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KRASNOSELSKII-TYPE APPROXIMATION SOLVABILITY OF A GENERALIZED CAYLEY INCLUSION PROBLEM IN SEMI-INNER PRODUCT SPACE

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Abstract. In this paper, we introduce and study a generalized Yosida approximation operator and generalized Cayley operator associated to $H(\cdot,\cdot)$-monotone operator and discuss some of their properties in semi-inner product spaces. An example is constructed in support of the various results proved. As an application we consider a generalized Cayley inclusion problem in semi-inner product spaces which is more general than the variational inclusion problem. Using the generalized resolvent operator, generalized Yosida approximation operator and generalized Cayley operator we develop an iterative algorithm to approximate the solution of generalized Cayley inclusion problem. Furthermore, an existence and convergence result is proved.

1. Introduction

Variational inclusions are useful and important extensions and generalizations of the variational inequalities with a wide range of applications in industry, mathematical finance, economics, decisions sciences, ecology, mathematical and engineering sciences. One of the important aspects in the theory of variational inequalities is the approximation solvability of the solution. In the recent past several researchers studied the approximation solvability of some important classes of variational inequalities. Among several other methods, the method based on the resolvent operator technique has been widely used to solve variational inclusions. It is known that the monotonicity of the underlying operator plays a prominent role in solving variational inclusion problems. In 2003, Fang and Huang [9] introduced and studied a new class of variational inclusions involving H-monotone operators in a Hilbert space. Using resolvent operator, they proposed an algorithm for solving the associated class of variational inclusions. Since then a number of researchers investigated variant forms of H-monotone operators for solving variational inclusions in different spaces, see for example [25, 19, 31, 14, 4, 20, 22, 35, 8, 10, 30, 33, 34, 6].

In 2014, Sahu et al. [29] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces. Moreover,

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they constructed an iterative algorithm for approximating the solution for this class of problems involving A-monotone and H-monotone operators by using the generalized resolvent operator technique. In the sequel, Kim and Bhat [16] and Bhat and Zahoor [5] considered and studied systems of variational inclusions in semi-inner product spaces using several classes of H-monotone operators. Recently, Kazmi and Furkan [15] considered and studied a system of split variational inequality problems in semi-inner product spaces.

It is also known that the monotone operators in abstract spaces can be regularized into single-valued Lipschitzian monotone operator through a process known as Yosida approximation, see [18, 27, 2, 7]. The Yosida approximation operators and the Cayley operators are advantageous to estimate the solution of variational inclusion problems using resolvent operators. In the recent past, many authors applied Yosida approximation operators and the Cayley operators to solve variational inclusions and system of variational inclusion problems, see [1, 7, 13, 18, 23, 26, 3] and the related references therein.

Motivated and inspired by the above works, in this paper we introduce and study a generalized Yosida approximation operator and generalized Cayley operator associated with $H(\cdot,\cdot)$-monotone operator and discuss some of their properties in semi-inner product spaces. Further, we construct an example in support of the various results proved for these classes of operators. As an application we consider a generalized Cayley inclusion problem in semi-inner product spaces which is more general than the variational inclusion problem. Using the generalized resolvent operator, generalized Yosida approximation operator and generalized Cayley operator we develop an iterative algorithm to approximate the solution of generalized Cayley inclusion problem. Furthermore, we give the existence and convergence analysis of the class of generalized Cayley inclusion problems.

2. Preliminaries

Throughout the paper, unless otherwise stated, X denotes a 2-uniformly smooth space equipped with norm $\|\cdot\|$ and semi-inner product $[\cdot,\cdot]$. 2^X denotes the power set of a nonempty set X and $CB(X)$ denotes the the family of all nonempty closed and bounded subsets of X. The metric induced by the norm is denoted by d and the Hausdörff metric on $CB(X)$ by $\mathcal{D}(\cdot,\cdot)$.

First, we recall some known definitions and results which are important to achieve the goal of this paper.

Definition 2.1 [21]**.** Let X be a vector space over the field $\mathbb{F}$ of real or complex numbers. A functional $[\cdot,\cdot] : X \times X \to \mathbb{F}$ is called a semi-inner product if it satisfies the following conditions:

(i) $[x+y,z] = [x,z] + [y,z],\ \forall\ x,y,z \in X;$
(ii) $[\lambda x,y] = \lambda[x,y],\ \forall\ \lambda \in \mathbb{F}$ and $x,y \in X;$
(iii) $[x,x] > 0,$ for $x \neq 0;$
(iv) $\big|[x,y]\big|^2 \leq [x,x][y,y].$

The pair $(X,[\cdot,\cdot])$ is said to be a semi-inner product space.

It can be seen that $\|x\| = [x,x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can define semi-inner product in infinitely many ways. Giles [11] had proved that if the underlying space X is a uniformly convex smooth Banach space, then it is possible to define a unique semi-inner product. Also the unique semi-inner product has the following nice properties:

(i) y is orthogonal to x if and only if $[x, y] = 0$, that is if and only if $\|y\| \leq \|y + \lambda x\|$, for all scalers λ.

(ii) **Generalized Riesz representation theorem:** If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y]$, for all $x \in X$.

(iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $\text{Re}[y, x + \lambda y] \to \text{Re}[y, x]$ as $\lambda \to 0$.

Since the sequence space $\ell^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces, we can define a semi-inner product on these spaces, uniquely.

Example 2.1 [29]**.** The real Banach space ℓ^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x, y] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \ x, y \in \ell^p.$$

Example 2.2 [11]**.** The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn\,(g(x))\; d\mu, \ f, g \in L^p.$$

Definition 2.2 [32]**.** Let X be a real Banach space. Then

(i) The *modulus of smoothness* of X is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = t, t > 0 \right\};$$

(ii) X is said to be *uniformly smooth*, if $\lim\limits_{t \to 0} \dfrac{\rho_X(t)}{t} = 0$;

(iii) X is said to be *q-uniformly smooth*, if there exists a positive real constant c such that $\rho_X(t) \leq ct^q, \ q > 1$.

(iv) X is said to be 2-*uniformly smooth*, if there exists a positive real constant c such that $\rho_X(t) \leq ct^2$.

Lemma 2.3 [32]**.** Let $q > 1$ be a real number. Then the following statements are equivalent:

(i) X is 2-uniformly smooth;

(ii) There is a constant $c > 0$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, f_x \rangle + c\|y\|^2, \ \forall\, x, y \in X, \tag{1}$$

where $f_x \in J(x)$ and $J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}$, is the *normalized duality mapping*.

Remark 2.4. Every normed linear space X is a semi-inner product space [21]. In fact, by Hahn-Banach theorem, for each $x \in X$, there exists atleast one functional

$f_x \in X^*$ such that $\langle x, f_x \rangle = \|x\|^2$. Given any such mapping f from X into X^*, we can show that $[y, x] = \langle y, f_x \rangle$ defines a semi-inner product. Hence inequality (1) can be written as

$$\|x + y\|^2 \leq \|x\|^2 + 2[y, x] + c\|y\|^2, \ \forall x, y \in X. \tag{2}$$

The constant c is called constant of smoothness of X and is chosen with best possible minimum value.

Example 2.3. The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,

$$\|x + y\|^2 \leq \|x\|^2 + 2[y, x] + (p - 1)\|y\|^2,$$

where $(p - 1)$ is the constant of smoothness of L^p.

Definition 2.5 [24]**.** The *Hausdorff metric* $\mathcal{D}(\cdot, \cdot)$ on $CB(X)$ is defined by

$$\mathcal{D}(P, Q) = \max\left\{\sup_{u \in P} \inf_{v \in Q} d(u, v), \ \sup_{v \in Q} \inf_{u \in P} d(u, v)\right\}, \ P, Q \in CB(X),$$

where $d(\cdot, \cdot)$ is the induced metric on X.

Definition 2.6 [24]**.** A set-valued mapping $P : X \to CB(X)$ is said to be μ-$\mathcal{D}$-Lipschitz continuous, if there exists a constant $\mu > 0$ such that

$$\mathcal{D}\big(P(x), P(y)\big) \leq \mu\|x - y\|, \ \forall \ x, y \in X.$$

Definition 2.7. Let X be a 2-uniformly smooth Banach space. Let $A, B, T : X \to X$ and $H : X \times X \to X$ be single-valued mappings. Then

(i) T is said to be *monotone*, if

$$[Tx - Ty, x - y] \geq 0, \ \forall x, y \in X;$$

(ii) T is said to be *strictly monotone*, if it is monotone and equality holds if and only if $x = y$;

(iii) T is said to be *r-strongly monotone*, if there exists a constant $r > 0$ such that

$$[Tx - Ty, x - y] \geq r\|x - y\|^2, \ \forall \ x, y \in X;$$

(iv) T is said to be *m-relaxed monotone*, if there exists a constant $m > 0$ such that

$$[Tx - Ty, x - y] \geq (-m)\|x - y\|^2, \ \forall \ x, y \in X;$$

(v) T is said to be *s-Lipschitz continuous*, if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \ \forall \ x, y \in X;$$

(vi) $H(A, \cdot)$ is said to be α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$[H(Ax, u) - H(Ay, u), x - y] \geq \alpha\|x - y\|^2, \ \forall \ x, y, u \in X;$$

(vii) $H(\cdot, B)$ is said to be β-relaxed monotone, if there exists a constant $\beta > 0$ such that

$$[H(u, Bx) - H(u, By), x - y] \geq -\beta\|x - y\|^2, \ \forall \ x, y, u \in X;$$

(viii) $H(A,B)$ is said to be $\alpha\beta$-symmetric monotone, if $H(A,\cdot)$ is α-strongly monotone and $H(\cdot,B)$ is β-relaxed monotone with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y,\ \forall\ x,y \in X$;

(ix) $H(A,\cdot)$ is said to be τ_1-Lipschitz continuous, if there exists a constant $\tau_1 > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \leq \tau_1\|x-y\|,\ \forall\ x,y,u \in X;$$

(x) $H(\cdot,B)$ is said to be τ_2-Lipschitz continuous, if there exists a constant $\tau_2 > 0$ such that

$$\|H(u,Bx) - H(u,By)\| \leq \tau_2\|x-y\|,\ \forall\ x,y,u \in X.$$

Definition 2.8. Let X be a real 2-uniformly smooth Banach space. A set-valued mapping $M : X \to 2^X$ is said to be

(i) *monotone*, if

$$[u-v, x-y] \geq 0,\ \forall\ x,y \in X, u \in M(x), v \in M(y);$$

(ii) *r-strongly monotone*, if there exists a constant $r > 0$ such that

$$[u-v, x-y] \geq r\|x-y\|^2,\ \forall\ x,y \in X, u \in M(x), v \in M(y);$$

(iii) *m-relaxed monotone*, if there exists a constant $m > 0$ such that

$$[u-v, x-y] \geq (-m)\|x-y\|^2,\ \forall\ x,y \in X, u \in M(x), v \in M(y).$$

Definition 2.9. Let $A, B : X \to X, H : X \times X \to X$ be single-valued mappings. A set-valued mapping $M : X \to 2^X$ is said to be $H(\cdot,\cdot)$-monotone with respect to A and B (or simply $H(\cdot,\cdot)$-monotone in the sequel), if M is monotone and $(H(A,B) + \lambda M)(X) = X$, for all $\lambda > 0$.

Theorem 2.10. Let $A, B : X \to X, H : X \times X \to X$ be single-valued mappings such that $H(A,B)$ is $\alpha\beta$-symmetric monotone and $M : X \to 2^X$ be an $H(\cdot,\cdot)$-monotone operator. Then the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued.

Proof. For any given $u \in X$, let $x, y \in (H(A,B) + \lambda M)^{-1}$. It follows that

$$v_x = \frac{1}{\lambda}\big(u - H(Ax,Bx)\big) \in M(x) \text{ and } v_y = \frac{1}{\lambda}\big(u - H(Ay,By)\big) \in M(y).$$

Using the monotonicity of M and H, we have

$$\begin{aligned}
0 \leq [v_x - v_y, x-y] &= \frac{1}{\lambda}\big[H(Ay,By) - H(Ax,Bx), x-y\big] \\
&= -\frac{1}{\lambda}\big[H(Ax,Bx) - H(Ay,By), x-y\big] \\
&= -\frac{1}{\lambda}\big\{\big[H(Ax,Bx) - H(Ay,Bx), x-y\big] \\
&\quad + \big[H(Ay,Bx) - H(Ay,By), x-y\big]\big\} \\
&\leq -\frac{1}{\lambda}\left\{\alpha\|x-y\|^2 - \beta\|x-y\|^2\right\} \\
&= -\frac{1}{\lambda}(\alpha-\beta)\|x-y\|^2 \\
&\leq 0,\ \text{for } \alpha > \beta.
\end{aligned}$$

Thus, we have $x = y$ and so $(H(A,B) + \lambda M)^{-1}$ is single-valued. This completes the proof.

Based on Theorem 2.10, we define the generalized resolvent operator for $H(\cdot,\cdot)$-monotone operator M as follows:

Definition 2.11. Let $A, B : X \to X, H : X \times X \to X$ be single-valued mappings such that $H(A,B)$ is $\alpha\beta$-symmetric monotone and $M : X \to 2^X$ be an $H(\cdot,\cdot)$-monotone operator. Then the generalized resolvent operator $\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda} : X \to X$ is defined by

$$\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x) = \big(H(A,B) + \lambda M\big)^{-1}(x),\ \forall\ x \in X. \tag{3}$$

Theorem 2.12. The generalized resolvent operator $\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda} : X \to X$ is $\frac{1}{\alpha-\beta}$-Lipschitz continuous, that is

$$\left\|\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x) - \mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(y)\right\| \leq \frac{1}{\alpha-\beta}\|x-y\|,\ \forall\ x,y \in X.$$

Proof. For any $x, y \in X$. It follows from (3) that

$$\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x) = (H(A,B) + \lambda M)^{-1}(x) \text{ and } \mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(y) = (H(A,B) + \lambda M)^{-1}(y).$$

This implies that

$$v^*_x = \frac{1}{\lambda}\left[x - H\left(A(\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x)), B(\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x))\right)\right] \in M\left(\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x)\right)$$

and

$$v^*_y = \frac{1}{\lambda}\left[y - H\left(A(\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(y)), B(\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(y))\right)\right] \in M\left(\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(y)\right).$$

For the sake of brevity, let

$$Ux = \mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x) \text{ and } Uy = \mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(y).$$

Since M is monotone,

$$\begin{aligned}
&\left[v^*_x - v^*_y, Ux - Uy\right] \\
&\quad = \frac{1}{\lambda}\left[(x - H(A(Ux), B(Ux))) - (y - H(A(Uy), B(Uy))), Ux - Uy\right] \\
&\quad = \frac{1}{\lambda}\left[x - y - \{H(A(Ux), B(Ux)) - H(A(Uy), B(Uy))\}, Ux - Uy\right] \geq 0.
\end{aligned}$$

Therefore, we have

$$[x - y, Ux - Uy] \geq [H(A(Ux), B(Ux)) - H(A(Uy), B(Uy)), Ux - Uy].$$

Since $H(A,B)$ is $\alpha\beta$-symmetric monotone, it follows that

$$\begin{aligned}
\|x-y\| \cdot \|Ux - Uy\| &\geq [x - y, Ux - Uy] \\
&\geq [H(A(Ux), B(Ux)) - H(A(Uy), B(Ux)), Ux - Uy] \\
&\quad + [H(A(Uy), B(Ux)) - H(A(Uy), B(Uy)), Ux - Uy] \\
&\geq \alpha\|Ux - Uy\|^2 - \beta\|Ux - Uy\|^2 \\
&= (\alpha - \beta)\|Ux - Uy\|^2,
\end{aligned}$$

and so

$$\left\|\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\|\leq\frac{1}{\alpha-\beta}\|x-y\|,\ \forall\ x,y\in X.$$

This completes the proof.

Definition 2.13. The generalised Yosida approximation operator $\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}:X\to X$ associated with $H(\cdot,\cdot)$-monotone operator M is defined as:

$$\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x)=\frac{1}{\lambda}\left(H(A,B)-\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}\right)(x),\ \forall\ x\in X\text{ and }\lambda>0,\tag{4}$$

where $\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}$ is defined by (3).

Definition 2.14. The generalised Cayley operator $\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}:X\to X$ associated with $H(\cdot,\cdot)$-monotone operator M is defined as:

$$\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x)=\left(2\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}-H(A,B)\right)(x),\ \forall\ x\in X\text{ and }\lambda>0,\tag{5}$$

where $\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}$ is defined by (3).

In view of Theorem 2.10, $\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}$ and $\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}$ are both sinlge-valued.

Theorem 2.15. Let $H(A,\cdot)$ be τ_1-Lipschitz continuous and $H(\cdot,B)$ be τ_2-Lipschitz continuous. Then the generalised Yosida approximation operator defined by (4) is η-Lipschitz continuous, where $\eta=\frac{1}{\lambda}\left(\tau_1+\tau_2+\frac{1}{\alpha-\beta}\right)$.

Proof. Let $x,y\in X$, then using the fact that $H(A,\cdot)$ is τ_1-Lipschitz continuous and $H(\cdot,B)$ is τ_2-Lipschitz continuous and Theorem 2.12, we have

$$\begin{aligned}
&\left\|\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\|\\
&\quad=\frac{1}{\lambda}\left\|\left\{H(Ax,Bx)-\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x)\right\}-\left\{H(Ay,B(y))-\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\}\right\|\\
&\quad\leq\frac{1}{\lambda}\left\{\|H(Ax,Bx)-H(Ay,B(y))\|+\left\|\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\|\right\}\\
&\quad\leq\frac{1}{\lambda}\big\{\,\|H(Ax,Bx)-H(Ay,Bx)\|\\
&\qquad+\|H(Ay,Bx)-H(Ay,B(y))\|+\frac{1}{\alpha-\beta}\|x-y\|\big\}\\
&\quad\leq\frac{1}{\lambda}\left(\tau_1+\tau_2+\frac{1}{\alpha-\beta}\right)\|x-y\|\\
&\quad=\eta\|x-y\|,
\end{aligned}$$

that is,

$$\left\|\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\|\leq\eta\|x-y\|,\ \forall\ x,y\in X.$$

Thus, the generalized Yosida approximation operator $\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}$ is Lipschitz continuous.

Theorem 2.16. Let $H(A,B)$ be $\alpha\beta$-symmetric monotone. Then the generalized Yosida approximation operator $\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}$ is δ-strongly monotone, where $\delta = \dfrac{(\alpha-\beta)^2-1}{\lambda(\alpha-\beta)}$.

Proof. For any $x, y \in X$, using Theorem 2.12 and $\alpha\beta$-symmetric monotonicity of $H(A,B)$, we have

$$\begin{aligned}
&\left[\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(y), x-y\right] \\
&\quad = \frac{1}{\lambda}\left[\left(H(Ax,Bx) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x)\right) - \left(H(Ay,By) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right), x-y\right] \\
&\quad = \frac{1}{\lambda}\left\{\left[H(Ax,Bx) - H(Ay,By), x-y\right] - \left[\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y), x-y\right]\right\} \\
&\quad \geq \frac{1}{\lambda}\Big\{\left[H(Ax,Bx) - H(Ay,Bx), x-y\right] + \left[H(Ay,Bx) - H(Ay,By), x-y\right] \\
&\qquad - \left\|\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| \|x-y\|\Big\} \\
&\quad \geq \frac{1}{\lambda}\left\{\alpha\|x-y\|^2 - \beta\|x-y\|^2 - \frac{1}{\alpha-\beta}\|x-y\|^2\right\} \\
&\quad = \left(\frac{(\alpha-\beta)^2-1}{\lambda(\alpha-\beta)}\right)\|x-y\|^2,
\end{aligned}$$

that is,

$$\left[\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(y), x-y\right] \geq \delta\|x-y\|^2.$$

Thus, the generalized Yosida approximation operator $\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}$ is δ-strongly monotone.

Theorem 2.17. Let $H(A,\cdot)$ be τ_1-Lipschitz continuous and $H(\cdot,B)$ be τ_2-Lipschitz continuous. Then the generalized Cayley operator defined by (5) is γ-Lipschitz continuous, where $\gamma = \dfrac{2+(\tau_1+\tau_2)(\alpha-\beta)}{\alpha-\beta}$.

Proof. In view of (5) and Theorem 2.12, we have for any $x, y \in X$

$$\begin{aligned}
\left\|\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| &= \left\|\left(2\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - H(Ax,Bx)\right) - \left(2\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y) - H(Ay,By)\right)\right\| \\
&= \left\|2\left(\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right) - (H(Ax,Bx) - H(Ay,By))\right\| \\
&\leq 2\left\|\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| + \|(H(Ax,Bx) - H(Ay,By)\| \\
&\leq 2\left\|\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| + \|H(Ax,Bx) - H(Ay,Bx)\| \\
&\quad + \|H(Ay,Bx) - H(Ay,By)\| \\
&\leq \frac{2}{\alpha-\beta}\|x-y\| + \tau_1\|x-y\| + \tau_2\|x-y\| \\
&= \left(\frac{2+(\tau_1+\tau_2)(\alpha-\beta)}{\alpha-\beta}\right)\|x-y\|,
\end{aligned}$$

that is,

$$\left\|\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| \leq \gamma\|x - y\|.$$

Thus, the generalized Cayley operator $\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}$ is γ-Lipschitz continuous.

Corresponding to the above concepts and results, we generate the following examples.

Example 2.4. Let $X = \mathbb{R}$ and $A, B : \mathbb{R} \to \mathbb{R}$ be single-valued mappings defined by

$$A(x) = 4x + 5 \text{ and } B(x) = 3x - 2, \ \forall\, x \in \mathbb{R}.$$

Let $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping defined by $H(A(x), B(x)) = A(x) + B(x) = 7x + 3, \ \forall\, x \in \mathbb{R}$. Then for any $u \in \mathbb{R}$, we have

$$\begin{aligned}[H(A(x), u) - H(A(y), u), x - y] &= [A(x) - A(y), x - y] \\ &= 4\|x - y\|^2 \geq 3\|x - y\|^2.\end{aligned}$$

Hence, $H(A, \cdot)$ is 3-strongly monotone and

$$\begin{aligned}[H(u, B(x)) - H(u, B(y)), x - y] &= [B(x) - B(y), x - y] \\ &= 3\|x - y\|^2 \geq -1\|x - y\|^2.\end{aligned}$$

Hence, $H(\cdot, B)$ is 1-relaxed monotone. Thus, $H(A, B)$ is $\alpha\beta$-symmetric monotone with $\alpha = 3$ and $\beta = 1$. Also,

$$\begin{aligned}\|H(A(x), u) - H(A(y), u)\| &= \|A(x) - A(y)\| \\ &= 4\|x - y\| \leq 5\|x - y\|\end{aligned}$$

and

$$\begin{aligned}\|H(u, B(x)) - H(u, B(y))\| &= \|B(x) - B(y)\| \\ &= 3\|x - y\| \leq 4\|x - y\|.\end{aligned}$$

Thus, $H(A, \cdot)$ is τ_1-Lipschitz continuous and $H(\cdot, B)$ is τ_2-Lipschitz continuous with $\tau_1 = 4$ and $\tau_2 = 4$.

Let $M : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued mapping defined by $M(x) = \{5x + 2\}, \ \forall\, x \in \mathbb{R}$. It can be easily verified that M is monotone. Also, for any $x \in \mathbb{R}$ and $\lambda = 1$, we have

$$\left(H(A, B) + \lambda M\right)(x) = H(A(x), B(x)) + M(x) = 12x + 5.$$

Clearly the right hand side of above equation generates the whole space $\mathbb{R}$, i.e.,

$$\left(H(A, B) + \lambda M\right)(\mathbb{R}) = \mathbb{R}.$$

Hence, M is $H(\cdot, \cdot)$-monotone.

Now, for $\lambda = 1$, the resolvent operator, the Yosida approximation operator and the Cayley operator defined by (3), (4) and (5), respectively are given by

$$\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) = \frac{1}{12}(x - 5), \ \mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) = \frac{1}{12}(83x + 41), \ \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) = -\frac{1}{6}(41x + 23).$$

Further,

$$\begin{aligned}\left\|\mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| &= \frac{1}{12}\|x - y\| \\ &\leq \frac{1}{2}\|x - y\|, \text{ where } \frac{1}{\alpha - \beta} = \frac{1}{2}.\end{aligned}$$

This shows that the generalized resolvent operator is $\frac{1}{2}$-Lipschitz continuous. Also,

$$\begin{aligned}\left\|\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| &= \frac{83}{12}\|x-y\| \\ &\leq \frac{19}{2}\|x-y\|, \text{ where } \eta=\left(\tau_1+\tau_2+\frac{1}{\alpha-\beta}\right)=\frac{19}{2}.\end{aligned}$$

This shows that the generalized Yosida approximation operator is $\frac{19}{2}$-Lipschitz continuous and

$$\begin{aligned}\left[\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(y),x-y\right] &= \frac{83}{12}\|x-y\|^2 \\ &\geq \frac{3}{2}\|x-y\|^2, \text{ where } \delta=\frac{(\alpha-\beta)^2-1}{(\alpha-\beta)}=\frac{3}{2}.\end{aligned}$$

Thus the generalized Yosida approximation operator is $\frac{3}{2}$-strongly monotone. Finally,

$$\begin{aligned}\left\|\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x)-\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| &= \frac{41}{6}\|x-y\| \\ &\leq 10\|x-y\|, \text{ where } \gamma=\frac{2+(\tau_1+\tau_2)(\alpha-\beta)}{\alpha-\beta}=10.\end{aligned}$$

This shows that the generalized Cayley operator is 10-Lipschitz continuous.

3. Formulation of problem

Let $A,B:X\to X, H,F:X\times X\to X$ be single valued mappings and $P:X\to CB(X), M:X\to 2^X$ be set-valued mappings. Let $\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}$ be the generalized Cayley operator. We consider the following generalized Cayley variational inclusion problem (in short, GCVIP):

Find $x\in X, u\in P(x)$ such that

$$0\in F(x,u)+\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x)+M(x); \tag{6}$$

(1) By taking $F\equiv 0$ and A-monotonicity of the set-valued mapping M instead of $H(\cdot,\cdot)$-monotonicity, GCVIP (6) reduces to the problem of finding $x\in X$ such that

$$0\in\mathcal{C}_{M,\lambda}^{A}(x)+M(x).$$

This problem was considered and studied by Rais *et al.* [28] in the setting of uniformly smooth Banach spaces.

(2) In case $F\equiv 0$ and $\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}\equiv 0$, then GCVIP (6) reduces to the problem of finding $x\in X$ such that

$$0\in M(x),$$

which is a fundamental and celebrated problem in the theory of optimization and variational inequalities.

We remark that for suitable choices of different mappings and the underlying space of GCVIP (6) includes, as special cases, various classes of variational inclusions and variational inequalities, see for example [12, 16, 29] and the related references therein.

4. Existence of solution, iterative algorithm and convergence analysis

The following lemma is a fixed point formulation GCVIP (6) involving generalized resolvent operator, generalized Yosida approximation operator and generalized Cayley operator defined by (3), (4) and (5), respectively.

Lemma 4.1. Let $x \in X$ and $u \in P(x)$, then (x, u) is a solution of GCVIP (6) if and only if it satisfies the following equation:

$$x = \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)} \left\{ \lambda \left(\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) - F(x,u) \right) + \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) \right\}, \tag{7}$$

where $\lambda, \rho > 0$ is a constant.

Proof. Suppose (7) hold, then using the definitions of resolvent, Yosida and Cayley operators, we have

$$\begin{aligned} & x = \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)} \left\{ \lambda \left(\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) - F(x,u) \right) + \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) \right\} \\ \Longleftrightarrow \quad & x = (H(A,B) + \lambda M)^{-1} \left\{ \lambda \left(\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) - F(x,u) \right) + \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) \right\} \\ \Longleftrightarrow \quad & H(Ax,Bx) + \lambda M(x) \ni H(Ax,Bx) - \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \lambda \left(\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) + F(x,u) \right) \\ & \qquad\qquad + \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) \\ \Longleftrightarrow \quad & 0 \in F(x,u) + \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) + M(x). \end{aligned}$$

Lemma 4.1 along with Nadler [24] allow us to suggest the following Krasnoselskii-type iterative algorithm for finding the approximate solution of GCVIP (6).

Iterative Algorithm 4.2. Given $x_0 \in X, u_0 \in P(x_0)$, compute the sequences $\{x_n\}, \{u_n\}$ by the iterative scheme:

$$x_{n+1} = (1-\kappa)x_n + \kappa \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)} \left[\lambda \left(\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x_n) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x_n) - F(x_n,u_n) \right) + \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x_n) \right];$$

such that

$$u_n \in P(x_n) : \|u_{n+1} - u_n\| \leq \left(1 + (1+n)^{-1}\right) \mathcal{D}\left(P(x_{n+1}), P(x_n)\right).$$

for $n = 0, 1, 2, ...$ and $\kappa \in (0,1)$.

Now we prove the following existence and convergence result for GCVIP (6).

Theorem 4.3. Let $A, B : X \to X, H, F : X \times X \to X$ be single valued mappings such that $H(A, B)$ be $\alpha\beta$-symmetric monotone; $H(A, \cdot)$ be τ_1-Lipschitz continuous and $H(\cdot, B)$ be τ_2-Lipschitz continuous; F be σ_1-Lipschitz continuous in the first argument and σ_2-Lipschitz continuous in the second argument. Let $M : X \to 2^X$ be an be an $H(\cdot, \cdot)$-monotone set-valued mapping and $P : X \to CB(X)$ be μ-$\mathcal{D}$-Lipschitz continuous set-valued mapping. If the following condition is satisfied

$$0 < \varphi = (1-\kappa) + \frac{\kappa\lambda(\alpha-\beta)\left[1 + \gamma + \sqrt{1 - 2\delta + c\eta^2} + (\sigma_1 + \mu\sigma_2)\right] + \kappa}{(\alpha-\beta)^2} < 1, \tag{8}$$

with $1 + c\eta^2 > 2\delta$, where $\gamma = \frac{2+(\tau_1+\tau_2)(\alpha-\beta)}{\alpha-\beta}$, $\delta = \frac{(\alpha-\beta)^2-1}{\lambda(\alpha-\beta)}$, $\eta = \frac{1}{\lambda}\left(\tau_1 + \tau_2 + \frac{1}{\alpha-\beta}\right)$ and c is the constant of smoothness of 2-uniformly smooth Banach space X. Then (x, u) is a solution of GCVIP (6) and the sequences $\{x_n\}, \{u_n\}$ generated by the Iterative Algorithm 4.2 converge strongly to (x, u).

Proof. From Algorithm 4.2 and Theorem 2.12, we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \Big\|\Big\{(1-\kappa)x_n+\kappa\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}\Big[\lambda\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-F(x_n,u_n)\Big)+\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)\Big]\Big\} \\
&\quad -\Big\{(1-\kappa)x_{n-1}+\kappa\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}\Big[\lambda\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})-F(x_{n-1},u_{n-1})\Big) \\
&\quad +\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big]\Big\}\Big\| \\
&\le (1-\kappa)\|x_n-x_{n-1}\|+\frac{\kappa}{\alpha-\beta}\Big\|\lambda\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-F(x_n,u_n)\Big) \\
&\quad -\lambda\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})-F(x_{n-1},u_{n-1})\Big)+\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{R}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big\| \\
&\le (1-\kappa)\|x_n-x_{n-1}\|+\frac{\kappa\lambda}{\alpha-\beta}\Big\|\Big[\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big] \\
&\quad -\Big[\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big]\Big\|+\frac{\kappa\lambda}{\alpha-\beta}\|F(x_n,u_n)-F(x_{n-1},u_{n-1})\| \\
&\quad +\frac{\kappa}{(\alpha-\beta)^2}\|x_n-x_{n-1}\| \\
&= (1-\kappa)\|x_n-x_{n-1}\|+\frac{\kappa\lambda}{\alpha-\beta}\Big\|\Big[(x_n-x_{n-1})-\Big(\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big)\Big] \\
&\quad -\Big[(x_n-x_{n-1})-\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big)\Big]\Big\| \\
&\quad +\frac{\kappa\lambda}{\alpha-\beta}\|F(x_n,u_n)-F(x_{n-1},u_{n-1})\|+\frac{\kappa}{(\alpha-\beta)^2}\|x_n-x_{n-1}\| \\
&\le (1-\kappa)\|x_n-x_{n-1}\|+\frac{\kappa\lambda}{\alpha-\beta}\Big\|\Big[(x_n-x_{n-1})-\Big(\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{C}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big)\Big]\Big\| \\
&\quad +\frac{\kappa\lambda}{\alpha-\beta}\Big\|(x_n-x_{n-1})-\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big)\Big\| \\
&\quad +\frac{\kappa\lambda}{\alpha-\beta}\|F(x_n,u_n)-F(x_{n-1},u_{n-1})\|+\frac{\kappa}{(\alpha-\beta)^2}\|x_n-x_{n-1}\|. \qquad (9)
\end{aligned}$$

Since the Yosida operator $\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}$ is δ-strongly monotone and η-Lipschitz continuous, therefore using (2), we have

$$\begin{aligned}
&\Big\|(x_n-x_{n-1})-\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big)\Big\|^2 \\
&\qquad \le \|x_n-x_{n-1}\|^2-2\Big[\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1}),x_n-x_{n-1}\Big] \\
&\qquad\quad +c\Big\|\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big\|^2 \\
&\qquad \le \|x_n-x_{n-1}\|^2-2\delta\|x_n-x_{n-1}\|^2+c\eta^2\|x_n-x_{n-1}\|^2 \\
&\qquad = \big(1-2\delta+c\eta^2\big)\|x_n-x_{n-1}\|^2.
\end{aligned}$$

This implies

$$\Big\|(x_n-x_{n-1})-\Big(\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_n)-\mathcal{J}^{H(\cdot,\cdot)}_{M,\lambda}(x_{n-1})\Big)\Big\|\le\sqrt{1-2\delta+c\eta^2}\,\|x_n-x_{n-1}\|, \qquad (10)$$

where $\delta=\frac{(\alpha-\beta)^2-1}{\lambda(\alpha-\beta)}$ and $\eta=\frac{1}{\lambda}\Big(\tau_1+\tau_2+\frac{1}{\alpha-\beta}\Big)$.

Also, since $\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}$ is γ-Lipschitz continuous, therefore we have

$$\begin{aligned}&\left\|(x_n - x_{n-1}) - \left(\mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x_n) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x_{n-1})\right)\right\| \\ &\qquad \le \|x_n - x_{n-1}\| + \left\|\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x_n) - \mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x_{n-1})\right\| \\ &\qquad \le \|x_n - x_{n-1}\| + \gamma \|x_n - x_{n-1}\| \\ &\qquad = (1+\gamma)\|x_n - x_{n-1}\|, \end{aligned} \tag{11}$$

where $\gamma = \frac{2+(\tau_1+\tau_2)(\alpha-\beta)}{\alpha-\beta}$.
Further, since F is σ_1-Lipschitz continuous in the first argument and σ_2-Lipschitz continuous in the second argument, therefore we have

$$\begin{aligned}&\|F(x_n,u_n) - F(x_{n-1},u_{n-1})\| \\ &\quad = \|F(x_n,u_n) - F(x_{n-1},u_n) + F(x_{n-1},u_n) - F(x_{n-1},u_{n-1})\| \\ &\quad \le \|F(x_n,u_n) - F(x_{n-1},u_n)\| + \|F(x_{n-1},u_n) - F(x_{n-1},u_{n-1})\| \\ &\quad \le \sigma_1\|x_n - x_{n-1}\| + \sigma_2\|u_n - u_{n-1}\| \\ &\quad \le \sigma_1\|x_n - x_{n-1}\| + \sigma_2\left(1+n^{-1}\right)\mathcal{D}\left(P(y_{n+1}),P(y_n)\right) \\ &\quad \le \left(\sigma_1 + \mu\sigma_2\left(1+n^{-1}\right)\right)\|x_n - x_{n-1}\| \end{aligned} \tag{12}$$

Using (10)-(12) in (9), we have

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\le \left\{(1-\kappa) + \frac{\kappa\lambda}{\alpha-\beta}\left[(1+\gamma) + \sqrt{1-2\delta+c\eta^2} + \left(\sigma_1+\mu\sigma_2\left(1+n^{-1}\right)\right)\right]\right. \\ &\quad \left. + \frac{\kappa}{(\alpha-\beta)^2}\right\}\|x_n - x_{n-1}\| \\ &= \left\{(1-\kappa) + \frac{\kappa\lambda(\alpha-\beta)\left[1+\gamma+\sqrt{1-2\delta+c\eta^2}+\left(\sigma_1+\mu\sigma_2\left(1+n^{-1}\right)\right)\right]+\kappa}{(\alpha-\beta)^2}\right\} \\ &\quad \times \|x_n - x_{n-1}\|. \end{aligned}$$

This implies

$$\|x_{n+1} - x_n\| \le \varphi_n\|x_n - x_{n-1}\|, \tag{13}$$

where $\varphi_n = (1-\kappa) + \frac{\kappa\lambda(\alpha-\beta)\left[1+\gamma+\sqrt{1-2\delta+c\eta^2}+\left(\sigma_1+\mu\sigma_2\left(1+n^{-1}\right)\right)\right]+\kappa}{(\alpha-\beta)^2}$.
Let $\varphi = (1-\kappa) + \frac{\kappa\lambda(\alpha-\beta)\left[1+\gamma+\sqrt{1-2\delta+c\eta^2}+(\sigma_1+\mu\sigma_2)\right]+\kappa}{(\alpha-\beta)^2}$.

It is clear that $\varphi_n \to \varphi$ as $n \to \infty$ and by Condition (8), we have $0 < \varphi < 1$. Thus it follows from (13) that $\{x_n\}$ is a Cauchy sequence and consequently there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$, thanks to the completeness of X.

By Algorithm 4.2 and μ-$\mathcal{D}$-Lipschitz continuity of P, we have

$$\|u_{n+1} - u_n\| \le \mu\left(1+(1+n)^{-1}\right)\|x_{n+1} - x_n\|.$$

It follows that $\{u_n\}$ is also a Cauchy sequence in X and consequently there exists $u \in X$ such that $u_n \to u$ as $n \to \infty$.

Next, we claim $u \in P(x)$. Since $u_n \in P(x_n)$, we have

$$d(u,P(x)) \le \|u-u_n\| + d(u_n,P(x)) \le \|u-u_n\| + \mathcal{D}\left(P(x_n),P(x)\right)$$

$$\leq \|u - u_n\| + \mu\|x_n - x\| \longrightarrow 0 \text{ as } n \to \infty.$$

Since $P(x)$ is closed, it follows that $u \in P(x)$.

Now by continuity of mappings $F, \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}, \mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}, \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}$ and Ierative Algorithm 4.2, we have

$$x = \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)} \left\{ \lambda \left(\mathcal{J}_{M,\lambda}^{H(\cdot,\cdot)}(x) - \mathcal{C}_{M,\lambda}^{H(\cdot,\cdot)}(x) - F(x,u) \right) + \mathcal{R}_{M,\lambda}^{H(\cdot,\cdot)}(x) \right\}.$$

Thus in view of Lemma 4.1 we conclude that (x, u), where $x \in X, u \in P(x)$, is a solution of GCVIP (6). This completes the proof.

5. Conclusion

The results presented in this paper generalize many known results in the literature. The class of operators considered in this paper can be further exploited for other classes of varional inclusions problems under different setings. The techniques presented in this paper can be further exploited to study various classes of problems via Cayley operators, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 19, 20, 22, 23, 25, 27, 28, 29, 30, 31, 33, 34, 35].

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