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ANALYSIS OF SOME DYNAMICAL BEHAVIORS OF A SIXTH ORDER DIFFERENCE EQUATION

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Abstract. The main goal of this research paper is to analyze some behaviors including stability , period of solution , solution of a special case and numerical examples to illustrate the analysis for the following sixth-order difference equation

$$Z_{n+1} = \alpha Z_{n-2} + \frac{\beta Z_{n\_2}}{\gamma Z_{n-2} - \delta Z_{n-5}}, \quad n = 0, 1, 2, 3, ...,$$

where the initial conditions $z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}$ and z_0 are arbitrary real numbers and the values α, β, γ and δ are defined as positive constants.

1. Introduction

This research paper aims to demonstrate the qualitative dynamical behaviors of the following sixth-order difference equation:

$$Z_{n+1} = \alpha Z_{n-2} + \frac{\beta Z_{n\_2}}{\gamma Z_{n-2} - \delta Z_{n-5}}, \quad n = 0, 1, 2, 3, ..., \tag{1}$$

where the initial conditions $z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}$ and z_0 are arbitrary nonzero real numbers and the values α, β, γ and δ are defined as positive constants.

Difference equations are fundamental in many areas of applied mathematics including engineering, biology, etc. In the last few years, researchers have concentrated on the behaviors of the solution of difference equations such as the local and global attractivity, boundedness character and the periodicity of the solutions which perform to importance this field. Moreover, we can present some recent studies in this field as the following :

Alayachi et al.[1], highlighted the analysis of the following fourth order difference equation:

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}.$$

Almatrafi et al.[3], investigated the following fourth order difference equation:

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}.$$

In[7], they studied behaviors of the solution for the following difference equation

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$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

In[9], the authors described the solutions of the following difference equation

$$x_{n+1} = \frac{ax_n x_{n-2}}{bx_n + cx_{n-3}}.$$

In[19], the behavior of the following difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

were obtained by Simsek et al.

Additionally, In[22] they analyzed some behaviors of the solution of the following equation

$$x_{n+1} = \frac{(ax_{n-1} + bx_{n-2})}{(c + dx_{n-1}x_{n-2})}\ .$$

Other details on this aspect can be seen in refs. [1]-[23].

This paper is divided into six main sections. Section 1 contains the introduction. Section 2 proves the solutions of period two. Section 3 illustrates the local stability of the solutions. Section 4 states the global attractivity character of solutions of Eq.(1). Section 5 discusses the special case of Eq.(1). In Section 6 we verify our theoretical results by providing some numerical examples with figures.

2. Periodicity of The Solutions

This section illustrate a theorem that states Eq.(1) has a periodic solutions under sufficient conditions.

Theorem 2.1. *Eq.(1) has a positive period two solutions if and only if*

$$(*) \quad (\gamma + \delta)(\alpha + 1) > 4\delta, \ \alpha\gamma \neq \delta.$$

Proof: We claim that Eq.(1) has a period two solution

$$..., p, q, p, q, ...$$

and we need to prove the condition (*) holds.

From Eq.(1) we observe that

$$p = \alpha q + \frac{\beta q}{\gamma q - \delta p},$$

and

$$q = \alpha p + \frac{\beta p}{\gamma p - \delta q}.$$

Then

$$\gamma pq - \delta p^2 = \alpha\gamma q^2 - \alpha\delta pq + \beta q, \tag{2}$$

and

$$\gamma pq - \delta q^2 = \alpha\gamma p^2 - \alpha\delta pq + \beta p. \tag{3}$$

Subtracting (3) from (2) implies that

$$\delta(q^2 - p^2) = \alpha\gamma(q^2 - p^2) + \beta(q - p).$$

Indeed $p \neq q$, we conclude that

$$p + q = \frac{\beta}{\delta - \alpha\gamma}. \tag{4}$$

By adding (2) and (3) we get

$$2\gamma pq - \delta(p^2 + q^2) = \alpha\gamma(p^2 + q^2) - 2\alpha\delta pq + \beta(p + q). \tag{5}$$

By using

$$p^2+q^2=(p+q)^2-2pq \quad \text{for all} \;\; p,q\in R,$$

we obtain from (4), (5)

$$pq=\frac{\beta^2\delta}{(\delta-\alpha\gamma)^2(\gamma+\delta)(\alpha+1)}. \tag{6}$$

From (4) and (6) we observe that the p and q are two positive roots of the following quadratic equation

$$t^2-(p+q)t+pq=0. \tag{7}$$

Which implies that

$$t^2-\frac{\beta}{(\delta-\alpha\gamma)}t+\frac{\beta^2\delta}{(\delta-\alpha\gamma)^2(\gamma+\delta)(\alpha+1)}=0.$$

That is

$$(\delta-\alpha\gamma)t^2-\beta t+\frac{\beta^2\delta}{(\delta-\alpha\gamma)(\gamma+\delta)(\alpha+1)}=0,$$

and so

$$\beta^2>\frac{4\beta^2\delta}{(\gamma+\delta)(\alpha+1)},$$

thus

$$(\gamma+\delta)(\alpha+1)>4\delta.$$

Therefore the condition (*) holds.

Now,we assume that condition (*) is true. We shall prove Eq.(1) has a period two solution.
suppose

$$p=\frac{\beta+\lambda}{2(\delta-\alpha\gamma)},$$

and

$$q=\frac{\beta-\lambda}{2(\delta-\alpha\gamma)},$$

where $\lambda=\sqrt{\beta^2-\dfrac{4\beta^2\delta}{(\gamma+\delta)(\alpha+1)}}$.
We can see from condition (*) that

$$(\gamma+\delta)(\alpha+1)>4\delta \quad\Rightarrow\quad 1>\frac{4\delta}{(\gamma+\delta)(\alpha+1)},$$

then after multiplying by β^2 we get

$$\beta^2>\frac{4\beta^2\delta}{(\gamma+\delta)(\alpha+1)}.$$

we conclude that p and q are distinct positive real numbers.
Put

$$z_{-1}=p \;\; \text{and} \;\; z_0=q.$$

and we wish to get that

$$z_1=z_{-1}=p \quad \text{and} \quad z_2=z_0=q.$$

From Eq.(1) we have

$$z_1 = \alpha z_0 + \frac{\beta z_0}{\gamma z_0 - \delta z_{-1}} = \alpha q + \frac{\beta q}{\gamma q - \delta p} = \frac{\alpha\gamma q^2 - \alpha\delta pq + \beta q}{\gamma q - \delta p},$$

$$= \frac{\alpha\gamma\left[\frac{\beta-\lambda}{2(\delta-\alpha\gamma)}\right]^2 - \alpha\delta\left[\frac{\beta^2\delta}{(\delta-\alpha\gamma)^2(\gamma+\delta)(\alpha+1)}\right] + \beta\left[\frac{\beta-\lambda}{2(\delta-\alpha\gamma)}\right]}{\gamma\left[\frac{\beta-\lambda}{2(\delta-\alpha\gamma)}\right] - \delta\left[\frac{\beta+\lambda}{2(\delta-\alpha\gamma)}\right]}.$$

Multiplying both the denominator and the numerator by $4(\delta-\alpha\gamma)^2$ we get:

$$= \frac{\alpha\gamma\left[\beta^2 - 2\beta\lambda + \beta^2 - \frac{4\beta^2\delta}{(\gamma+\delta)(\alpha+1)}\right]\frac{4\alpha\beta^2\delta^2}{(\gamma+\delta)(\alpha+1)} + 2(\beta\delta - \alpha\beta\gamma)\left[\beta-\lambda\right]}{2(\delta-\alpha\gamma)\left\{\gamma\beta - \beta\delta - (\gamma+\delta)\lambda\right\}},$$

$$= \frac{2\beta^2\delta - \frac{4\alpha\beta^2\gamma\delta + 4\alpha\beta^2\delta^2}{(\gamma+\delta)(\alpha+1)} - 2\beta\delta\lambda}{2(\delta-\alpha\gamma)\left\{\gamma\beta - \beta\delta - (\gamma+\delta)\lambda\right\}}.$$

Multiplying both the denominator and numerator by $\{\gamma\beta - \beta\delta + (\gamma+\delta)\lambda\}$ we get:

$$z_1 = \frac{\left[\begin{array}{c} -4\beta^3\delta^2 - \frac{4\alpha\beta^3\gamma^2\delta - 4\alpha\beta^3\delta^3}{(\gamma+\delta)(\alpha+1)} - 4\beta^2\delta^2\lambda \\ -\left(\frac{(4\alpha\beta^2\gamma^2\delta + 8\alpha\beta^2\gamma\delta^2 + 4\alpha\beta^2\delta^3)}{(\gamma+\delta)(\alpha+1)}\right)\lambda + \frac{8\beta^3\gamma\delta^2 + 8\beta^3\delta^3}{(\gamma+\delta)(\alpha+1)} \end{array}\right]}{2(\delta-\alpha\gamma)\left\{-4\beta^2\gamma\delta + \frac{4\beta^2\gamma^2\delta + 4\beta^2\delta^3 + 8\beta^2\gamma\delta^2}{(\gamma+\delta)(\alpha+1)}\right\}}.$$

Multiplying both the denominator and numerator by $\{(\gamma+\delta)(\alpha+1)\}$

$$z_1 = \frac{\left[\begin{array}{c} -4\beta^3\delta^2(\gamma+\delta)(\alpha+1) - 4\alpha\beta^3\gamma^2\delta - 4\alpha\beta^3\delta^3 - 4\beta^2\delta^2(\gamma+\delta)(\alpha+1)\alpha \\ -(4\alpha\beta^2\gamma^2\delta + 8\alpha\beta^2\gamma\delta^2 + 4\alpha\beta^2\delta^3)\lambda + 8\beta^3\gamma\delta^2 + 8\beta^3\delta^3 \end{array}\right]}{2(\delta-\alpha\gamma)\left\{4\beta^2\gamma\delta^2 + 4\beta^2\delta^3 - 4\alpha\beta^2\gamma^2\delta - 4\alpha\beta^2\gamma\delta^2\right\}},$$

$$= \frac{\left[\begin{array}{c} \left(4\beta^3\delta^3 + 4\beta^3\gamma\delta^2 - 4\alpha\beta^3\gamma^2\delta - 4\alpha\beta^3\gamma\delta^2\right) \\ + \left(4\beta^2\gamma\delta^2 + 4\beta^2\delta^3 - 4\alpha\beta^2\gamma^2\delta - 4\alpha\beta^2\gamma\delta^2\right)\lambda \end{array}\right]}{2(\delta-\alpha\gamma)\left\{4\beta^2\gamma\delta^2 + 4\beta^2\delta^3 - 4\alpha\beta^2\gamma^2\delta - 4\alpha\beta^2\gamma\delta^2\right\}}.$$

Now, Dividing both the denominator and numerator by $\left\{4\beta^2\gamma\delta^2 + 4\beta^2\delta^3 - 4\alpha\beta^2\gamma^2\delta - 4\alpha\beta^2\gamma\delta^2\right\}$gives that

$$z_1 = \frac{\beta+\lambda}{2(\delta-\alpha\gamma)} = p.$$

Similarly as before we can conclude

$$z_2 = q.$$

So by induction we get that

$$z_{2n} = q \quad \text{and} \quad z_{2n+1} = p \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$...,p,q,p,q,...$$

Hence, the proof is complete.

3. The Local Stability

In this part , the discussion centres on studying the local stability of the equilibrium point of Eq.(1).

Eq.(1) has a unique fixed point given by the following

$$\overline{z} = \alpha\overline{z} + \frac{\beta\overline{z}}{\gamma\overline{z} - \delta\overline{z}}.$$

If $\gamma \neq \delta,\ \alpha \neq 1$, then the unique fixed point of Eq.(1) is

$$\overline{z} = \frac{\beta}{(\gamma-\delta)(1-\alpha)}.$$

Let us define the function $f : (0,\infty)^2 \longrightarrow (0,\infty)$ by

$$f(u,v) = \alpha u + \frac{\beta u}{\gamma u - \delta v}.$$

Thus

$$\begin{aligned} \frac{\partial f(u,v)}{\partial u} &= \alpha - \frac{\beta\delta v}{(\gamma u-\delta v)^2}, \\ \frac{\partial f(u,v)}{\partial v} &= \frac{\beta\delta u}{(\gamma u-\delta v)^2}. \end{aligned}$$

Then we see that at equilibrium point $\overline{z}$,

$$\begin{aligned} \frac{\partial f(\overline{z},\overline{z})}{\partial u} &= \alpha - \frac{\delta(1-\alpha)}{(\gamma-\delta)} = p_0, \\ \frac{\partial f(\overline{z},\overline{z})}{\partial v} &= \frac{\delta(1-\alpha)}{(\gamma-\delta)} = p_1. \end{aligned}$$

Then by linearization of Eq.(1) about $\overline{z}$ we have

$$y_{n+1} - p_0 y_{n-2} - p_1 y_{n-5} = 0.$$

Theorem 3.1. *Suppose that*

$$|\alpha\gamma - \delta| + |\alpha\delta - \delta| < |\gamma - \delta|.$$

Then the fixed point of Eq.(1) is locally asymptotically stable.

Proof: We see from Theorem A in [13], Eq.(1) is asymptotically stable if

$$|p_0| + |p_1| < 1.$$

$$\left|\alpha + \frac{\delta(1-\alpha)}{(\gamma-\delta)}\right| + \left|-\frac{\delta(1-\alpha)}{(\gamma-\delta)}\right| < 1.$$

Which can be rewrite as follows:

$$|\alpha(\gamma-\delta) + \delta(1-\alpha)| + |-\delta(1-\alpha)| < |\gamma - \delta|.$$

Therefore

$$|\alpha\gamma - \delta| + |\alpha\delta - \delta| < |\gamma - \delta|.$$

Which proved the require.

4. The Global Attractivity

This section is devoted to present the global stability of the Eq.(1).

Theorem 4.1. *The fixed point $\overline{z}$ of Eq.(1) is global attractor if $\alpha\gamma < \delta$.*

Proof: Assume that a and b are two real numbers and let us define the function $g : (a,b)^2 \longrightarrow (a,b)$ by

$$g(u,v) = \alpha u + \frac{\beta u}{\gamma u - \delta v}.$$

Then, we have

$$\frac{\partial g(u,v)}{\partial u} = \alpha - \frac{\beta\delta v}{(\gamma u - \delta v)^2}, \tag{8}$$

$$\frac{\partial g(u,v)}{\partial v} = \frac{\beta\delta u}{(\gamma u - \delta v)^2}. \tag{9}$$

So, we have to cases to consider:

Case(1): If $\alpha - \frac{\beta\delta v}{(\gamma u-\delta v)^2} > 0$ then from (8) and (9), the function $g(u,v)$ is increasing in both u and v.

Assume that (m,M) is a solution of the system

$$m = g(m,m) \quad and \quad M = g(M,M).$$

Then we have from Eq.(1)

$$m = \alpha m + \frac{\beta m}{\gamma m - \delta m}, \quad M = \alpha M + \frac{\beta M}{\gamma M - \delta M}.$$

This result gives

$$(M - m) = \alpha(M - m), \quad \alpha \neq 1.$$

Thus

$$M = m.$$

Which gives by Theorem B in [13] that $\overline{z}$ is a global attractor of Eq.(1).

Case(2): If $\alpha - \frac{\beta\delta v}{(\gamma u-\delta v)^2} < 0$ then from (8) and (9), the function $g(u,v)$ is decreasing in u and increasing v.

Assume that (m,M) is a solution of the system

$$m = g(M,m) \quad and \quad M = g(m,M).$$

Then we have from Eq.(1)

$$m = \alpha M + \frac{\beta M}{\gamma M - \delta m}, \quad M = \alpha m + \frac{\beta m}{\gamma m - \delta M}.$$

Thus

$$\gamma Mm - \alpha\gamma M^2 - \delta m^2 + \alpha\delta Mm = \beta M,$$
$$\gamma Mm - \alpha\gamma m^2 - \delta M^2 + \alpha\delta Mm = \beta m.$$

This result gives

$$(M^2 - m^2) = (\delta - \alpha\gamma) = \beta(M - m), \quad \alpha\gamma > 1.$$

Which implies that

$$M = m.$$

Thus, Theorem B in [13] gives that $\overline{z}$ is a global attractor of Eq.(1).
Which completed the proof.

5. Special case of Equation

This section will demonstrate a special case of Eq.(1) as the follows

$$Z_{n+1} = Z_{n-2} + \frac{Z_{n-2}}{Z_{n-2} - Z_{n-5}} \quad n = 0, 1, 2, 3, ..., \tag{10}$$

where $z_{-5}, \ , z_{-4}, \ z_{-3}, \ z_{-2}, \ z_{-1}, \ z_0$ are arbitrary real numbers with $z_{-5} \neq z_{-3} \neq z_{-1}$ and $z_{-4} \neq z_{-2} \neq z_0$.

Theorem 5.1. *Let $\{x_n\}_{n=-5}^{\infty}$ is a solution of Eq.(10) satisfying $z_{-5} = r, z_{-4} = l, z_{-3} = t, z_{-2} = s, z_{-1} = k, z_0 = h$. Then for $n = 0, 1, ...$*

$$\begin{aligned}
z_{6n-5} &= ns - (n-1)r + n(n-1) + \frac{ns}{s-r}, \\
z_{6n-4} &= nk - (n-1)l + n(n-1) + \frac{nk}{k-l}, \\
z_{6n-3} &= nh - (n-1)t + n(n-1) + \frac{nh}{h-t}, \\
z_{6n-2} &= (n+1)s - nr + n^2 + \frac{ns}{s-r}, \\
z_{6n-1} &= (n+1)k - nl + n^2 + \frac{nk}{k-l}, \\
z_{6n} &= (n+1)h - nt + n^2 + \frac{nh}{h-t}.
\end{aligned}$$

Proof: *For $n = 0$ the result holds. Next assume that $n > 0$ and our assumption satisfies for $n - 1$. That is,*

$$\begin{aligned}
z_{6n-11} &= (n-1)s - (n-2)r + (n-1)(n-2) + \frac{(n-1)s}{s-r}, \\
z_{6n-10} &= (n-1)k - (n-2)l + (n-1)(n-2) + \frac{(n-1)k}{k-l}, \\
z_{6n-9} &= (n-1)h - (n-2)t + (n-1)(n-2) + \frac{(n-1)h}{h-t},
\end{aligned}$$

$$z_{6n-8} = ns-(n-1)r+(n-1)^2+\frac{(n-1)s}{s-r},$$

$$z_{6n-7} = nk-(n-1)l+(n-1)^2+\frac{(n-1)k}{k-l},$$

$$z_{6n-6} = nh-(n-1)t+(n-1)^2+\frac{(n-1)h}{h-t}.$$

Now, it follows from Eq.(10) that

$$\begin{aligned}
Z_{6n-5} &= Z_{6n-8}+\frac{Z_{6n-8}}{Z_{6n-8}-Z_{6n-11}}\\
&= ns-(n-1)r+(n-1)^2+\frac{(n-1)s}{s-r}\\
&+\frac{ns-(n-1)r+(n-1)^2+\frac{(n-1)s}{s-r}}{ns-(n-1)r+(n-1)^2+\frac{(n-1)s}{s-r}-((n-1)s-(n-2)r+(n-1)(n-2)+\frac{(n-1)s}{s-r})}\\
&= ns-(n-1)r+(n-1)^2+\frac{(n-1)s}{s-r}+\frac{(ns-nr+r)(s+n-r+1)}{(s-r)(s+n-r+1)}\\
&= ns-(n-1)r+(n-1)^2+\frac{2ns-s-nr+r}{s-r}\\
&= ns-(n-1)r+\frac{n(ns-nr+r)}{s-r}\\
&= ns-(n-1)r+n(n-1)+\frac{ns}{s-r}.
\end{aligned}$$

Next, we obtain from Eq.(10) that

$$\begin{aligned}
Z_{6n-4} &= Z_{6n-7}+\frac{Z_{6n-7}}{Z_{6n-7}-Z_{6n-10}}\\
&= nk-(n-1)l+(n-1)^2+\frac{(n-1)k}{k-l}\\
&+\frac{nk-(n-1)l+(n-1)^2+\frac{(n-1)k}{k-l}}{nk-(n-1)l+(n-1)^2+\frac{(n-1)k}{k-l}-((n-1)k-(n-2)l+(n-1)(n-2)+\frac{(n-1)k}{k-l})}\\
&= nk-(n-1)l+(n-1)^2+\frac{(n-1)k}{k-l}+\frac{(nk-nl+l)(k+n-l+1)}{(k-l)(k+n-l+1)}\\
&= nk-(n-1)l+(n-1)^2+\frac{2nk-k-nl+l}{k-l}\\
&= nk-(n-1)l+\frac{n(nk-nl+l)}{k-l}\\
&= nk-(n-1)l+n(n-1)+\frac{nk}{k-l}.
\end{aligned}$$

Next, from Eq(10) we have

$$
\begin{aligned}
Z_{6n-3} &= Z_{6n-6} + \frac{Z_{6n-6}}{Z_{6n-6} - Z_{6n-9}} \\
&= nh - (n-1)t + (n-1)^2 + \frac{(n-1)h}{h-t} \\
&+ \frac{nh - (n-1)t + (n-1)^2 + \frac{(n-1)h}{h-t}}{nh - (n-1)t + (n-1)^2 + \frac{(n-1)h}{h-t} - ((n-1)h - (n-2)t + (n-1)(n-2) + \frac{(n-1)h}{h-t})} \\
&= nh - (n-1)t + (n-1)^2 + \frac{(n-1)h}{h-t} + \frac{(nh - nt + t)(h + n - t + 1)}{(h-t)(h + n - t + 1)} \\
&= nh - (n-1)t + (n-1)^2 + \frac{2nh - h - nt + t}{h-t} \\
&= nh - (n-1)t + \frac{n(nh - nt + t)}{h-t} \\
&= nh - (n-1)t + n(n-1) + \frac{nh}{h-t}.
\end{aligned}
$$

Also, we get from Eq.(10)

$$
\begin{aligned}
Z_{6n-2} &= Z_{6n-5} + \frac{Z_{6n-5}}{Z_{6n-5} - Z_{6n-8}} \\
&= ns - (n-1)r + n(n-1) + \frac{ns}{s-r} \\
&+ \frac{ns - (n-1)r + n(n-1) + \frac{ns}{s-r}}{ns - (n-1)r + n(n-1) + \frac{ns}{s-r} - (ns - (n-1)r + (n-1)^2 + \frac{(n-1)s}{s-r})} \\
&= ns - (n-1)r + n(n-1) + \frac{ns}{s-r} + \frac{(s-n-r)(ns - nr + r)(s-r)}{(ns - nr + r)(s-r)} \\
&= ns - (n-1)r + n(n-1) + \frac{ns}{s-r} + s - n - r \\
&= (n+1)s - nr + n^2 + \frac{ns}{s-r}.
\end{aligned}
$$

We also obtain from Eq.(10) that

$$
\begin{aligned}
Z_{6n-1} &= Z_{6n-4} + \frac{Z_{6n-4}}{Z_{6n-4} - Z_{6n-7}} \\
&= nk - (n-1)l + n(n-1) + \frac{nk}{k-l} \\
&+ \frac{nk - (n-1)l + n(n-1) + \frac{nk}{k-l}}{nk - (n-1)l + n(n-1) + \frac{nk}{k-l} - (nk - (n-1)l + (n-1)^2 + \frac{(n-1)k}{k-l})} \\
&= nk - (n-1)l + n(n-1) + \frac{nk}{k-l} + \frac{(k-n-l)(nk - nl + l)(k-l)}{(nk - nl + l)(k-l)} \\
&= nk - (n-1)l + n(n-1) + \frac{nk}{k-l} + k - n - l \\
&= (n+1)k - nl + n^2 + \frac{nk}{k-l}.
\end{aligned}
$$

Finally, from Eq.(10) we prove that

$$\begin{aligned} Z_{6n} &= Z_{6n-3} + \frac{Z_{6n-3}}{Z_{6n-3} - Z_{6n-6}} \\ &= nh - (n-1)t + n(n-1) + \frac{nh}{h-t} \\ &+ \frac{nh - (n-1)t + n(n-1) + \frac{nh}{h-t}}{nh - (n-1)t + n(n-1) + \frac{nh}{h-t} - (nh - (n-1)t + (n-1)^2 + \frac{(n-1)h}{h-t})} \\ &= nh - (n-1)t + n(n-1) + \frac{nh}{h-t} + \frac{(h-n-t)(nh-nt+t)(h-t)}{(nh-nt+t)(h-t)} \\ &= nh - (n-1)t + n(n-1) + \frac{nh}{h-t} + h - n - t \\ &= (n+1)h - nt + n^2 + \frac{nh}{h-t}. \end{aligned}$$

Hence, the proof is completed.

6. The Numerical Solutions

In this part, we provide some numerical examples in order to verify our results.

Example 1: In this example, we show the local stability behaviour of our problem when $Z_{-5} = 6, Z_{-4} = 0.5, Z_{-3} = 3, Z_{-2} = 2, Z_{-1} = 1.5, Z_0 = 1, \alpha = 0.2, \beta = 1, \gamma = 5$ and $\delta = 0.5$. See Fig.1.

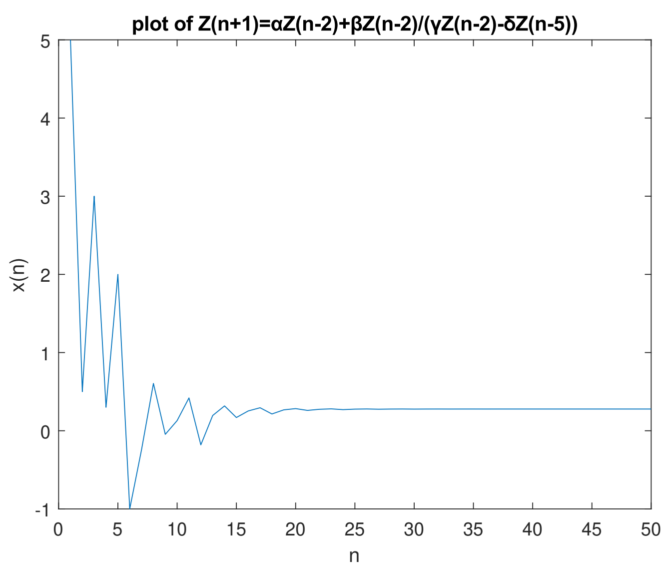

Figure 1. Local Stability of The Equilibrium Point.

Example 2: Suppose that $Z_{-5} = 5, Z_{-4} = 6, Z_{-3} = -5, Z_{-2} = -2, Z_{-1} = 3, Z_0 = -1, \alpha = 0.5, \beta = 1, \gamma = 2$ and $\delta = 9$. Then this example demonstrate the global stability behaviour of Eq.(1). See Fig.2.

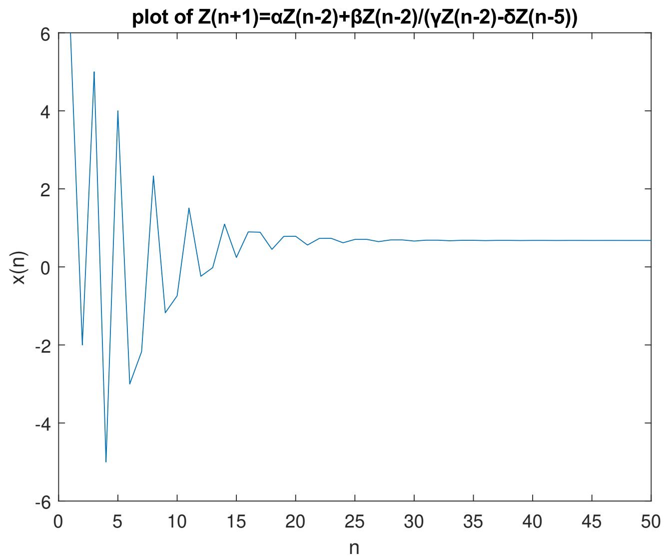

FIGURE 2. Global Stability of The Equilibrium Point.

Example 3: In this example we present the plot of the solution when we have $Z_{-5} = 5, Z_{-4} = 0.3, Z_{-3} = -2, Z_{-2} = 1, Z_{-1} = 4, Z_0 = 0.5, \alpha = 1, \beta = 1, \gamma = 0.5$ and $\delta = 9$. See Fig 3.

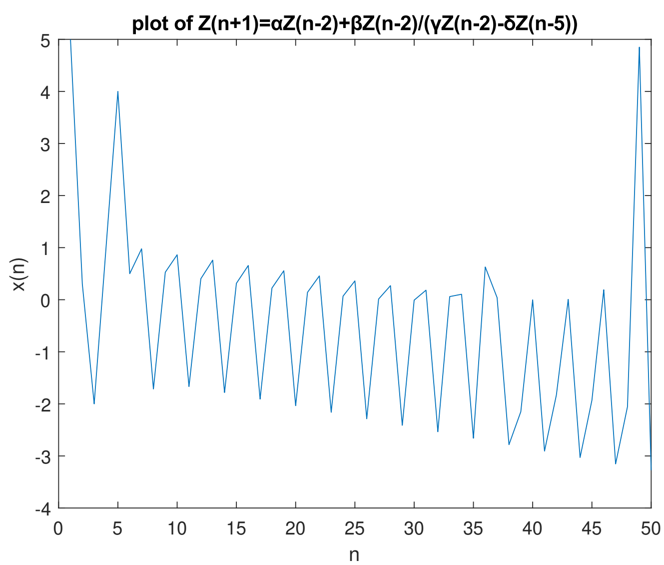

FIGURE 3. Solution of Eq.(1)

Example 4: Here, we also present the plot of the behavior of the solution under $Z_{-5} = 4, Z_{-4} = 0.5, Z_{-3} = 1, Z_{-2} = 3, Z_{-1} = 6, Z_0 = 8, \alpha = 1, \beta = 1, \gamma = 0.5$ and $\delta = 9$. See Fig 4.

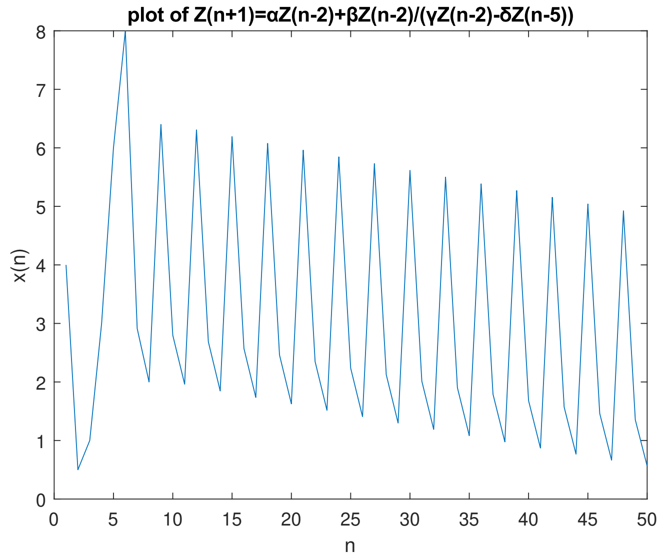

FIGURE 4. Solution of Eq.(1)

Example 5: This example will provide the behavior of the solution in the special case with $Z_{-5} = 5, Z_{-4} = 0.1, Z_{-3} = 1, Z_{-2} = 0.5, Z_{-1} = 4, Z_0 = 8, \alpha = 1 = \beta = \gamma = \delta$. See Fig 5.

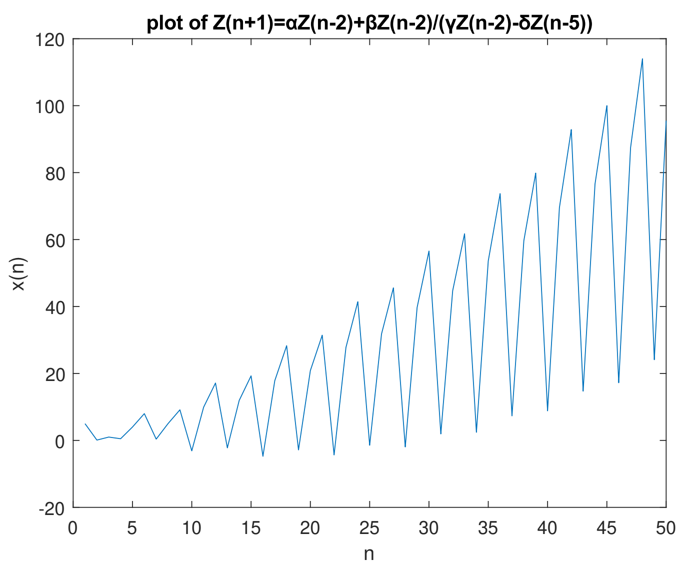

FIGURE 5. Solution of The Special Case Equation

Example 6: Here, we confirm that our equation has a periodic solution of period two , when $\alpha = 2, \beta = 6, \gamma = 8, \delta = 20$ and $z_{-1} = 0.914$ ($p = \frac{\beta+\lambda}{2(\delta-\alpha\gamma)}$) , $z_0 = 0.586$ ($q = \frac{\beta-\lambda}{2(\delta-\alpha\gamma)}$) , $\lambda = \sqrt{\beta^2 - \frac{4\beta^2\delta}{(\gamma+\delta)(\alpha+1)}}$. See Fig 6.

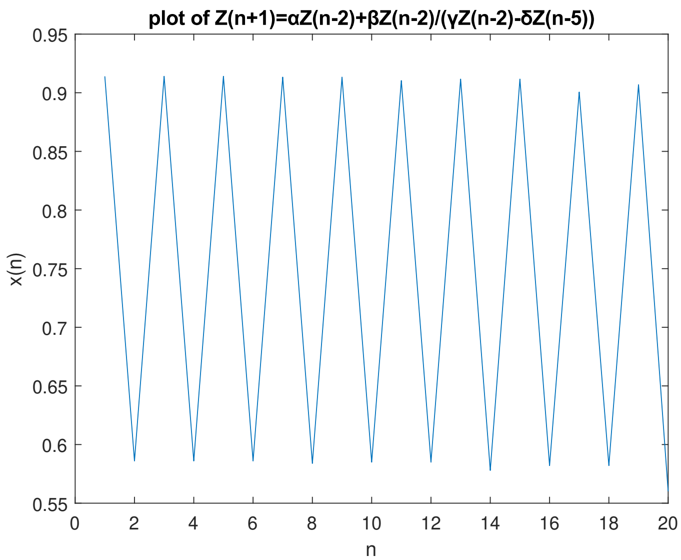

Figure 6. Periodic Solution of Period Two

References

[1] H. S. Alayachi, M. S. Noorani and E. M. Elsayed, Qualitative analysis of a fourth order difference equation, Journal of Applied Analysis and Computation, 10 (4)(2020),1343-1354.

[2] H. S. Alayachi, M. S. M. Noorani, A. Q. Khan and M. B. Almatrafi, Analytic solutions and stability of sixth order difference equations, Mathematical Problems in Engineering, Volume 2020(2020), Article ID 1230979, 12 pages.

[3] M. B. Almatrafi, E. M. Elsayed and F. Alzahrani, Investigating some properties of a fourth order difference equation, Journal of Computational Analysis and Applications, 28(2)(2020), 243-253.

[4] M. Aloqeili, Dynamics of a Rational Difference Equation, Appl. Math. Comp., 176 (2) (2006), 768-774.

[5] A. Alshareef, F. Alzahrani and A. Khan, Dynamics and Solutions Expressions of a Higher-Order Nonlinear Fractional Recursive Sequence, Mathematical Problems in Engineering, Volume 2021(2021), Article ID 1902473, 12 pages.

[6] C. Cinar, On The Positive Solutions of The Difference Equation $x_{n+1} = ax_{n-1}/(1 + bx_nx_{n-1})$, Applied Mathematics and Computation, 156 (2004) 587-590.

[7] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On The Difference Equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Advances in Difference Equations, 2006 (2006), 1-10.

[8] M. M. El-Dessoky and M. El-Moneam, On The Higher Order Difference Equation $x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + (x_{n-k}) = (Dx_{n-s} + Ex_{n-t})$, J. Computational Analysis and Applications, 25 (2) (2018), 342-354.

[9] E. M. Elsayed, A. Alotaibi and H. A. Almaylabi, The Behavior and Closed Form of The Solutions of Some Difference Equations, Journal of Computational and Theoretical Nanoscience, 13 (1-10) (2016).

[10] T. Ibrahim, On The Third Order Rational Difference Equation$x_{n+1} = (x_nx_{n-2}) = (x_{n-1}(a + bx_nx_{n-2}))$, Int.J. Contemp. Math. Sciences, 4 (27)(2009), 1321-1334.

[11] T. Ibrahim, A. Khan, and A. Ibrahim, Qualitative Behavior of a Nonlinear Generalized Recursive Sequence with Delay, Mathematical Problems in Engineering, Volume 2021(2021), Article ID 6162320, 8 pages.

[12] R. Karatas, Global Behavior of a Higher Order Difference Equation, International Journal of Contemporary Mathematical Sciences, 12 (3)(2017),133-138.

[13] A. Khaliq, F. Alzahrani and E. M. Elsayed, Global Attractivity of a Rational Difference Equation of Order Ten, Journal of Nonlinear Science and Applications, 9 (2016), 4465-4477.

[14] A. Q. Khan, M. S. Noorani and H. S. Alayachi, Global dynamics of higher order exponential systems of difference equations, Discrete Dynamics in Nature and Society, Volume 2019(2019), Article ID 3825927, 19 pages.

[15] Y. Kostrov, On a Second-Order Rational Difference Equation with a Quadratic Term, International Journal of Difference Equations, 11 (2)(2016), 179-202.

[16] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.

[17] K. Liu, P. Li, F. Han, and W. Zhong, Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + x_n/x_{n-1}x_{n-2}$, Journal of Computational Analysis and Applications, 24 (6)(2018), 1125-1132.

[18] M. Saleh and M. Aloqeili, On The Difference Equation $y_{n+1} = A + y_n/y_{n-k}$, Appl. Math. Comput., 176 (1),(2006), 359 363.

[19] D. Simsek, C. Cinar and I. Yalcinkaya, On The Recursive Sequence $x_{n+1} = x_{n-3}/1 + x_{n-1}$, Int. J. Contemp. Math.Sci., 1 (10) (2006), 475-480.

[20] Y. Su and W. Li, Global Asymptotic Stability of a Second-Order Nonlinear Difference Equation, Applied Mathematics and Computation, 168 (2005), 981-989.

[21] D. Tolly, Y. Yazlik and N. Taskara, Behavior of Positive Solutions of a Difference Equation, 35 (3-4) (2017),217-230.

[22] X. Yang, W. Su, B. Chen, G. M. Megson and D. J. Evans, On The Recursive Sequence $x_{n+1} = (ax_{n-1} + bx_{n-2})/(c + dx_{n-1}x_{n-2})$; Appl. Math. Comp., 162 (2005), 1485-1497.

[23] E. Zayed and M. El-Moneam, On The Rational Recursive Sequence $x_{n+1} = Ax_n + Bx_{n-k} + (\beta x_n + \gamma x_{n-k})/(Cx_n + Dx_{n-k})$, Acta Applicandae Mathematicae, 111 (3)(2010), 287-301.

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