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STABILITY OF COPULAS UNDER SURVIVAL TRANSFORM

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Abstract. It is useful in the literature to consider that the survival copula $\widehat{C}(.,.)$ associated to a given archimedean one $C(.,.)$ is also an archimedean copula. Here we study deeply survival copulas, give a corrigendum for this misconception and discuss conditions under which a suitable conclusion is possible. Stability of some classical families of copulas under survival transform is also discussed. An application to homogeneous copulas is detailed.

1. Introduction

Over the past five decades, statisticians and other researchers often use *copulas* as an efficient tool to study scale-free measures of dependence. Copulas are mobilized also to construct families of multivariate copulas using some adequate methods such as *compatibility* with bivariate copulas. For a simple and understandable presentation showing interest of copulas in several areas of statistics and economic studies, it is recommendable to see [4] or browse the introduction of [9] where are cited some bridges between our topic and some classical problems in economy and sociology such as Arrow's impossibility theorem [1] or [2] mainly when ordering problems are raised. Historically, the word *copula* is attributed to Sklar who was the first to use it in the statistical sense [8] meaning the manner to join the multivariate distribution to its mono-dimensional marginal distribution functions. For more precision, we recall a version of this revolutionary theorem which gave the bridge between theoretical probability and statistical applications:

Theorem 1 (Sklar's theorem). *Let H be two-dimensional distribution function on a probability space (Ω, p) with continuous marginal distribution functions F and G. Then there exists unique copula C such that*

$$H(x, y) = C(F(x), G(y)). \tag{1}$$

It is almost impossible to deal with copulas without evoking masterpiece Nelsen's book [4]. Therein, pages 32-36 were devoted to survival copulas which describe the probability of pair (U, V) to live or survive beyond times (u, v)(i.e $P(U > u, V > v)$). But recently many researches have extensively used survival copulas to study and make clearer several notions associated with copulas. We cite a list of references,

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far being exhaustive, as a familiarizing tool with this concept [5],[6],[7],[13]...etc.
The interest of survival copulas consists on one hand in their use for exponential models and on the other hand in its characterization of the initial copulas. Indeed, as we restrict ourselves here to the bivariate case, a given copula is related to its survival one by the well known identity: $\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. Hence, as explained in [4, page 42], a perfect knowledge of $\widehat{C}$ allows to simulate the associated copula C thanks to a relatively simple algorithm. Furthermore, for particular cases, C and $\widehat{C}$ characterize some geometrical properties such as *radial symmetry* stated in [4, Theorem 2.7.3].
The framework of current paper is archimedean copulas in association with their correspondent survival ones. We focus on the generators of latter copulas and exhibit relations between the analytic properties of copulas and regularity of their generators.
In the literature, many of papers treating survival copulas of archimedean ones try to express and characterize the generator of $\widehat{C}$ without taking care of its existence. In other words, they assume that the corresponding survival copula $\widehat{C}$ is automatically archimedean. It turns out that this likely hypothesis is far from being true. We explain the result in an elementary way, and give some sufficient conditions to ensure the archimedean character of a given survival copula...
This paper is organized as follows: after this current brief introduction, we recall in section 2 fundamental results on copulas, mainly in the bivariate case, and define suitably survival copulas. After dealing with archimedean copulas we explain the interest and role of generator in a big reduction of concordance and dependence ratios and measurement, like Kendall-τ and ρ-coefficient of Spearman, we pursue to emphasize, via an appropriate example, the corrigendum of the misconception on automatic archimedean property of the survival copula when the initial one is archimedean. We refer mainly and remarkably to [6] and [7] where the examined copulas arise from a utility functions. The third and last section is devoted to discuss some hypothesis of automatic existence of survival copula generator although the archimedean character of the associate one by invoking the general theory of associative functions. At last, we discuss stability of some classical parametrized families of copulas under the survival transform. We close by a synthetic application to homogeneous copulas.

2. Preliminaries: Copulas and survival copulas

As mentioned above, the framework is restricted to the bivariate case. It is not a loss of generality but it is a nice field to introduce results susceptible to be generalized with some further cautions to higher dimensions. Let $I^2 = [0,1] \times [0,1]$ be the unit closed square.

Definition 1. *A copula C is a bifunction on I^2 into $I = [0,1]$ which satisfies the following conditions for all u, v, u_1, v_1, u_2, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$:*

1. *Vanishing property on borders:* $C(0,v) = C(u,0) = 0$.
2. *Uniform margins:* $C(1,v) = v$ *and* $C(u,1) = u$.
3. *the 2-increasing property:* $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$.

It is well known that all bivariate copulas are framed between two bifunctions denoted W and M given by $W(u,v) = \max(u+v-1, 0)$ and $M(u,v) = \min(u,v)$.

Precisely, we have

$$\forall (x,y) \in I^2: \quad W(x,y) \leq C(x,y) \leq M(x,y).$$

In the literature, the bifunctions W and M are called respectively the lower and upper Fréchet-Hoeftding bounds of the set of all copulas. It is worth to recall that the fact W is a copula characterize uniquely bivariate copulas since for all $n \geq 3$, the third item in definition 1 fails in general. In a probabilistic and statistical points of view, Fréchet-Hoeftding bound M models the co-monotonicity of random or empirical variables X and Y "coupled" by M while the lower bound W models anti-monotonicity of X and Y. At the middle point, the copula $\Pi: \ (x,y) \in I^2 \mapsto xy$ models the complete independence between the two variables. For soft independence cases, one may consider small perturbations of Π as treated partially at the end of section 2 of the current paper and detailed in [16].

Many copulas may be associated to a given one, among others *co-copula*, *transpose copula*, *scaled copula*...etc. We are concerned here with *survival copula*. Let us make precise this latter notion which will be the hard core of the current paper

Definition 2. *Let (X,Y) be a random vector with copula C, joint distribution function H and with marginal distribution functions F, G, respectively. The marginal survival functions $\bar{F}$, $\bar{G}$ and joint survival function $\bar{H}$ of the vector (X,Y) are given by $\bar{F}(x) = P[X > x]$, $\bar{G}(y) = P[Y > y]$ and $\bar{H}(x,y) = P[X > x, Y > y]$ respectively. The function $\widehat{C}$ which joins (or couples) $\bar{H}$ to its survival marginal functions $\bar{F}$ and $\bar{G}$ is called the survival copula associated to the initial copula C.*

It is not hard to prove that $\widehat{C}$ is indeed a copula. In addition C and $\widehat{C}$ satisfy

$$\forall (u,v) \in I^2: \quad \widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

The aim of this paper is to study some properties of the survival copula. In particular, a big care and attention will be devoted to generators of such copulas when they are archimedean.

Definition 3. *An Archimedean copula is a function C from I^2 to I given by $C(u,v) = \varphi^{(-1)}(\varphi(u) + \varphi(v))$, where φ (the generator of C) is a continuous strictly decreasing convex function from I to $[0,+\infty]$ such that $\varphi(1) = 0$, and where $\varphi^{(-1)}$ denotes the "pseudo-inverse" of φ*

$$\begin{cases} \varphi^{(-1)}(t) = \varphi^{-1}(t) \text{ for } t \in [0, \varphi(0)] \\ \varphi^{(-1)}(t) = 0 \text{ for } t \geq \varphi(0) \end{cases}$$

When $\varphi(0) = \infty$, φ and C are said to be strict (and $\varphi^{(-1)} = \varphi^{-1}$); when $\varphi(0) < \infty$, φ and C are said non-strict.

The nomenclature goes back to a classical result which looks like the known archimedean property in *real analysis* stated in [4, Theorem 4.3.1].
Before giving a characterization of this important class of copulas, we explain briefly the benefit to have at its disposal the generator ϕ in terms of ratios which summarize and describe the dependence between copula margins. We allude here to τ of Kendall and ρ of Spearman coefficients known to be the two most common

measures of dependence. For a continuous and analytically smooth copula C, these coefficients may be written in terms of the copula as follows:

$$\tau = 4 \iint_{I^2} C(u,v) dC(u,v) - 1,$$

and

$$\rho = 12 \iint_{I^2} uv dC(u,v) - 3 = 12 \iint_{I^2} C(u,v) du dv - 3$$

We give here, as illustration of the importance of archimedean property the following classical and technical tool to determine Kendall's τ coefficient:

Proposition 1. *For random variables X and Y with an Archimedean copula C generated by ϕ. The population version τ_C of Kendall's τ for X and Y is given by*

$$\tau_C = 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt.$$

Now we characterize the archimedean property in the following proposition:

Proposition 2. *An archimedean copula C satisfies the following statements:*

(1) *C is associative meaning $(\forall u, v, w \in I,\ C\left(C(u,v), w\right) = C\left(u, C(v,w)\right)$.*
(2) *$\forall u \in (0,1),\ \delta_C(u) < u$ where $\delta_C(u) = C(u,u)$.*

Remark 1.
(1) *Consider the Farlie-Gumbel-Morgenstern parametrized copulas $K_\theta = uv + \theta uv(1-u)(1-v)$. To highlight the importance of associativity, it is enough to mention that family of copulas K_θ is not archimedean, except for $\theta = 0$ for which $K_\theta = \Pi$. The lack of associativity is an easy efficient tool to establish the result as done in* [4, Page 131].
(2) *The survival copula of an archimedean one satisfies the property* 2 *since it is itself a copula but it is not in general associative.Thus the property of being archimedean is not necessary preserved by the survival copula.*
(3) *If a copula is associative and satisfies the second item in Proposition 2 then it is an archimedean one. For the proof see* [10].

The last point in the remark above is crucial since it seems to be a source of confusion in the literature. For example, Spreuw in his paper entitled "Archimedean copulas derived from utility functions" (see [7]) tried to determine the survival copula generator regardless of its existence. He considers any survival archimedean copula as also an archimedean one. It turns out that this result which seems obvious is far for being true as an immediate consequence of the following purposes:

Let C be an archimedean copula with the generator φ then for $u \in (0,1)$ we have

$$\begin{array}{rcl} \delta_{\widehat{C}}(u) & = & 2u - 1 + \delta_C(u) \\ & < & 2u - 1 + 1 - u \\ & < & u. \end{array}$$

So the first condition in the converse statement of Proposition 2 is always satisfied for all survival copula which are different from the upper bound M. Unfortunately, the second item is not so trivial. In fact, we establish the following result as a corrigendum

Proposition 3. *The copula $\frac{\Pi}{\sum - \Pi}$: $(u,v) \mapsto \frac{uv}{u+v-uv}$ is archimedean but its survival copula is not.*

Proof. The fact that copula $C = \frac{\Pi}{\sum - \Pi}$ is archimedean is easy to verify since the function $\varphi(t) = \frac{1}{t} - 1$ serves as an obvious generator of C.
The survival copula of C is given by

$$\begin{aligned} \widehat{C}(u) &= u + v - C(1-u, 1-v) \\ &= u + v - 1 - \tfrac{(1-u)(1-v)}{1-u+1-v-(1-u)(1-v)} \\ &= \tfrac{uv(2-u-v)}{1-uv}. \end{aligned}$$

We have

$$\widehat{C}\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{2 - \frac{5}{6}}{5} = \frac{7}{30}$$

Then

$$\widehat{C}\left(\widehat{C}\left(\frac{1}{2}, \frac{1}{3}\right), \frac{1}{4}\right) = \widehat{C}\left(\frac{7}{30}, \frac{1}{4}\right) = \frac{637}{6780}$$

On the other hand

$$\widehat{C}\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{2 - \frac{7}{12}}{11} = \frac{17}{132}$$

$$\widehat{C}\left(\frac{1}{2}, \widehat{C}\left(\frac{1}{3}, \frac{1}{4}\right)\right) = \widehat{C}\left(\frac{1}{2}, \frac{17}{132}\right) = \frac{3077}{32604}$$

So

$$\widehat{C}\left(\widehat{C}\left(\frac{1}{2}, \frac{1}{3}\right), \frac{1}{4}\right) \neq \widehat{C}\left(\frac{1}{2}, \widehat{C}\left(\frac{1}{3}, \frac{1}{4}\right)\right)$$

Thus the copula $\widehat{C}$ is not associative hence it is not archimedean.

□

3. Main result

Results in previous section lead to two natural questions:

Question 1. *What are archimedean copulas for which the survival copulas remain archimedean?*

and

Question 2. *If a copula C is a member of a parameterized family C_α, is it the same for its survival copula?*

We start by treating the first question. Taking profit of the converse property stated by Ling [10] and taken up by Kraus [11] and [12] which restricts the archimedean property to a simple question of associativity, Alsina et.al in [13] characterize copulas which are simultaneously, with their survival ones, associative. let us make more precise this important result. To this end, it will be fruitful to locate the notion of *archimedean copulas* in a general framework of *associative functions*. The natural bridge is the well known Aczel's Theorem for which we recall an adapted version

Theorem 2 (Aczel's representation theorem)**.** *Let $I = [0,1]$ and consider the function $C : I^2 \longrightarrow J$ which is associative, continuous in each argument (ie $x \mapsto C(x,y)$ and $y \mapsto C(x,y)$ are continuous), and cancellative on I^2 if and only if it admits the representation*

$$C(x,y) = \phi^{-1}(\phi(x) + \phi(y)),$$

where $\phi : I \longrightarrow \mathbb{R}$ *is continuous and strictly monotonic.*

For an original and complete version of Aczel's representation theorem, one may see [13, Theorem 2.7.1, page 82]. It is worth to mention that a bifunction is *cancellative* means simply that fixing an argument, the partial mono-argument is injective. Precisely

$$\text{C is cancellative} \iff \text{the mappings } x \mapsto C(x,y) \text{ and } y \mapsto C(x,y) \text{ are injective.}$$

The transcription of *cancellability* in terms of archimedean copulas is equivalent to existence of a strict generator. When this hypothesis is satisfied, the key to ensure archimedean property of a given copula is the *associativity*. This yields a simple characterization of copulas C answering the question 1, ie for which we have:

$$C \quad \text{is archimedean} \iff \widehat{C} \quad \text{is archimedean.}$$

At this state, it is enough to recall the following theorem (see [13, Theorem 3.1.2, page 104]).

Theorem 3. *A copula C and its survival copula $\hat{C}$ are simultaneously associative if and only if C is a member of the family C_α below or an ordinal sum of members of this family.*

The family C_α is defined for all real $\alpha \neq 0$ by

$$C_\alpha(x,y) = -\frac{1}{\alpha}\ln[1 + \frac{(e^{-\alpha x}-1)(e^{-\alpha y}-1)}{(e^{-\alpha}-1)}]$$

and the *comprehensive* limits (see [4]) for extremal values of the parameter α:

$$C_{-\infty} = W \quad C_0 = \pi \quad C_{+\infty} = M.$$

The theorem above gives a satisfying answer to the question 1 since archimedean copulas are exactly those which are associative with diagonal entirely under the first bisector. This explains especially the counter example given above. In fact, it is easy to verify that the copula $\frac{\Pi}{\sum - \Pi}$ does not belong to C_α family.

Comment 1. *Mc Neil and J. Neslehovà in their attempt to characterize multivariate archimedean copulas have proved that this class of copulas coincides exactly with the class of survival copulas of $d-$dimensional l^1-norm symmetric distributions that does not affect any weight to the origin. This latter type of distributions is known to be* simplex distributions. *For more information and developments, one may see* [14] *and* [15].

Let us now come back to question 2. To make a first idea on survival copula expression, we present a list, far to be exhaustive, of copulas and their survival distributions

The family of Ali Mikhael Haq

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)} \; \theta \in [-1,1]$$

and its survival copula is

$$\widehat{C}(u,v) = u + v - 1 + \frac{(1-u)(1-v)}{1-\theta uv}$$

- The Gumbel-Hougaard family

$$C(u,v) = exp\left[-\left((-\ln u)^{\theta} + (-\ln v)^{\theta}\right)^{\frac{1}{\theta}}\right]$$

and its survival

$$\widehat{C}(u,v) = u + v - 1 + exp\left[-\left((-\ln(1-u))^{\theta} + (-\ln(1-v))^{\theta}\right)^{\frac{1}{\theta}}\right]$$

The Frank family

$$C(u,v) = -\frac{1}{\theta}\left(\ln(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1})\right)$$

Its survival is

$$\widehat{C}(u,v) = u + v - 1 - \frac{1}{\theta}\left(\ln(1 + \frac{(e^{-\theta(1-u)}-1)(e^{-\theta(1-v)}-1)}{e^{-\theta}-1})\right)$$

These examples show that, in general, the copula and its survival distribution does not belong to the same family.
Although this negative answer to question 2, there are parametrized classes of copulas which are stable under the survival transform. A wide class of a particular interest is the one of $\Pi-$ perturbed copulas. For details and more general expounding on bivariate copulas generated by perturbations, we refer to [16].
For our purpose, we consider just the perturbations of $\Pi-$copula in the particular form $C_\alpha(x,y) = \Pi(x,y) + \alpha f(x)f(y) = xy + \alpha f(x)f(y)$. To highlight the role of perturbation function f, C_α is denoted Π_f. In the literature, this family is known to be *Amblard-Girard* copulas (see among others [17]). It is easy to prove that for every member of this family, the corresponding survival copula remains of Amblard-Girard type. The following calculation shows this statement

$$\begin{aligned}
\widehat{C}(x,y) &= x + y + C(1-x, 1-y) \\
&= x + y - 1 + (1-x)(1-y) + \alpha f(1-x)f(1-y)) \\
&= \Pi(x,y) + \alpha\widehat{f}(x)\widehat{f}(y).
\end{aligned}$$

where we have put $\widehat{f}(t) = f(1-t)$. So the survival copula belongs to Amblard-Girard and $\widehat{C} = \Pi_{\widehat{f}}$.

Question 2 is a kind of global stability of a given family of copulas under survival transform. Let us now examine the punctual stability. Let $\mathcal{S}$ denote the set of all copula C satisfying $C = \widehat{C}$. We postpone the determination elements of $\mathcal{S}$ in details to a future work. We give instead a topological property of $\mathcal{S}$

Theorem 4. *The set $\mathcal{S}$ of all 2-copulas which equal to their survival ones is closed and convex.*

Proof. Let (C_n) be a sequence of elements of $\mathcal{S}$ which converges uniformly to a 2-copulas C.
Using the expression $\widehat{C}_n(u,v) = u + v - 1 + C_n(1-u, 1-v)$ we have that $(\widehat{C}_n)$ converges to $\widehat{C}$. The uniqueness of the limit gives $\widehat{C} = C$. Thus $C \in \mathcal{S}$

For the convexity, let $(C_k)_{1\leq k\leq p}$ be a finite family of elements of $\mathcal{S}$ and $(\alpha_k)_{1\leq k\leq n}$ be a family of strictly positive reals such that $\sum_{k=1}^{n}\alpha_k = 1$. Then

$$\left(\widehat{\sum_{k=1}^{n}\alpha_k C_k}\right)(u,v) = u+v-1+\sum_{k=1}^{n}\alpha_k C_k(1-u,1-v) = \sum_{k=1}^{n}\alpha_k \widehat{C_k}(u,v)$$

□

Corollary 1. *All members of a family of Frchet-Mardia are in $\mathcal{S}$.*

Proof. It is enough to recall that Frchet-Mardia family consists on convex combinations of the three copulas W,Π and M which are in $\mathcal{S}$ and conclude with theorem 4 above. □

4. Application

Here we give an application to homogeneous copulas. Let us first recall the definition as adopted in [4].

Definition 4. *Let λ a strictly positive number and $\alpha \geq 0$. A copula C is homogeneous of degree α if for all $(x,y)\in I^2$, and all $\lambda \in I$ we have $C(\lambda x,\lambda y) = \lambda^{\alpha}C(x,y)$.*

The definition is a particular case of homogeneous several entries functions. It is then possible to characterize homogeneous smooth (differentiable) copulas via the following classical Euler's equivalence

$$C \text{ is a homogeneous copula} \quad \Longleftrightarrow \quad \forall(x,y)\in I^2 : x\frac{\partial C}{\partial x}(x,y) + y\frac{\partial C}{\partial y} = \alpha C(x,y).$$

Unfortunately such attempts become less interesting after this important theorem which states that Cuadras-Augé family is the unique class homogeneous copulas

Theorem 5. [4, Theorem 3.4.2] *A copula C is homogeneous of degree α if and only if C is a member of the $\theta-$parametrized Cuadras-Augé family*

$$C_\theta(x,y) = M^\theta(x,y)\Pi^{1-\theta}(x,y) = [\min(x,y)]^\theta(xy)^{1-\theta}.$$

More precise statement consists in saying that the degree is exactly $\alpha = 2-\theta$. Thus the degree of homogeneity satisfies $1 \leq \alpha \leq 2$. This remarkable result is a main tool to characterize homogeneous copulas stable with survival transform. Indeed a combination of Theorem 3 and Theorem 5 leads to the main characterization

Proposition 4. *Except the independence copula Π, there is no archimedean homogeneous copula which coincides with its survival copula.*

Proof. Let C be such copula. From theorem 4, there is a parameter θ such that C is a $\theta-$geometric mean of M and Π. That means for each $(x,y)\in I^2$, $C(x,y) = M^\theta(x,y)\Pi^{1-\theta}(x,y)$. To conclude, it will be enough to prove that this latter copula differs from its survival one for all $\theta \in [0,1[$. Assume the converse. So since on the diagonal we will have for all $x \in I$:

$$\delta_{\widehat{C}}(x) = 2x-1+(1-x)^{2-\theta},$$

then

$$\delta_{\widehat{C}}(x) = \delta_C(x) \quad \text{is equivalent to} \quad x^{2-\theta} = 2x-1+(1-x)^{2-\theta}.$$

It is elementary to verify that this equivalence is false except for extreme values of θ (i.e $\theta = 0$ or $\theta = 1$). But for $\theta = 1$, $C_\theta = M$ which is not archimedean and for $\theta = 0$ one retrieves easily the independence copula Π. This shows the claim. $\square$

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