b) The cases where $k = 2, 3, 4$ and $\ell = 1$ are proved by Tran in [1]. Moreover each of the inequality constraint relating to the real part of rational function is bounded. In a follow-up paper [2], Tran proves a similar result for any positive integer k and $\ell = 1$.

6. FINAL REMARKS

Problem. Give a complete proof of Conjecture 1 as formulated by B. Shapiro. This still remains unknown to the author and it should be a project for a future work. Additionally to give a complete proof of Conjecture 2 as formulated in this paper.

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