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FIXED POINT RESULTS VIA A CONTROL FUNCTION IN SUPER METRIC SPACE

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Abstract. In the present paper, we generalize the results of Karapinar and Khojasteh [7], Karapinar and Fulga [6] in super metric space by using the control function and weakly compatible mappings.

1. Introduction and Preliminaries

Fixed points are the points which remain invariant under a map or transformation. Fixed points give us the idea of points that are not moved by the transformation. Geometrically, the fixed points of a curve are the point of intersection of the curve with the line $y = x$. A map can have one fixed point, two fixed points, infinitely many fixed points and no fixed point. The mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x$, for all $x \in \mathbb{R}$ has a unique fixed point $x = 0$. The mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, for all $x \in \mathbb{R}$ has a two fixed points $x = 0$ and $x = 1$. The identity mapping has infinitely many fixed point where as the translation mapping has no fixed point.

Metric fixed point theory involves the study of fixed points depending on the mapping conditions on the spaces under consideration. There is a revolution in metric fixed point theory with the escalation of the Banach contraction principle which states that "every contraction mapping on a complete metric space has a unique fixed point."

Fixed point theory has grown into a full branch of mathematics within the span of more than hundred years. It has very fruitful applications in control theory, game theory, and many other areas. In particular, fixed point techniques have been applied in various diverse fields such as biology, chemistry, physics, economics and engineering. The fixed point theorems are mainly used in existence theory of random differential equations, numerical methods;

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2. Picard's existence theorem,
3. Existence of solution of integral equations and a system of linear equations.

The notion of distance function was proposed by Fréchet [4] which is, in present, known as Euclidean metric or metric.

Definition 1.1. Let X be a nonempty set and $d : X \times X \to [0, +\infty)$ be a mapping which satisfies

(d_1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
(d_2) $d(x, y) = d(y, x)$ for all $x, y, \in X$,
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. (triangular inequality)

Then, the pair (X, d) is called a Euclidean metric space or a metric space.

The concept of metric has been generalized and extended by various authors. Some of the most interesting generalizations of metric space are: partial metric space [9], semi-metric space [11], b-metric space [3], G-metric space [10], Fuzzy metric space [12]. Recently in 2022, Karapinar and Khojasteh [7] have introduced a new extension of metric space and named it as super metric space.

Definition 1.2 ([7])**.** Let X be a nonempty set and $m : X \times X \to [0, +\infty)$ be a mapping satisfying

(m_1) if $m(x, y) = 0$, then $x = y$ for all $x, y \in X$,
(m_2) $m(x, y) = m(x, y)$ for all $x, y \in X$,
(m_3) there exists $s \geq 1$ such that for all $y \in X$, there exist distinct sequences $\{x_n\}, \{y_n\} \subset X$, with $m(x_n, y_n) \to 0$ when n tends to infinity, such that

$$\lim_{n\to\infty} \sup m(y_n, y) \leq s \lim_{n\to\infty} \sup m(x_n, y).$$

Then, the pair (X, m) is called a super metric space.

Definition 1.3 ([7])**.** Let (X, m) be a super metric space and let $\{x_n\}$ be a sequence in X. We say

(i) $\{x_n\}$ converges to x in X if and only if $m(x_n, x) \to 0$, as $n \to \infty$.
(ii) $\{x_n\}$ is a Cauchy sequence in X if and only if $\lim_{n\to\infty} \sup\{m(x_n, x_m) : m > n\} = 0$.
(iii) (X, m) is a complete super metric space if and only if every Cauchy sequence is convergent in X.

Proposition 1.4 ([6])**.** On a super metric space, the limit of a convergent sequence is unique.

Definition 1.5 ([1])**.** Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 1.6 ([5])**.** A pair (f, g) of self mappings of metric space (X, d) is said to be weakly compatible if the mappings commute at all of their coincidence points, that is, $fx = gx$ for some $x \in X$ implies $fgx = gfx$.

Proposition 1.7 ([1])**.** Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

In 1977, Mathkowski [8] introduced the Φ-map as the following: Let Φ be the set of all functions φ such that $\varphi : [0, +\infty) \to [0, +\infty)$ is a non decreasing function satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\varphi \in \Phi$, then φ is called a Φ-map. Furthermore, if φ is a Φ-map, then

(i) $\varphi(t) < t$ for all $t \in (0, \infty)$,
(ii) $\varphi(0) = 0$.

From now on, unless otherwise stated, φ is meant the Φ-map.

In the present paper, we use the control function φ and generalize the results of Karapinar and Khojasteh [7], Karapinar and Fulga [6].

2. Main Results

Theorem 2.1. *Let (X, m) be a complete super metric space and the mappings $f, g : X \to X$ satisfy*

$$m(fx, fy) \le \varphi[m(gx, gy)], \tag{2.1}$$

for all $x, y \in X$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point of X. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. In this way, we can construct two distinct sequences $\{fx_n\}$ and $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, we have $gx_n = gx_{n+1}$, then f and g have a point of coincidence. On the contrary, let $gx_n \ne gx_{n+1}$ for all $n \in \mathbb{N}$.

Thus, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n) \\ &\le \varphi[m(gx_{n-1}, gx_n)] \\ &\le \varphi^2[m(gx_{n-2}, gx_{n-1})] \\ &\vdots \\ &\le \varphi^n[m(gx_0, gx_1)]. \end{aligned} \tag{2.2}$$

Our aim is to prove that $\{gx_n\}$ is Cauchy sequence. Let $\epsilon > 0$.

Since $\lim_{n\to\infty} \varphi^n m(gx_0, gx_1) = 0$, there exists $N \in \mathbb{N}$ such that

$$\varphi^n[m(gx_0, gx_1)] < \epsilon \quad \text{for all } n \ge N.$$

Therefore, using (2.2) for all $n \ge$ N

$$m(gx_n, gx_{n+1}) < \epsilon. \tag{2.3}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$m(gx_n, gx_m) < \epsilon \quad \text{for all } m \ge n \ge N. \tag{2.4}$$

Now from (2.4), we get that the result is true for $m = n + 1$. If $x_n = x_m$, (2.4) is trivially true.

Without loss of generality, we can take $x_n \ne x_m$. Suppose (2.4) is true for $m = k$ i.e.

$$\lim_{n\to\infty} \sup m(gx_n, gx_k) = 0.$$

Therefore, by using (2.1) for $m = k+1$ we have

$$\begin{aligned} m(gx_n, gx_{k+1}) &= m(fx_{n-1}, fx_k) \\ &\leq \varphi[m(gx_{n-1}, gx_k)]. \end{aligned}$$

Taking $n \to \infty$,

$$\lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) \leq \varphi[\lim_{n\to\infty} \sup m(gx_{n-1}, gx_k)].$$

Using (m_3), we get

$$\begin{aligned} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &\leq s\,\varphi[\lim_{n\to\infty} \sup m(fx_{n-1}, gx_k)] \\ &= s\,\varphi[\lim_{n\to\infty} \sup m(gx_n, gx_k)]. \end{aligned}$$

Hence, by induction $\lim\limits_{n\to\infty} \sup m(gx_n, gx_{k+1}) = 0$, since $\varphi(t) < t$ and $s \geq 1$ is finite.

This shows that $\{gx_n\}$ is a Cauchy sequence. By completeness of $g(X)$, we get that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$, such that $gp = q = \lim\limits_{n\to\infty} gx_n$. We will show that $gp = fp$.

We have, by using (2.1) and (m_3),

$$\begin{aligned} m(gp, fp) &= m(q, fp) \\ &= \lim_{n\to\infty} m(gx_n, fp) \\ &= \lim_{n\to\infty} m(fx_{n-1}, fp) \\ &\leq \varphi[\lim_{n\to\infty} \sup m(gx_{n-1}, gp)] \\ &\leq s\,\varphi[\lim_{n\to\infty} \sup m(gx_n, gp)] \\ &= 0. \end{aligned} \tag{2.5}$$

Therefore $gp = fp$. We will now show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. By applying (2.1), it follows that

$$m(gp, gq) = m(fp, fq) \leq \varphi[m(gp, gq)] < m(gp, gq),$$

which is a contradiction. Therefore, we have $m(gp, gq) = 0$, which gives $gp = gq$.

This implies that f and g have a unique point of coincidence. By Proposition 1.7, we conclude that f and g have a unique common fixed point. □

Corollary 2.2 ([7, Theorem 2.6])**.** Let (X, m) be a complete super metric space and let $T : X \to X$ be a mapping. Suppose that $0 < k < 1$ such that

$$m(Tx, Ty) \leq km(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = kt$. Therefore, φ is a non decreasing function and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t \in (0, +\infty)$. It follows that the contractive conditions of Theorem 2.1 are now satisfied. This completes the proof. □

Remark 2.3. In order to apply Corollary 2.2, an example [7, Example 2.7] is proposed, where $X = [2, 3]$ and $T : X \to X$ is defined as

$$Tx = \begin{cases} 2, & x \neq 3, \\ \frac{3}{2}, & x = 3. \end{cases}$$

But the mapping T is not a valid mapping on $X = [2, 3]$. Thus, the main motto of the example is forfeited.

Remark 2.4. In [7, Example 2.7], there seems to be no typographical error in writing the set $X = [2, 3]$, since [7, Theorem 2.6] is verified for $2 \leq x < 3$.

Example 2.5. *Let* $X = [1, 3]$ *and define*

$$m(x, y) = \begin{cases} xy, & x \neq y, \\ 0, & x = y. \end{cases}$$

It has been shown in [7] *that* (X, m) *is a super metric space. Further let* $\varphi(t) = \frac{t}{2}$, *which is clearly a* Φ*-map.*

Now consider $f, g : X \to X$ *as follows*

$$fx = \begin{cases} 2, & x \neq 3, \\ \frac{3}{2}, & x = 3 \end{cases} \quad \text{and} \quad gx = 4 - x.$$

Here $g(X) = [1, 3]$, $f(X) \subset g(X)$ *and* $g(X)$ *is complete space.*

We obtain that f *and* g *satisfy the contractive conditions of Theorem 2.1. Indeed for* $x \neq 3$, $y = 3$ *and* $s = 6$, *we obtain*

$$m(fx, fy) = m\left(2, \frac{3}{2}\right) = 2 \times \frac{3}{2} = 3,$$

and $\varphi[m(gx, gy)] = \frac{1}{2}m(gx, 1) = \frac{1}{2}gx$, *where* $gx \in (1, 3]$.

The other cases are straightforward. Now for $x = 2$, $fx = gx$ *and* $fgx = gfx$. *Thus* 2 *is the unique point of coincidence of* f *and* g. *Therefore,* 2 *is the unique common fixed point by Theorem 2.1. But note that if we consider the metric* $d(x, y) = |x - y|$, *then for all* $\varphi(t) = \frac{t}{2}$, *if* $x_n = 3 - \frac{1}{n}$ *and* $y = 3$, *we have*

$$|fx - fy| = \left|2 - \frac{3}{2}\right| = \frac{1}{2} > \varphi\left|4 - \left(3 - \frac{1}{n}\right) - 1\right| = \frac{\varphi}{n},$$

for all $n \geq 1$. *Thus,* f *is not a Banach contraction with respect to* g *in* (X, d).

Theorem 2.6. *Let* (X, m) *be a complete super metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$m(fx, fy) \leq \varphi\left[\max\left\{m(gx, gy), \frac{m(gx, fx)m(gy, fy)}{m(gx, gy) + 1}\right\}\right], \tag{2.6}$$

for all $x, y \in X$. *If* $f(X) \subset g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. Inductively, we can construct two distinct sequences $\{fx_n\}$ and $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus, we can suppose that

$gx_n \neq gx_{n+1}$, for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n) \\ &\leq \varphi\left[\max\left\{m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_n, fx_n)}{m(gx_{n-1}, gx_n)+1}\right\}\right] \\ &= \varphi\left[\max\left\{m(gx_{n-1}, gx_n), \frac{m(gx_{n-1}, gx_n)\ m(gx_n, gx_{n+1})}{m(gx_{n-1}, gx_n)+1}\right\}\right] \\ &\leq \varphi\left[\max\{m(gx_{n-1}, gx_n), m(gx_n, gx_{n+1})\}\right]. \end{aligned}$$

If $\max\{m(gx_{n-1}, gx_n), m(gx_n, gx_{n+1})\} = m(gx_n, gx_{n+1})$, then

$$m(gx_n, gx_{n+1}) \leq \varphi[m(gx_n, gx_{n+1})] < m(gx_n, gx_{n+1}),$$

which leads to a contradiction. This implies that

$$m(gx_n, gx_{n+1}) \leq \varphi[m(gx_{n-1}, gx_n)].$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} m(gx_n, gx_{n+1}) &= m(fx_{n-1}, fx_n) \\ &\leq \varphi[m(gx_{n-1}, gx_n)] \\ &\leq \varphi^2[m(gx_{n-2}, gx_{n-1})] \\ &\ \vdots \\ &\leq \varphi^n[m(gx_0, gx_1)]. \end{aligned}$$

We will show that $\{gx_n\}$ is a Cauchy sequence.

Since $\lim_{n\to\infty} \varphi^n[m(gx_0, gx_1)] = 0$, then there exists $N \in \mathbb{N}$, such that

$$\varphi^n[m(gx_0, gx_1)] < \epsilon \quad \text{for all } n \geq \text{N}.$$

This implies that

$$m(gx_n, gx_{n+1}) < \epsilon \quad \text{for all } n \geq N. \tag{2.7}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$m(gx_n, gx_m) < \epsilon \quad \text{for all } m \geq n \geq N \tag{2.8}$$

by induction on m. From (2.8), the result is true for $m = n+1$. Suppose that (2.8) holds for $m = k$. Therefore, for $m = k+1$, we have

$$\begin{aligned} m(gx_n, gx_{k+1}) &= m(fx_{n-1}, fx_k) \\ &\leq \varphi\left[\max\left\{m(gx_{n-1}, gx_k), \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k)}{m(gx_{n-1}, gx_k)+1}\right\}\right]. \end{aligned}$$

Case I. If $\max\left\{m(gx_{n-1}, gx_k), \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k)}{m(gx_{n-1}, gx_k)+1}\right\} = m(gx_{n-1}, gx_k)$, then

$$m(gx_n, gx_{k+1}) \leq \varphi[m(gx_{n-1}, gx_k)] < m(gx_{n-1}, gx_k).$$

Using (m_3),

$$\begin{aligned} \lim_{n\to\infty} \sup m(gx_n, gx_{k+1}) &< s \lim_{n\to\infty} \sup m(fx_{n-1}, gx_k) \\ &= s \lim_{n\to\infty} \sup m(gx_n, gx_k) \\ &= 0. \end{aligned}$$

Hence

$$m(gx_n, gx_{k+1}) < \epsilon. \tag{2.9}$$

Case II. If $\max\left\{m(gx_{n-1}, gx_k), \dfrac{m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k)}{m(gx_{n-1}, gx_k)+1}\right\}$

$$= \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k)}{m(gx_{n-1}, gx_k)+1},$$

then,

$$\begin{aligned} m(gx_n, gx_{k+1}) &\le \varphi\left[\frac{m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k)}{m(gx_{n-1}, gx_k)+1}\right] \\ &< \frac{m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k)}{m(gx_{n-1}, gx_k)+1} \\ &\le m(gx_{n-1}, fx_{n-1})\ m(gx_k, fx_k) \\ &= m(gx_{n-1}, gx_n)\ m(gx_k, fx_k). \end{aligned}$$

Using (m_3),

$$\begin{aligned} \lim_{n\to\infty}\sup m(gx_n, gx_{k+1}) &\le s \lim_{n\to\infty}\sup m(fx_{n-1}, gx_n)\ m(gx_k, fx_k) \\ &= s \lim_{n\to\infty}\sup m(gx_n, gx_n) m(gx_k, fx_k) \\ &= 0, \qquad \text{since } s \ge 1 \text{ is finite.} \end{aligned}$$

Therefore,

$$m(gx_n, gx_{k+1}) < \epsilon. \tag{2.10}$$

Thus (2.8) holds for all $m \ge n \ge N$. It follows that $\{gx_n\}$ is a Cauchy sequence. By the completeness of $g(X)$, we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \ne fp$. By (2.6), we have

$$\begin{aligned} m(gx_n, fp) &= m(fx_{n-1}, fp) \\ &\le \varphi\left[\max\left\{m(gx_{n-1}, gp), \frac{m(gx_{n-1}, fx_{n-1})\ m(gp, fp)}{m(gx_{n-1}, gp)+1}\right\}\right]. \end{aligned}$$

Case I. If $\max\left\{m(gx_{n-1}, gp), \dfrac{m(gx_{n-1}, fx_{n-1})\ m(gp, fp)}{m(gx_{n-1}, gp)+1}\right\} = m(gx_{n-1}, gp)$ then,

$$\begin{aligned} m(gx_n, fp) &\le \varphi[m(gx_{n-1}, gp)] \\ &< m(gx_{n-1}, gp). \end{aligned}$$

Taking $n \to \infty$ and using (m_3),

$$\begin{aligned} \lim_{n\to\infty} m(gx_n, fp) &< \lim_{n\to\infty}\sup m(gx_{n-1}, gp) \\ &\le s \lim_{n\to\infty}\sup m(fx_{n-1}, gp) \\ &= s \lim_{n\to\infty}\sup m(gx_n, gp) \\ &= 0, \end{aligned}$$

that is, $m(gp, fp) = 0$, giving $gp = fp$.

Case II. If $\max\left\{m(gx_{n-1},gp), \dfrac{m(gx_{n-1},fx_{n-1})\ m(gp,fp)}{m(gx_{n-1},gp)+1}\right\}$

$$= \frac{m(gx_{n-1},fx_{n-1})\ m(gp,fp)}{m(gx_{n-1},gp)+1},$$

then,

$$\begin{aligned}
m(gx_n,fp) &\le \varphi\left[\frac{m(gx_{n-1},fx_{n-1})\ m(gp,fp)}{m(gx_{n-1},gp)+1}\right]\\
&< \frac{m(gx_{n-1},fx_{n-1})\ m(gp,fp)}{m(gx_{n-1},gp)+1}\\
&\le m(gx_{n-1},fx_{n-1})\ m(gp,fp)\\
&= m(gx_{n-1},gx_n)\ m(gp,fp).
\end{aligned}$$

Taking $n\to\infty$ and using (m_3),

$$\begin{aligned}
\lim_{n\to\infty}\sup m(gx_n,fp) &\le \lim_{n\to\infty}\sup m(gx_{n-1},gx_n)\ m(gp,fp)\\
&\le s\lim_{n\to\infty}\sup m(fx_{n-1},gx_n)\ m(gp,fp)\\
&= s\lim_{n\to\infty}\sup m(gx_n,gx_n)m(gx_k,fp)\\
&= 0 \qquad \text{since } s\ge 1 \text{ is finite.}
\end{aligned}$$

So, $(gp,fp)=0$, giving $gp=fp$.

We now show that f and g have a unique point of coincidence. Let $fq=gq$ for some $q\in X$.

Assume that $gp\ne gq$. By applying (2.6), it follows that

$$m(gp,gq)=m(fp,fq)\le\varphi\left[\max\left\{m(gp,gq),\frac{m(gp,fp)\ m(gq,fq)}{m(gp,gq)+1}\right\}\right].$$

But

$$\max\left[m(gp,gq),\frac{m(gp,fp)\ m(gq,fq)}{m(gp,gq)+1}\right]=m(gp,gq),$$

since $gp=fp$.

Therefore, $m(gp,gq)\le\varphi m(gp,gq)<m(gp,gq)$ which leads to a contradiction. Hence $gp=gq$.

This implies that f and g have a unique point of coincidence. By Proposition 1.7, we can conclude that f and g have a unique common fixed point. □

Corollary 2.7 ([6, Theorem 1])**.** Let (X,m) be a complete super metric space and let $T:X\to X$ be a mapping such that there exists $k\in(0,1)$ and

$$m(Tx,Ty)\le k\left[\max\left\{m(x,y),\frac{m(x,Tx)\ m(y,Ty)}{m(x,y)+1}\right\}\right].$$

Then, T has a unique fixed point.

Proof. Define $\varphi:[0,\infty)\to[0,\infty)$ by $\varphi(t)=kt$. Therefore φ is a nondecreasing function and $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t\in(0,+\infty)$. It follows that the contractive conditions in Theorem 2.6 are now satisfied. This completes the proof. □

Remark 2.8. Our Example 2.5 surely satisfies the conditions (2.6), since for $x \neq 3$, $y = 3$, $s = 6$ and $\varphi(t) = \frac{t}{2}$, we have

Case I. If $\max\left\{m(gx, gy), \dfrac{m(gx, fx)\ m(gy, fy)}{m(gx, gy) + 1}\right\} = m(gx, gy)$, we are through due to Theorem 2.1.

Case II. If $\max\left\{m(gx, gy), \dfrac{m(gx, fx)\ m(gy, fy)}{m(gx, gy) + 1}\right\} = \dfrac{m(gx, fx)\ m(gy, fy)}{m(gx, gy) + 1}$, then

$$\frac{m(gx, fx)\ m(gy, fy)}{m(gx, gy) + 1} = \frac{m(gx, 2)\ m\left(1, \frac{3}{2}\right)}{m(gx, 1) + 1} = \frac{3gx}{gx + 1},$$

where $gx \in (1, 3]$.

Therefore $m(fx, fy) \leq \varphi[m(gx, gy)]$.

Hence all the conditions of Theorem 2.6 are satisfied. Therefore we conclude that the mappings f and g have a unique common fixed point; that is, $x = 2$.

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