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NEW OSCILLATION RESULTS TO THIRD ORDER DAMPED DELAY DIFFERENCE EQUATIONS

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Abstract. This paper deals with the oscillation of certain class of third order nonlinear delay difference equations with damping. Some new criteria of oscillation of the third order equation in terms of oscillation of a related second order linear difference equation without damping are obtained. Examples are provided to illustrate the main results.

1. Introduction

Consider nonlinear third order delay difference equation of the form

$$\Delta(a_n\Delta(b_n(\Delta y_n)^\alpha)) + p_n(\Delta y_{n+1})^\alpha + q_n f(y_{n-k}) = 0,\ n \geq n_0, \tag{1}$$

where $n_0 \in \mathbb{N}$ is a fixed integer, and $\alpha \geq 1$ is a ratio of odd positive integers. We assume that
(H_1) $\{a_n\}$, $\{b_n\}$, $\{p_n\}$ and $\{q_n\}$ are real positive sequences for all $n \geq n_0$, and k is a positive integer;
(H_1) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(u) > 0$, and $\frac{f(u)}{u^\beta} \geq M > 0$, for all $u \neq 0$, where $\beta \leq \alpha$ is a ratio of odd positive integers.
By a solution of equation (1), we mean a nontrivial sequence $\{y_n\}$ defined for all $n_0 - k$, and satisfies equation (1) for all $n \geq n_0$. Clearly, if $y_n = c_n$ for $n = n_0 - k, n_0 - \sigma + 1, ..., n_0 - 1$ are given, then equation (1) has a unique solution satisfying the above initial conditions. A solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. A difference equation is called nonoscillatory if all its solutions are nonoscillatory.
Recently, there has been a great interest in determining the oscillation criteria for various types of second order difference equations, see for example [1] and the references cited therein. However, the study of oscillatory behavior of third order difference equations has considerably received less attention eventhough such equations have wide applications in the fields such as economics, mathematical biology, and many other areas of mathematics in which discrete models are used, see for example [5, 7, 11].

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The study of oscillation and asymptotic behavior of equation (1) in the continuous case (third-order delay differential equations with damping) and on time scales (third-order delay dynamic equations on time scales) has been investigated in the following papers [12, 13, 26].

We note that the analog equation of (1) in the continuous case is the functional differential equation

$$(a(t)(b(t)(y'(t))^{\alpha})')' + p(t)(y'(t))^{\alpha} + q(t)f(y(t-k)) = 0$$

where a, b, p and q are positive real continuous functions, k is a positive constant and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$. For related works regarding the oscillation of some special cases of equation (1), we refer to the papers [3, 6, 9], and the references cited therein.

From the review of literature, one can see that most of the oscillation results are for the third order difference equations without damping term, see for example [1, 2, 8, 10, 14, 15], and the references cited therein and very few results available for the equation with damping term [16, 18, 19]. Recently in [5], the authors considered the equation (1) with $\alpha = \beta = 1$, and established some sufficient conditions which ensure that all solutions of equation (1) are either oscillatory or tend to zero as $n \to \infty$.

The purpose of this paper to improve and generalize the results in [2, 4, 8, 10, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], and present some sufficient conditions which ensure that any solution of equation (1) oscillates when the related second order linear difference equation without delay

$$\Delta(a_n \Delta z_n) + \frac{p_n}{b_{n+1}} z_{n+1} = 0 \tag{2}$$

is nonoscillatory.

The paper is organized as follows. In Section 2, we present some preliminary lemmas which are used to prove the main results, and in Section 3, we state and prove oscillation theorems. Finally in Section 4, we provide some examples to illustrate the main results.

2. Preliminary Results

For the sake of convenience, we denote $L_0(y_n) = y_n,\ L_1(y_n) = b_n(\Delta(L_0(y_n)))^{\alpha}$, $L_2(y_n) = a_n\Delta(L_1(y_n))$ and $L_3(y_n) = \Delta(L_2(y_n))$ for all $n \geq n_0$. Hence, equation (1) can be written as

$$L_3(y_n) + \frac{p_n}{b_{n+1}} L_1(y_{n+1}) + q_n f(y_{n-k}) = 0,\ n \geq n_0.$$

If $\{y_n\}$ is a solution of equation (1), then $\{z_n\} = \{-y_n\}$ is a solution of the equation

$$L_3(z_n) + \frac{p_n}{b_{n+1}} L_1(z_{n+1}) + q_n f^*(z_{n-k}) = 0,\ n \geq n_0$$

where $f^*(z_{n-k}) = -f(-z_{n-k})$ and $uf^*(u) > 0$ for $u \neq 0$. Thus, concerning nonoscillatory solutions of equation (1) we can restrict our attention only to solutions which are positive for all large n.

Define

$$R_1(n, N) = \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}},\ R_2(n, N) = \sum_{s=N}^{n-1} \frac{1}{a_s},$$

and

$$R_3(n,N)=\sum_{s=N}^{n-1}\left(\frac{R_2(s,N)}{b_s}\right)^{\frac{1}{\alpha}},\ n\geq N\geq n_0.$$

We assume that

$$R_1(n,N)\to\infty \text{ as } n\to\infty, \tag{3}$$

and

$$R_2(n,N)\to\infty \text{ as } n\to\infty. \tag{4}$$

We begin with the following lemma given in [5].

Lemma 1 Suppose that equation (2) is nonoscillatory. If $\{y_n\}$ is a nonoscillatory solution of equation (1) for all $n\geq n_0$, then there exists an integer $N\geq n_0$ such that either $y_nL_1(y_n)>0$ or $y_nL_1(y_n)<0$ for all $n\geq N$.

Lemma 2 Let $\{y_n\}$ be a nonoscillatory solution of equation (1) with $y_nL_1(y_n)>0$ for $n\geq N\geq n_0$. Then

$$L_1(y_n)\geq R_2(n,N)L_2(y_n),\ n\geq N, \tag{5}$$

and

$$y_n\geq R_3(n,N)L_2^{1/\alpha}(y_n),\ n\geq N. \tag{6}$$

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1), say $y_n>0,\ y_{n-k}>0$, and $L_1(y_n)>0$ for all $n\geq N$. Since

$$\Delta(a_n\Delta(b_n(\Delta y_n)^\alpha))=-\frac{p_n}{b_{n+1}}L_1(y_{n+1})-q_nf(y_{n-k})\leq 0,\ n\geq N,$$

we have that $a_n\Delta(b_n(\Delta y_n)^\alpha)$ is nonincreasing for all $n\geq N$, and hence

$$\begin{aligned} L_1(y_n)=b_n(\Delta y_n)^\alpha &= b_N(\Delta y_N)^\alpha+\sum_{s=N}^{n-1}\Delta(L_1(y_s))\geq\sum_{s=N}^{n-1}\Delta(L_1(y_s))\\ &\geq a_n\Delta(b_n(\Delta y_n)^\alpha)\sum_{s=N}^{n-1}\frac{1}{a_s}=R_2(n,N)L_2(y_n). \end{aligned}$$

It follows from the last inequality that

$$\Delta y_n\geq\left(\frac{R_2(n,N)}{b_n}\right)^{1/\alpha}L_2^{1/\alpha}(y_n).$$

Now, summing this inequality from N to $n-1$, and then using the fact that $L_2(y_n)$ is nonincreasing, we obtain

$$\begin{aligned} y_n &= y_N+\sum_{s=N}^{n-1}\Delta y_s\geq\sum_{s=N}^{n-1}\Delta y_s\\ &\geq \sum_{s=N}^{n-1}\left(\frac{R_2(s,N)}{b_s}\right)^{1/\alpha}L_2^{1/\alpha}(y_s)\\ &\geq R_3(n,N)L_2^{1/\alpha}(y_n), \end{aligned}$$

for all $n\geq N$. This completes the proof.

Next, consider the second order delay difference equation

$$\Delta(a_n\Delta x_n)=Q_nx_{n-l} \tag{7}$$

where $\{a_n\}$ is same as in equation (1), $\{Q_n\}$ is a positive real sequence, and l is a positive integer.

Lemma 3 If condition (3) and

$$\lim_{n\to\infty}\sup\sum_{s=n-l}^{n-1} Q_s R_2(n-l, s-l) > 1, \tag{8}$$

are satisfied, then every bounded solution of equation (7) is oscillatory.

Proof. Let $\{x_n\}$ be a bounded nonoscillatory solution of equation (7), say $x_n > 0$, and $x_{n-l} > 0$ for all $n \geq N \geq n_0$. By (7), we have that $\{a_n \Delta x_n\}$ is strictly increasing for all $n \geq N$. Hence for any $N_1 \geq N$, we obtain

$$\begin{aligned} x_n &= x_{N_1} + \sum_{s=N_1}^{n-1} \Delta x_s = x_{N_1} + \sum_{s=N_1}^{n-1} \frac{a_s \Delta x_s}{a_s} \\ &> x_{N_1} + a_{N_1}\Delta x_{N_1} \sum_{s=N_1}^{n-1} \frac{1}{a_s} \\ &= x_{N_1} + a_{N_1}\Delta x_{N_1} R_2(n, N_1). \end{aligned}$$

So $\Delta x_{N_1} < 0$, as otherwise (3) would imply that $x_n \to \infty$ as $n \to \infty$, a contradiction to the boundedness of $\{x_n\}$. Therefore

$$x_n > 0,\ \Delta x_n < 0,\ \text{and}\ \Delta(a_n \Delta x_n) > 0,\ n \geq N. \tag{9}$$

Now for $j \geq s \geq N$, we have

$$\begin{aligned} x_s &> x_s - x_j = -\sum_{t=s}^{j-1} \Delta x_t = -\sum_{t=s}^{j-1} \frac{a_t \Delta x_t}{a_t} \\ &\geq -a_j \Delta x_j \sum_{t=s}^{j-1} \frac{1}{a_t} = -R_2(j, s) a_j \Delta x_j. \end{aligned} \tag{10}$$

For $n \geq t \geq N_1$, setting $s = i - l$ and $j = n - l$ in (10), we obtain

$$x_{i-l} > -R_2(n-l, i-l) a_{n-l} \Delta x_{n-l}.$$

Summing the equation (7) from $n - l$ to $n - 1$, and then using the last inequality we obtain

$$\begin{aligned} -a_{n-l}\Delta x_{n-l} &> a_n \Delta x_n - a_{n-l}\Delta x_{n-l} \\ &= \sum_{s=n-l}^{n-1} Q_s x_{s-l} \\ &> -\left[\sum_{s=n-l}^{n-1} Q_s R_2(n-l, s-l)\right] a_{n-l}\Delta x_{n-l}. \end{aligned}$$

That is,

$$1 > \sum_{s=n-l}^{n-1} Q_s R_2(n-l, s-l). \tag{11}$$

Taking limit supremum as $n \to \infty$ on both sides of (11) yields a contradiction to (8). This completes the proof.

Lemma 4 If condition (3) and

$$\lim_{n\to\infty}\sup\sum_{s=n-l}^{n-1}\left(\frac{1}{a_s}\sum_{t=s}^{n-1}Q_t\right)>1, \tag{12}$$

are satisfied, then every bounded solution of equation (7) is oscillatory.

Proof. Let $\{x_n\}$ be a bounded nonoscillatory solution of equation (7), say $x_n>0$, and $x_{n-l}>0$ for all $n\ge N\ge n_0$. As in Lemma 3, we obtain (9). Summing the equation (7) from s to $n-1$, we obtain

$$\begin{aligned}-a_s\Delta x_s &> a_n\Delta x_n-a_s\Delta x_s=\sum_{t=s}^{n-1}Q_tx_{t-l}\\ &\ge \left[\sum_{t=s}^{n-1}Q_t\right]x_{n-l}.\end{aligned}$$

That is,

$$-\Delta x_s>\left(\frac{1}{a_s}\sum_{t=s}^{n-1}Q_t\right)x_{n-l}.$$

Summing the last inequality from $n-l$ to $n-1$, we have

$$x_{n-l} > x_{n-l}-x_n>\left[\sum_{s=n-l}^{n-1}\left(\frac{1}{a_s}\sum_{t=s}^{n-1}Q_t\right)\right]x_{n-l},$$

or

$$1>\sum_{s=n-l}^{n-1}\left(\frac{1}{a_s}\sum_{t=s}^{n-1}Q_t\right). \tag{13}$$

Taking limit supremum as $n\to\infty$ on both sides of (13) yields a contradiction with (12), and the proof is completed.

3. Oscillation Results

Now, we begin to present the main results.

Theorem 1 Let conditions (3) and (4) hold. Suppose that equation (2) is nonoscillatory. If there exists a positive real sequence $\{\rho_n\}$, and a positive integer l such that $l\le k$, and

$$\lim_{n\to\infty}\sup\sum_{s=N_1}^{n-1}\left[M\rho_sq_s-\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{A_s^{\alpha+1}}{B_s^\alpha}\right]=\infty \tag{14}$$

where

$$\begin{aligned}A_n &= \frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\rho_n}{\rho_{n+1}}\frac{p_n}{b_{n+1}}R_2(n+1,N),\\ B_n &= c^*\beta(R_3(n+1-k,N))^{\frac{\beta}{\alpha}-1}\frac{R_2^{1/\alpha}(n-k,N)}{b_{n-k+1}^{1/\alpha}},\end{aligned}$$

and (8) or (12) holds with

$$Q_n=\left[cMq_nR_1^\beta(n-l,n-k)-\frac{p_n}{b_{n+1}}\right]\ge 0 \text{ for all } n\ge N_1\ge N,$$

and $c, c^* > 0$, then every solution $\{y_n\}$ or $\{L_2(y_n)\}$ of equation (1) is oscillatory.
Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $y_n > 0$, and $y_{n-k} > 0$ for all $n \geq N \geq n_0$. Then, it follows from Lemma 1 that $L_1(y_n) > 0$ or $L_1(y_n) < 0$ for all $n \geq N$. First we assume that $L_1(y_n) > 0$ for all $n \geq N$. From equation (1), we see that $L_2(y_n)$ is decreasing for all $n \geq N$. Hence for any integer $N_1 \geq N$, we have

$$\begin{aligned} L_1(y_n) &= L_1(y_{N_1}) + \sum_{s=N_1}^{n-1} \Delta(L_1(y_s)) = L_1(y_{N_1}) + \sum_{s=N_1}^{n-1} \frac{L_2(y_s)}{a_s} \\ &\leq L_1(y_{N_1}) + L_2(y_{N_1}) R_2(n, N_1), \end{aligned}$$

so $L_2(y_{N_1}) > 0$, as otherwise (4) would imply that $L_1(y_n) \to -\infty$ as $n \to \infty$, a contradiction to the positivity of $L_1(y_n)$. Therefore $L_2(y_n) > 0$ for all $n \geq N_1$. Define

$$w_n = \rho_n \frac{a_n \Delta(b_n(\Delta y_n)^\alpha)}{y_{n-k}^\beta}, \ n \geq N_1. \tag{15}$$

Then $w_n > 0$ for all $n \geq N_1$, and

$$\begin{aligned} \Delta w_n &= \rho_n \frac{\Delta(L_2(y_n))}{y_{n-k}^\beta} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} \frac{w_{n+1} \Delta y_{n-k}^\beta}{y_{n-k}^\beta} \\ &\leq -M \rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} \frac{\Delta y_{n-k}^\beta}{y_{n-k}^\beta} w_{n+1} \\ &\quad - \frac{\rho_n}{\rho_{n+1}} \frac{p_n}{b_{n+1}} R_2(n+1, N) w_{n+1}. \end{aligned} \tag{16}$$

Using Mean value theorem, we have

$$\frac{\Delta y_{n-k}^\beta}{y_{n-k}^\beta} \geq \beta \frac{\Delta y_{n-k}}{y_{n+1-k}}, \ \beta > 0. \tag{17}$$

In view of (17), (5), and the fact that $L_2(y_n)$ is decreasing, and y_n is increasing, we have from (16) that

$$\begin{aligned} \Delta w_n &\leq -M \rho_n q_n + \left(\frac{\Delta \rho_n}{\rho_{n+1}} - \frac{\rho_n}{\rho_{n+1}} \frac{p_n}{b_{n+1}} R_2(n+1, N) \right) w_{n+1} \\ &\quad - \beta \frac{\rho_n}{\rho_{n+1}^{1+1/\alpha}} \frac{R_2^{1/\alpha}(n-k, N)}{b_{n-k+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha} y_{n+1-k}^{\beta/\alpha - 1}. \end{aligned} \tag{18}$$

It follows from $L_3(y_n) < 0$ that $0 < L_2(y_n) \leq c_1 < \infty$ for all $n \geq N_1 \geq N$. Hence for $n \geq N_1$, we have

$$\begin{aligned} b_n(\Delta y_n)^\alpha &= L_1(y_n) = L_1(y_N) + \sum_{s=N}^{n-1} \Delta(L_1(y_s)) \\ &\leq L_1(y_N) + c_1 R_2(n, N) \\ &= \left[\frac{L_1(y_N)}{R_2(n, N)} + c_1 \right] R_2(n, N) \leq c_2 R_2(n, N) \end{aligned}$$

holds where $c_2 = c_1 + \frac{L_1(y_N)}{R_2(N_1,N)}$. Therefore, we have for all $n \geq N_2 \geq N_1$, that

$$\begin{aligned} y_n &= y_{N_1} + \sum_{s=N_1}^{n-1} \Delta y_s \leq y_{N_1} + \sum_{s=N_1}^{n-1} \left(\frac{c_2 R_2(s,N)}{b_s}\right)^{1/\alpha} \\ &\leq y_{N_1} + c_2^{1/\alpha} R_3(n,N_1) \\ &= \left[\frac{y_{N_1}}{R_3(n,N_1)} + c_2^{1/\alpha}\right] R_3(n,N_1) \leq c_3 R_3(n,N_1) \end{aligned}$$

holds where $c_3 = c_2^{1/\alpha} + \frac{y_{N_1}}{R_3(N_2,N_1)}$. Thus, we have

$$y_{n+1-k}^{\beta/\alpha-1} \geq c_3^{\beta/\alpha-1}(R_3(n+1-k,N_1))^{\beta/\alpha-1} \text{ for all } n \geq N_2.$$

Thus from (18), we obtain

$$\Delta w_n \leq -M\rho_n q_n + A_n w_{n+1} - B_n w_{n+1}^{1+1/\alpha},\ n \geq N_2. \tag{19}$$

Using the inequality $Cu - Du^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{C^{\alpha+1}}{D^\alpha}$, for $D > 0$ in (19) with $C = A_n$ and $D = B_n$, we obtain

$$\Delta w_n \leq -M\rho_n q_n + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A_n^{\alpha+1}}{B_n^\alpha},\ n \geq N_2.$$

Summing the last inequality from N_2 to $n-1$, we have

$$\sum_{s=N_2}^{n-1} \left(M\rho_s q_s - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A_s^{\alpha+1}}{B_s^\alpha} \right) \leq w_{N_2}. \tag{20}$$

Taking limit supremum as $n \to \infty$ in (20), we obtain a contradiction to (14).
Next consider the function $L_2(y_n)$. The case $L_2(y_n) \leq 0$ cannot hold for all large n, say $n \geq N_2 \geq N_1$, since by the summation of

$$\Delta y_n = \left(\frac{L_1(y_n)}{b_n}\right)^{1/\alpha} \leq \left(\frac{L_1(y_{N_2})}{b_n}\right)^{1/\alpha},\ n \geq N_2,$$

we have from (3) that $y_n < 0$ for large n, which is a contradiction. Thus, assume $y_n > 0$, $L_1(y_n) < 0$, and $L_2(y_n) > 0$ for all large n, say $n \geq N_3 \geq N_2$. Now for $j \geq s \geq N_3$, we obtain

$$\begin{aligned} y_s - y_j &= -\sum_{t=s}^{j-1} \frac{(b_t(\Delta y_t)^\alpha)^{1/\alpha}}{b_t^{1/\alpha}} \geq (-L_1(y_j))^{1/\alpha} \left(\sum_{t=s}^{j-1} \frac{1}{b_t^{1/\alpha}}\right) \\ &= R_1(j,s)(-L_1(y_j))^{1/\alpha}. \end{aligned}$$

Setting $s = n-k$ and $j = n-l$, we obtain

$$y_{n-k} \geq R_1(n-l,n-k)(-L_1(y_{n-l}))^{1/\alpha} = R_1(n-l,n-k)x_{n-l}$$

for all $n \geq N_3$, where $x_n = (-L_1(y_n)) > 0$ for $n \geq N_3$. From equation (1), and the fact that $\{x_n\}$ is decreasing, and $n-l \leq n-k < n$, we obtain

$$\Delta(a_n \Delta z_n) + \left(\frac{p_n}{b_{n+1}}\right) z_{n+1-l} \geq M q_n (R_1(n-l,n-k))^\beta z_{n-l} z_{n-l}^{\beta/\alpha-1}$$

where $z_n = x_n^\alpha$. Since $\{z_n\}$ is decreasing and $\alpha \geq \beta$, there exists a constant $c_4 > 0$ such that $z_n^{\beta/\alpha-1} \geq c_4$ for $n \geq N_3$. Thus

$$\Delta(a_n \Delta z_n) \geq Q_n z_{n-l},\ n \geq N_3.$$

Proceeding exactly as in the proof of Lemmas 3 and 4, we arrive at the desired conclusion. This completes the proof.
From the above theorem, we obtain the following corollary.
Corollary 2 Let conditions (3) and (4) hold. Suppose that equation (2) is nonoscillatory, and $A_n \leq 0$, for all $n \geq n_0$, where A_n is defined as in Theorem 1. If there exists a positive function $\{\rho_n\}$, and a positive integer l such that $l \leq k$, and

$$\sum_{n=N}^{\infty} \rho_n q_n = \infty \tag{21}$$

for any $N \geq n_0$, and (8) or (12) holds with Q_n as in Theorem 1, then every solution $\{y_n\}$ or $\{L_2(y_n)\}$ of equation (1) is oscillatory.
For $n \geq N \geq n_0$, we set

$$P_n = \frac{p_n}{b_{n+1}} R_2(n+1, N),\ R_n = M q_n R_3^{\beta}(n-l, N),$$

and

$$E_n = \prod_{s=N}^{n-1} (1 + P_s).$$

In the following, we present comparison results for the oscillation of equation (1).
Theorem 3 Let conditions (3) and (4) hold. Suppose that equation (2) is nonoscillatory. Further assume that there exists a positive integer l such that $l \leq k$ for $n \geq n_0$, and (8) or (12) holds with Q_n as in Theorem 1. If every solution of the first order delay difference equation

$$\Delta u_n + E_{n-k}^{1-\beta/\alpha} R_n u_{n-k}^{\beta/\alpha} = 0 \tag{22}$$

is oscillatory, then every solution $\{y_n\}$ or $\{L_2(y_n)\}$ of equation (1) is oscillatory.
Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $y_n > 0$, and $y_{n-k} > 0$ for all $n \geq N \geq n_0$. Further, it follows from Lemma 1 that $L_1(y_n) > 0$ or $L_1(y_n) < 0$ for all $n \geq N$. First assume $L_1(y_n) > 0$. Choose an integer $N_1 \geq N$ such that $n-k \geq N$ for all $n \geq N_1 \geq N+k$. Using (5) and (6) in equation (1), we obtain

$$\Delta(L_2(y_n)) + \frac{p_n}{b_{n+1}} R_2(n+1, N) L_2(y_{n+1}) + M q_n R_3^{\beta}(n-k, N) L_2^{\beta/\alpha}(y_{n-k}) \leq 0$$

for all $n \geq N_1$, which can be rewritten as

$$\Delta w_n + P_n w_{n+1} + R_n w_{n-k}^{\beta/\alpha} \leq 0,\ n \geq N_1$$

where $w_n = L_2(y_n)$, that is,

$$\Delta(E_n w_n) + E_n R_n w_{n-k}^{\beta/\alpha} \leq 0,\ n \geq N_1.$$

Setting $u_n = E_n w_n > 0$ in the last inequality, and noting that $E_{n-k} \leq E_n$, we obtain

$$\Delta u_n + E_{n-k}^{1-\beta/\alpha} R_n u_{n-k}^{\beta/\alpha} \leq 0.$$

This difference inequality has a positive solution, and by Lemma 2.7 of [25], the corresponding difference equation (22) has a positive solution, which is a contradiction. The case $L_2(y_n) < 0$ for $n \geq N$ is similar to that of Theorem 1, and hence is omitted. This completes the proof.
From the above theorem, we obtain the following corollary.

Corollary 4 Let conditions (3) and (4) hold. Suppose that equation (2) is nonoscillatory. Further, assume that there exists a positive integer l such that $l \leq k$ for all $n \geq n_0$, and (8) or (12) holds with Q_n as in Theorem 1. If

$$\lim_{n\to\infty} \inf \sum_{s=n-k}^{n-1} R_s > \left(\frac{k}{k+1}\right)^{k+1} \text{ when } \alpha = \beta,$$

or

$$\sum_{s=N}^{\infty} E_{n-k}^{1-\beta/\alpha} R_s = \infty \text{ when } \alpha > \beta,$$

holds, then every solution $\{y_n\}$ or $\{L_2(y_n)\}$ of equation (1) is oscillatory.

Proof. The proof follows from Theorem 3 and Theorem 7.5.1 of [11] for $\alpha = \beta$ or Theorem 1 of [22] for the case $\alpha > \beta$.

For $\alpha = 1$, we derive the following new oscillation criteria for equation (1).

Theorem 5 Let conditions (3), (4) and $\alpha = 1$ hold. Suppose that equation (2) is nonoscillatory. If there exists a positive real sequence $\{\rho_n\}$ and a positive integer l such that $k \geq l$, and

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n-1} \left[\prod_{t=N}^{s-1} (1 + \rho_t - A_t)\right] (M\rho_s q_s - \frac{\rho_s^2}{B_s}) = \infty \tag{23}$$

for every $N \geq n_0$. If (8) or (12) holds with Q_n as in Theorem 1, then every solution $\{y_n\}$ or $\{L_2(y_n)\}$ of equation (1) is oscillatory.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $y_n > 0$, and $y_{n-k} > 0$ for all $n \geq N \geq n_0$. Proceeding as in the proof of Theorem 1, we obtain (19), that is,

$$\Delta w_n \leq -M\rho_n q_n + A_n w_{n+1} - B_n w_{n+1}^2,$$

and so

$$\Delta w_n \leq -M\rho_n q_n + (A_n - \rho_n) w_{n+1} + \rho_n w_{n+1} - B_n w_{n+1}^2,$$

or

$$\Delta w_n + (\rho_n - A_n) w_{n+1} + (M\rho_n q_n - \frac{\rho_n^2}{B_n}) \leq 0, n \geq N.$$

It follows that

$$\sum_{s=N}^{n} \left[\prod_{t=N}^{s-1} (1 + \rho_t - A_t)\right] (M\rho_s q_s - \frac{\rho_s^2}{B_s}) \leq w_N.$$

Hence

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \left[\prod_{t=N}^{s-1} (1 + \rho_t - A_t)\right] (M\rho_s q_s - \frac{\rho_s^2}{B_s}) \leq w_N,$$

which contradicts (23). The rest of the proof is similar to that of Theorem 1, and hence is omitted. This completes the proof.

4. Examples

In this section, we provide some examples to illustrate the main results.

Example 1 Consider the delay difference equation

$$\Delta^2((\Delta y_n)^3) + \frac{1}{5n^2}(\Delta y_{n+1})^3 + (8 - \frac{2}{5n^2})y_{n-2} = 0, \ n \geq 1. \tag{24}$$

Here $a_n = b_n = 1,\ p_n = \frac{1}{5n^2},\ q_n = 8 - \frac{2}{5n^2},\ \alpha = 3,\ \beta = 1$ and $k = 2$. Note that the corresponding second order difference equation $\Delta^2 z_n + \frac{1}{5n^2} z_{n+1} = 0$ is nonoscillatory by [1, Theorem 1.14.2]. Taking $l = 1$ and $\rho_n = 1$, we have $A_n = \frac{-1}{5n} < 0$ for all $n \geq 1$, and

$$\sum_{n=N}^{\infty} \rho_n q_n = \sum_{n=1}^{\infty}(8 - \frac{2}{5n^2}) = \infty.$$

Further

$$Q_n = c(8 - \frac{2}{5n^2}) - \frac{1}{5n^2} > 0 \text{ for } n \geq 1,$$

and

$$\begin{aligned}
\lim_{n\to\infty}\sup \sum_{s=n-l}^{n-1} Q_s R_2(n-l, s-1) &= \lim_{n\to\infty}\sup \sum_{s=n-1}^{n-1} [c(8 - \frac{2}{5s^2}) - \frac{1}{5s^2}](n-s+1) \\
&= \lim_{n\to\infty}\sup [c(8 - \frac{2}{5(n-1)^2}) - \frac{1}{5(n-1)^2}]2 \\
&= 16c > 1
\end{aligned}$$

for $c > \frac{1}{16} > 0$. Thus, all conditions of Corollary 2 are satisfied, and hence every solution of equation (24) is oscillatory. In fact $\{y_n\} = \{\frac{(-1)^n}{2}\}$ is one such oscillatory solution of equation (24).

Example 2 Consider the third order delay difference equation

$$\Delta^2((\Delta y_n)^3) + \frac{9}{2^{n+1}}(\Delta y_{n+1})^3 + \left(32 - \frac{36}{2^n}\right) y_{n-2}^3 = 0, \ n \geq 1. \tag{25}$$

Here $a_n = b_n = 1,\ p_n = \frac{9}{2^{n+1}},\ q_n = 32 - \frac{36}{2^n},\ \alpha = \beta = 3$, and $k = 2$. Note that the corresponding second order difference equation $\Delta^2 z_n + \frac{9}{2^{n+1}} z_{n+1} = 0$ is nonoscillatory by [1, Theorem 1.14.2]. The other conditions of Corollary 4 are satisfied, and hence every solution of equation (25) is oscillatory. In fact $\{y_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (25).

Example 3 Consider the third order delay difference equation

$$\Delta^3 y_n + \frac{1}{6n^2}\Delta y_{n+1} + (8 - \frac{1}{3n^2})y_{n-2}^{1/3} = 0, \ n \geq 1. \tag{26}$$

Here $a_n = b_n = 1,\ p_n = \frac{1}{6n^2},\ q_n = 8 - \frac{1}{3n^2},\ \alpha = 1,\ \beta = \frac{1}{3}$, and $k = 2$. Note that the corresponding second order difference equation $\Delta^2 z_n + \frac{1}{6n^2} z_{n+1} = 0$ is nonoscillatory by [1, Theorem 1.14.2]. With $\rho_n = 1$ it is easy to see that all conditions of Theorem 5 are satisfied, and hence every solution of equation (26) is oscillatory. One oscillatory solution of equation (26) is $\{y_n\} = \{(-1)^n\}$.

we conclude this paper with the following remark.

Remark 1 The results presented in this paper are new and high degree of generality. We note that the results in [1, 2, 4, 18, 19] are applicable only when $\alpha = \beta = 1$, and therefore the results obtained in this paper are complement and generalize to that of in [2, 4, 8, 10, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. It would be interesting to consider equation (1) and try to obtain some oscillation criteria if $p_n < 0$ and $q_n < 0$ for all $n \geq n_0$. This has been left to future research.

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