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PROPERTIES OF THE DIRICHLET KERNEL

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Abstract. We prove several statements concerning the Dirichlet kernel which, as far as we know, are not proved in the literature. In particular, we show that the Dirichlet kernel does not satisfy the third condition in Stein's and Shakarchi's definition of good kernel.

1. Introduction

Let us consider a family of integral operators of the form

$$f \to \int_{-\pi}^{\pi} \mathcal{K}_n (x-t) f(t)\, dt \tag{1}$$

where the function $\mathcal{K}_n : \mathbb{R} \to \mathbb{R}$, called the kernel of the operator, is 2π-periodic and continuous for each $n = 1, 2, ...,$ and the function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and Riemann integrable on $[-\pi, \pi]$.

Elias M. Stein and Rami Shakarchi formulated in [7] three integral conditions on $\mathcal{K}_n$ that, jointly, imply the existence of

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \mathcal{K}_n (x-t) f(t)\, dt = f(x)$$

at each point x where f is continuous. A function $\mathcal{K}_n$ is called a good kernel if it satisfies those three conditions.

When $\mathcal{K}_n$ is the Dirichlet kernel $\mathcal{D}_n$ the operator (1) returns the nth partial sum $S_n(x)$ of the Fourier series for f. It is known that $\mathcal{D}_n$ is not a good kernel since, as it is proved in the literature, it does not satisfy the second of Stein's and Shakarchi's conditions. The first condition is a normalization requirement which $\mathcal{D}_n$ satisfies, while $\mathcal{D}_n$ does not satisfy the third condition, a fact rarely mentioned in the literature, and always without a proof as far as we know. This article is dedicated, in part, to give such a proof which, although uses basic tools, turns out to be rather subtle. We also state and prove sharp lower and upper bounds for $\mathcal{D}_n$ and we show directly that $\mathcal{D}_n$ does not satisfy a pointwise estimate due to Antoni Zygmund [8].

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The organization of our article is as follows. In Section 2 we briefly discuss historic background material. To be sure, the history of how to represent periodic functions using trigonometric series is long and laborious, so we look only at those parts that are relevant to our purposes. Section 3 is dedicated to Stein's and Shakarchi's work on good kernels. In Section 4 we develop a formula for the Dirichlet kernel $\mathcal{D}_n$ and we show that it is not a good kernel. Finally, in Section 5 we go over our main result, namely that $\mathcal{D}_n$ does not satisfy the third of Stein's and Shakarchi's conditions. Also in this last section, we state and prove sharp lower and upper bounds for $\mathcal{D}_n$ as well as we show that the kernel $\mathcal{D}_n$ does not satisfy Zygmund's pointwise estimate.

2. Where Fourier meets Cauchy

Before Joseph Fourier, the nature of heat was not well understood. For instance, in 1736, the French Academy called for essays on the topic "The nature and the propagation of 'fire'", where the word 'fire' was meant to signify 'heat'. All the submissions, including Euler's, missed the point and attempted to explain how fires develop ([4], p. 5).

Nevertheless, according to Umberto Bottazzini ([1], p. 59), by the end of the eighteenth century, heat was starting to be perceived as a form of energy that could aid in production. However, "if it is the *practical interests* that are best expressed in the English textile mills, it is the *theoretical aspects* that particularly engaged the French scientists." ([1], p. 59).

Under the title *The Analytical Theory of Heat*, Fourier published in 1822 two pieces, written in 1807 and 1811. In a radical departure from the work of others, Fourier developed a mathematical model for the propagation of heat, a differential equation known as heat equation. In the 1811 piece, he appropriately included a quotation attributed to Plato: "Also heat is governed by numbers." ([4], p. 6).

To solve the heat equation, Fourier used certain series, now called Fourier series. At the time, the heat equation was viewed as Fourier's crowning achievement, while the series "were considered a disgrace." ([4], p. 6).

The topic of Fourier series basically rests upon the formulas

$$f(x) = a_0 + \sum_{n\geq 1} \left(a_n \cos nx + b_n \sin nx\right), \tag{2}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx, \tag{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\; dx,\ n = 1, 2, ..., \tag{4}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\; dx,\ n = 1, 2, ... \tag{5}$$

Fourier shows in several cases that the series converges to the function f, meaning that the series converges pointwise to $f(x)$, for each x, in the sense of Augustin-Louis Cauchy's definition. Then, he proceeds to state that "all the series converge". Later on, he says "we must remark that our demonstration applies to an entirely arbitrary function." ([4], p. 12).

In spite of, or perhaps because of, these rather exuberant statements, the topic of representing a function as (2) has been the catalyst for many developments in analysis, such as the Riemann integral and the Lebesgue integral, and it was the inspiration for Georg Cantor's theory of the transfinite.

Now, given a function f, let us calculate a_n and b_n, if the integrals exist in some sense, and then let us form the series appearing on the right-hand side of (2). Of particular interest in our context, is the following question: Does the series converge to $f(x)$ for all x? According to Fourier, the answer is always yes. However, after several mathematicians of the time, including Cauchy, produced more or less faulty proofs, Peter Gustav Lejeune Dirichlet showed pointwise convergence under rather general conditions. The work was published in 1829 in *J. reine und angew. Math.* (Journal de Crelle). Here is Dirichlet's result:

Theorem 1. *(Dirichlet) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function that is continuous and has a bounded continuous derivative, except, possibly, at a finite number of points. Then, the equality (2) holds at every $x \in \mathbb{R}$ where f is continuous.*

In Jean-Pierre Kahane's words ([4], p. 31), "The article of Dirichlet on Fourier series is a turning point in the theory and also in the way mathematical analysis is approached and written. Its intention is simply to give a correct statement and a correct proof of the convergence of Fourier series. The result is a paradigm of what is correctness in analysis."

Kahane reproduces the full article in pp. 36-46 of [4].

After Dirichlet's result, it was natural to wonder about the necessity of its assumptions. A counterexample produced by Paul du Bois-Reymond in 1873, showed that the hypotheses on f could not be relaxed indefinitely. Indeed,

Theorem 2. *(du-Bois Reymond) (for a proof see, for instance,* [6]*, p. 67, Theorem 18.1) There is a function $g : \mathbb{R} \to \mathbb{R}$, 2π-periodic and continuous, for which*

$$\limsup_{n\to\infty} S_n(0) = \infty,$$

where $S_n(0)$ denotes the partial sum of the Fourier series for g, evaluated at $x = 0$.

The realization that Theorem 1 is not generally true when f is only continuous, closed the door forever on Fourier's paradise.

The example by du-Bois Reymond seemed, for a long time, to open the possibility of finding a continuous function whose Fourier series diverges at every point. However, Lennart Carleson, in the 1960s, proved the impossibility of such a function, when he showed the following:

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic and continuous function. Then, there is a set $E \subset \mathbb{R}$ of Lebesgue measure zero so that*

$$a_0 + \sum_{n\geq 1} (a_n \cos nx + b_n \sin nx) = f(x)$$

for $x \in \mathbb{R} \backslash E$.

Actually, Carleson proved, among other things, that the conclusion of Theorem 3 holds for a 2π-periodic function that is just square integrable on $[-\pi, \pi]$, in the sense of Lebesgue [2]. In the *Mathematical Reviews*, **MR**199631, Kahane refers to the results in Carleson's article as "spectacular" and catalogs the proofs as "very difficult" and "very delicate". We mention Carleson's convergence result, just for the sake of completeness. Its proof is, indeed, of a great complexity, well beyond the scope of our exposition.

The proof of Theorem 1 is, for instance, in ([6], p. 56, Section 15) for everywhere continuous functions and in ([6], p. 59, Section 16) for the general case.

Let us point out that, for our purposes, it will suffice to work with Riemann integrable functions.

3. The importance of being a good kernel

We begin with the following definition.

Definition 1. *([7], p. 48) Given an integral operator of the form*

$$f \to \int_{-\pi}^{\pi} \mathcal{K}_n (x-t) f(t) \, dt,$$

the kernel $\mathcal{K}_n$ is called a good kernel if it satisfies the following conditions:

(1)
$$\int_{-\pi}^{\pi} \mathcal{K}_n (t) \, dt = 1$$
for all $n \geq 1$.

(2) *There is $C > 0$ so that*
$$\int_{-\pi}^{\pi} |\mathcal{K}_n (t)| \, dt \leq C$$
for all $n \geq 1$.

(3) *For each $0 < \delta < \pi$ fixed, there is*
$$\lim_{n\to\infty} \int_{\delta \leq |t| \leq \pi} |\mathcal{K}_n (t)| \, dt = 0.$$

The significance of this definition is shown in the result that follows.

Theorem 4. *([7], p. 49, Theorem 4.1) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, that is Riemann integrable on $[-\pi, \pi]$. Then,*

a): *if $\mathcal{K}_n$ is a good kernel, there is*
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \mathcal{K}_n (x-t) f(t) \, dt = f(x)$$
at each $x \in \mathbb{R}$ where the function f is continuous, and

b): *the limit is uniform on $x \in \mathbb{R}$, when f is continuous everywhere.*

Proof. Since

$$\int_{-\pi}^{\pi} \mathcal{K}_n (x-t) f(t) \, dt \underset{t\to s=x-t}{=} -\int_{x+\pi}^{x-\pi} \mathcal{K}_n (s) f(x-s) \, ds$$
$$= \int_{x-\pi}^{x+\pi} \mathcal{K}_n (s) f(x-s) \, ds,$$

the 2π-periodicity of $\mathcal{K}_n$ and f implies that the above is equal to

$$\int_{-\pi}^{\pi} \mathcal{K}_n (s) f(x-s) \, ds. \tag{6}$$

Therefore,

$$\left|\int_{-\pi}^{\pi} \mathcal{K}_n(x-t) f(t)\, dt - f(x)\right| \underset{(i)}{=} \left|\int_{-\pi}^{\pi} \mathcal{K}_n(s) f(x-s)\, ds - f(x)\int_{-\pi}^{\pi} \mathcal{K}_n(s)\, ds\right|$$
$$= \left|\int_{-\pi}^{\pi} \mathcal{K}_n(s)\left(f(x-s) - f(x)\right) ds\right|,$$

where we have used, in (i), condition 1).

If the function f is continuous at x, given $\varepsilon > 0$, there is $\delta = \delta(x, \varepsilon) > 0$, which we can choose smaller than π, so that

$$|f(x-s) - f(x)| \leq \varepsilon$$

for $|s| < \delta$.

Then, we can write

$$\begin{aligned}\left|\int_{-\pi}^{\pi} \mathcal{K}_n(s)\left(f(x-s) - f(x)\right) ds\right| &\leq \varepsilon \int_{|s|<\delta} |\mathcal{K}_n(s)|\, ds \\ &\quad + 2\sup_{|t|\leq\pi} |f(t)| \int_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(s)|\, ds \\ &\underset{(ii)}{\leq} C\varepsilon + 2B\int_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(s)|\, ds,\end{aligned}$$

where $B = \sup_{|t|\leq\pi} |f(t)|$ and we have used, in the first term of (ii), condition 2).

Finally, condition 3) tells us that there is $N = N(\varepsilon) \geq 1$ so that

$$\int_{\delta\leq|t|\leq\pi} |\mathcal{K}_n(s)|\, ds \leq \varepsilon$$

for $n \geq N$.

This completes the proof of a).

As for b), we only need to observe that when f is continuous everywhere, it is uniformly continuous on $[-\pi, \pi]$, and also on $\mathbb{R}$ because f is periodic. Then, δ can be chosen independently of x and, therefore,

$$\sup_{x\in\mathbb{R}} \left|\int_{-\pi}^{\pi} \mathcal{K}_n(s)\left(f(x-s) - f(x)\right) ds\right| \leq (C + 2B)\,\varepsilon.$$

So, we have proved b).

This completes the proof of the theorem. □

Remark 1. *The proof of Theorem 4 appears in ([7], pp. 49-50). We have included it here to illustrate how the conditions in Definition 1 are used to prove convergence results.*

4. The Dirichlet kernel is not a good kernel, part I

To prove Dirichlet's result on pointwise convergence, the function $x \to S_n(x)$ is written as an integral operator of the form

$$\int_{-\pi}^{\pi} \mathcal{D}_n(x-t) f(t)\, dt$$

where $\mathcal{D}_n : \mathbb{R} \to \mathbb{R}$ is called the Dirichlet kernel. As we will see, the impossibility of having pointwise convergence everywhere of the sequence $\{S_n(x)\}_{n\geq 0}$ for every 2π-periodic and continuous function f, rests upon the nature of $\mathcal{D}_n$.

In the lemma that follows we calculate a formula for $\mathcal{D}_n$. We assume that the function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, and Riemann integrable on $[-\pi, \pi]$.

Lemma 5. *The nth partial sum S_n can be written as*

$$S_n(x) = \int_{-\pi}^{\pi} \mathcal{D}_n(x-t) f(t)\, dt,$$

where the function $\mathcal{D}_n(t)$, called the Dirichlet kernel, is

$$\begin{matrix} \frac{1}{2\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} & \text{if} \quad t \neq 0 \\ \frac{2n+1}{2\pi} & \text{if} \quad t = 0 \end{matrix} . \tag{7}$$

Proof. Let us write

$$S_n(x) = \sum_{j=0}^{n} \left(a_j \cos jx + b_j \sin jx\right),$$

where we agree that $b_0 = 0$. For convenience, we will use complex exponentials, although "they were not used in Fourier series until well into the twentieth century" ([4], p. 2). The identities

$$\cos jx = \frac{e^{ijx} + e^{-ijx}}{2},$$
$$\sin jx = \frac{e^{ijx} - e^{-ijx}}{2i}$$

give

$$S_n(x) = \sum_{j=0}^{n} \frac{1}{2}\left(a_j + \frac{b_j}{i}\right) e^{ijx} + \sum_{j=0}^{n} \frac{1}{2}\left(a_j - \frac{b_j}{i}\right) e^{-ijx}, \tag{8}$$

where

$$\frac{1}{2}\left(a_j + \frac{b_j}{i}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ijt} f(t)\, dt,$$
$$\frac{1}{2}\left(a_j - \frac{b_j}{i}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ijt} f(t)\, dt,$$

for $j \geq 1$.

Therefore,

$$S_n(x) = \int_{-\pi}^{\pi} \left[\frac{1}{2\pi}\sum_{j=-n}^{n} e^{ij(x-t)}\right] f(t),$$

while

$$\sum_{j=-n}^{n} e^{ij(x-t)} = e^{-in(x-t)} \sum_{j=0}^{2n} e^{ij(x-t)} \underset{(i)}{=} e^{-in(x-t)} \frac{1 - e^{i(2n+1)(x-t)}}{1 - e^{i(x-t)}}$$
$$= \frac{e^{-in(x-t)} - e^{i(n+1)(x-t)}}{e^{i(x-t)/2}\left[e^{-i(x-t)/2} - e^{i(x-t)/2}\right]} = \frac{\sin\left(n+\frac{1}{2}\right)(x-t)}{\sin\frac{(x-t)}{2}},$$

for $x \neq t$, where we have used in (i) the formula for the sum of the $2n$ first terms of a geometric progression.

So, we arrive at (7) for $t \neq 0$. An application of L'Hôpital's rule when $t \to 0$, gives the value of the kernel at $t = 0$.

This completes the proof of the lemma. □

Let us observe that, technically speaking, we should refer to $\mathcal{D}_n$ as the nth Dirichlet kernel.

Remark 2. *The function $\mathcal{D}_n : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, even, and it is continuous with continuous derivatives of all orders. Moreover, it satisfies condition 1) in Definition 1 since*

$$\int_{-\pi}^{\pi} \mathcal{D}_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=-n}^{n} e^{ijt}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt = 1.$$

Proposition 6. *The kernel $\mathcal{D}_n$ is not a good kernel.*

Proof. First, we go over a quick roundabout way of showing that $\mathcal{D}_n$ is not a good kernel: du-Bois Reymond's counterexample tells us that the conclusion of Theorem 4 cannot hold generally when $\mathcal{K}_n = \mathcal{D}_n$. Therefore, $\mathcal{D}_n$ cannot be a good kernel.

Next, we show directly that $\mathcal{D}_n$ is not a good kernel, by proving that it does not satisfy condition 2) in Definition 1 ([7], p. 66, Problem 2).

$$\begin{aligned}
\int_{-\pi}^{\pi} |\mathcal{D}_n(t)|\,dt &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| dt = \int_{0}^{\pi}\frac{1}{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{\sin\frac{t}{2}}dt \\
&\geq \frac{2}{\pi}\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}dt \underset{t\to s=\left(n+\frac{1}{2}\right)t}{=} \frac{2}{\pi}\int_0^{\left(n+\frac{1}{2}\right)\pi}\frac{|\sin s|}{s}ds \\
&\geq \frac{2}{\pi}\int_0^{n\pi}\frac{|\sin s|}{s}ds = \frac{2}{\pi}\sum_{j=0}^{n-1}\int_{j\pi}^{(j+1)\pi}\frac{|\sin s|}{s}ds \\
&\geq \frac{2}{\pi^2}\sum_{j=0}^{n-1}\frac{1}{j+1}\int_{j\pi}^{(j+1)\pi}|\sin s|\;ds \\
&\geq \frac{2}{\pi^2}\sum_{j=0}^{n-1}\frac{1}{j+1}\left|\int_{j\pi}^{(j+1)\pi}\sin s\,ds\right| \\
&= \frac{4}{\pi^2}\sum_{j=1}^{n}\frac{1}{j}. \qquad (9)
\end{aligned}$$

If we compare the sum in (9) with the integral of $f(t) = \frac{1}{t}$ over the interval $[1, n]$, we conclude that

$$\int_{-\pi}^{\pi} |\mathcal{D}_n(t)|\,dt \geq \frac{4}{\pi^2}\ln n. \qquad (10)$$

This completes the proof of the proposition. □

Remark 3. *Although (10) suffices for now, in the next, and last, section we will go over a much improved version that gives sharp lower and upper bounds for the integral.*

Remark 4. *If $\mathcal{K}_n$ is a good kernel, given $0 < \delta < \pi$ fixed,*

$$\int_0^{\delta}|\mathcal{K}_n(t)|\,dt = \int_0^{\pi}|\mathcal{K}_n(t)|\,dt - \int_{\delta}^{\pi}|\mathcal{K}_n(t)|\,dt$$

can be made arbitrarily close to $\frac{C}{2}$, for n large enough, where C is the constant in condition 2) of Definition 1.

This observation does not hold for the Dirichlet kernel.

Indeed,

$$\int_{\delta}^{\pi}|\mathcal{D}_n(t)|\,dt \le \frac{1}{\pi\sin\frac{\delta}{2}}\int_{\delta}^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|dt \le \frac{\pi-\delta}{\pi\sin\frac{\delta}{2}},$$

for every $n \ge 1$.

Therefore, using (10),

$$\begin{aligned}\int_{0}^{\delta}|\mathcal{D}_n(t)|\,dt &= \int_{0}^{\pi}|\mathcal{D}_n(t)|\,dt - \int_{\delta}^{\pi}|\mathcal{D}_n(t)|\,dt \\ &\ge \frac{2}{\pi^2}\ln n - \frac{\pi-\delta}{\pi\sin\frac{\delta}{2}} \underset{n\to\infty}{\to} \infty.\end{aligned}$$

Remark 5. *If we write the arithmetic mean of* $\mathcal{D}_0, \mathcal{D}_1, ..., \mathcal{D}_n$,

$$\frac{1}{n+1}\sum_{j=0}^{n}\mathcal{D}_j(t) = \frac{1}{2\pi(n+1)}\sum_{j=0}^{n}\frac{\sin\left(j+\frac{1}{2}\right)t}{\sin\frac{t}{2}},$$

after a few calculations we arrive at the Fejér kernel

$$\mathcal{F}_n(t) = \begin{cases} \frac{1}{2\pi(n+1)}\frac{\sin^2\frac{n+1}{2}t}{\sin^2\frac{t}{2}} & \text{if} \quad t \ne 0 \\ \frac{n+1}{2\pi} & \text{if} \quad t = 0 \end{cases}.$$

This kernel is named after Leopold Fejér, who proved the summability of a Fourier series by arithmetic means, a method due to Ernesto Cesàro. As evidence of the heuristic principle "averaging might make for better behavior", two improvements over the Dirichlet kernel are readily apparent: First the Fejér kernel is non-negative, second the presence of the factor $\frac{1}{n+1}$ *assures that* $\mathcal{F}_n(t) \to 0$ *as* $n \to \infty$ *for each* $t \ne 0$. *But not only that,* $\mathcal{F}_n$ *is a good kernel (see, for instance, [7], p. 53). Therefore, applying Theorem 4 to the operator with kernel* $\mathcal{F}_n$*, proves immediately the convergence of the arithmetic means for any* 2π*-periodic and continuous function, in contrast with Theorem 2. Much more can be said, but since our interest is in convergence à la Cauchy and not summability, we will say no more.*

5. The Dirichlet kernel is not a good kernel, part II

As promised in Remark 3, the following result gives sharp bounds for the integral $\int_{-\pi}^{\pi}|\mathcal{D}_n(t)|\,dt$.

Theorem 7. *For each* $n \ge 1$,

$$\frac{2}{\pi} + \frac{4}{\pi^2}\ln\left(n+\frac{1}{2}\right) \le \int_{-\pi}^{\pi}|\mathcal{D}_n(t)|\,dt \le 1 + \frac{2}{\pi} + \frac{4}{\pi^2}\ln(2n). \tag{11}$$

In particular,

$$\int_{-\pi}^{\pi}|\mathcal{D}_n(t)|\,dt = \frac{4}{\pi^2}\ln n + O(1),$$

where $O(1)$*, in the "big o" notation, indicates a quantity that remains bounded as* $n \to \infty$. *More specifically,*

$$0 \le O(1) \le 1 + \frac{2}{\pi} + \frac{4}{\pi^2}\ln 2.$$

Proof.

$$\begin{aligned}\int_{-\pi}^{\pi}\left|\mathcal{D}_n(t)\right|dt &= \frac{1}{\pi}\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{\sin\frac{t}{2}}dt \\ &= \frac{1}{\pi}\int_0^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|\left(\frac{1}{\sin\frac{t}{2}}-\frac{1}{\frac{t}{2}}\right)dt+\frac{2}{\pi}\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}dt. \end{aligned} \tag{12}$$

The function $\frac{1}{\sin t}-\frac{1}{t}$ is positive, and it is increasing for $0<t\leq\frac{\pi}{2}$ as it can be seen by manipulating its derivative. Moreover, an application of L'Höpital's rule shows that the function is continuous on $\left[0,\frac{\pi}{2}\right]$ if we assign the value zero at $t=0$. Therefore,

$$(12)\leq\frac{1}{\pi}\left(1-\frac{2}{\pi}\right)\int_0^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|dt+\frac{2}{\pi}\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}dt.$$

First, we consider

$$\begin{aligned}&\frac{1}{\pi}\left(1-\frac{2}{\pi}\right)\int_0^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|dt \underset{s=\left(n+\frac{1}{2}\right)t}{=} \frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)\int_0^{\left(n+\frac{1}{2}\right)\pi}\left|\sin s\right|ds \\ &= \frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)\int_0^{n\pi}\left|\sin s\right|ds+\frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)\int_{n\pi}^{\left(n+\frac{1}{2}\right)\pi}\left|\sin s\right|ds \\ &\underset{(i)}{=} \frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)n\int_0^{\pi}\sin s\,ds+\frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)\int_0^{\frac{\pi}{2}}\sin s\,ds \\ &= \frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)2n+\frac{2}{(2n+1)\pi}\left(1-\frac{2}{\pi}\right)=\frac{2}{\pi}\left(1-\frac{2}{\pi}\right),\end{aligned}$$

where (i) holds because the function $|\sin s|$ is π-periodic.

Next,

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}dt &\underset{s=\left(n+\frac{1}{2}\right)t}{=} \frac{2}{\pi}\int_0^{(2n+1)\frac{\pi}{2}}\frac{|\sin s|}{s}ds \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin s}{s}ds+\frac{2}{\pi}\sum_{j=1}^{2n}\int_{j\frac{\pi}{2}}^{(j+1)\frac{\pi}{2}}\frac{|\sin s|}{s}ds \\ &\leq 1+\frac{2}{\pi}\sum_{j=1}^{2n}\frac{1}{j\frac{\pi}{2}}\int_{j\frac{\pi}{2}}^{(j+1)\frac{\pi}{2}}|\sin s|\,ds \underset{(ii)}{=} 1+\frac{4}{\pi^2}\left(\int_0^{\frac{\pi}{2}}\sin s\,ds\right)\sum_{j=1}^{2n}\frac{1}{j} \\ &\leq 1+\frac{4}{\pi^2}\left(1+\int_1^{2n}\frac{dt}{t}\right)=1+\frac{4}{\pi^2}+\frac{4}{\pi^2}\ln(2n),\end{aligned}$$

where the equality (ii) holds because $|\sin s|$ is π-periodic and the graph is locally symmetric with respect to each of the lines $s=\pi$, $s=2\pi$, $s=3\pi$, etc.

Finally,

$$\begin{aligned}\int_{-\pi}^{\pi}\left|\mathcal{D}_n(t)\right|dt &\leq \frac{2}{\pi}\left(1-\frac{2}{\pi}\right)+1+\frac{4}{\pi^2}+\frac{4}{\pi^2}\ln 2n \\ &= 1+\frac{2}{\pi}+\frac{4}{\pi^2}\ln(2n).\end{aligned}$$

As for the lower bound, according to (12),

$$\int_{-\pi}^{\pi} |\mathcal{D}_n(t)|\, dt \geq \frac{2}{\pi}\int_0^{\pi} \frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t} dt \underset{s=\left(n+\frac{1}{2}\right)t}{=} \frac{2}{\pi}\int_0^{(2n+1)\frac{\pi}{2}} \frac{|\sin s|}{s} ds$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin s}{s} ds + \frac{2}{\pi}\sum_{j=1}^{2n}\int_{j\frac{\pi}{2}}^{(j+1)\frac{\pi}{2}} \frac{|\sin s|}{s} ds.$$

By L'Höpital's rule, $\frac{\sin s}{s} \to 1$ as $s \to 0$. Therefore, the function

$$f(s) = \begin{cases} \frac{\sin s}{s} & \text{for} \quad 0 < s \leq \frac{\pi}{2} \\ 1 & \text{for} \quad s = 0 \end{cases}$$

is continuous and positive on $\left[0, \frac{\pi}{2}\right]$. So,

$$\min_{0 \leq s \leq \frac{\pi}{2}} f(s) > 0.$$

Comparing the graphs of $f(s)$ and $\frac{2}{\pi}s$ on $\left[0, \frac{\pi}{2}\right]$, we conclude that, for $0 \leq s \leq \frac{\pi}{2}$,

$$f(s) \geq f\left(\frac{\pi}{2}\right) = \frac{2}{\pi},$$

which is usually called Jordan's inequality, for the mathematician Marie Ennemond Camille Jordan.

Hence,

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin s}{s} ds \geq \frac{2}{\pi}.$$

Next, we consider

$$\frac{2}{\pi}\sum_{j=1}^{2n}\int_{j\frac{\pi}{2}}^{(j+1)\frac{\pi}{2}} \frac{|\sin s|}{s} ds \geq \frac{2}{\pi}\sum_{j=1}^{2n}\frac{2}{\pi}\frac{1}{j+1}\int_{j\frac{\pi}{2}}^{(j+1)\frac{\pi}{2}} |\sin s|\, ds$$

$$= \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \sin s ds \sum_{j=1}^{2n}\frac{1}{j+1}$$

$$= \frac{4}{\pi^2}\sum_{j=2}^{2n+1}\frac{1}{j} \geq \frac{4}{\pi^2}\int_2^{2n+1}\frac{dt}{t} = \frac{4}{\pi^2}\ln\left(n+\frac{1}{2}\right).$$

Therefore, we can write

$$\int_{-\pi}^{\pi} |\mathcal{D}_n(t)|\, dt \geq \frac{2}{\pi} + \frac{4}{\pi^2}\ln\left(n+\frac{1}{2}\right).$$

This completes the proof of the theorem. □

In page 14 of [5], Yitzhak Katznelson states, without proof, that the Dirichlet kernel does not satisfy the following equivalent version of condition 3) in Definition 1 ([5], p. 9):

For each $0 < \delta < \pi$, there is

$$\lim_{n\to\infty}\int_{\delta}^{2\pi-\delta} |\mathcal{K}_n(t)|\, dt = 0. \tag{13}$$

We do not know of any reference where this claim is proved. Likewise, we do not know of any reference where a similar claim concerning condition 3) in Definition 1 is stated and proved. Therefore, we present below our own proof of the statement.

Theorem 8. *For each $0 < \delta \leq \frac{\pi}{2}$, the expression*

$$\int_{\delta \leq |t| \leq \pi} |\mathcal{D}_n(t)| \, dt$$

does not go to zero as $n \to \infty$.

Proof. Let us fix $0 < \delta \leq \frac{\pi}{2}$. Then,

$$\begin{aligned} \int_{\delta \leq |t| \leq \pi} |\mathcal{D}_n(t)| \, dt &= 2 \int_{\delta}^{\pi} |\mathcal{D}_n(t)| \, dt \geq 2 \int_{\frac{\pi}{2}}^{\pi} |\mathcal{D}_n(t)| \, dt \\ &\underset{s=t-\frac{\pi}{2}}{=} \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{\left|\sin\left(n+\frac{1}{2}\right)\left(s+\frac{\pi}{2}\right)\right|}{\sin \frac{s+\frac{\pi}{2}}{2}} dt. \end{aligned}$$

Now,

$$\begin{aligned} \sin\left(n+\frac{1}{2}\right)\left(s+\frac{\pi}{2}\right) &= \sin\left(n+\frac{1}{2}\right) s \cos\left(n+\frac{1}{2}\right)\frac{\pi}{2} + \cos\left(n+\frac{1}{2}\right) s \sin\left(n+\frac{1}{2}\right)\frac{\pi}{2} \\ &= \sin\left(n+\frac{1}{2}\right) s \cos(2n+1)\frac{\pi}{4} + \cos\left(n+\frac{1}{2}\right) s \sin(2n+1)\frac{\pi}{4}. \end{aligned}$$

For $n = 4k$, with $k = 1, 2, 3, \ldots$,

$$(2n+1)\frac{\pi}{4} = (8k+1)\frac{\pi}{4} = 2k\pi + \frac{\pi}{4},$$

so

$$\begin{aligned} \cos(2n+1)\frac{\pi}{4} &= \cos\left(2k\pi + \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \\ \sin(2n+1)\frac{\pi}{4} &= \sin\left(2k\pi + \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Furthermore, if $0 \leq s \leq \frac{\pi}{2}$, then $0 \leq \frac{s}{2} \leq \frac{\pi}{4}$, so

$$\frac{\pi}{4} \leq \frac{s+\frac{\pi}{2}}{2} \leq \frac{\pi}{2}$$

and we can write

$$\frac{\sqrt{2}}{2} \leq \sin\frac{s+\frac{\pi}{2}}{2} \leq \frac{s}{2} + \frac{\pi}{4}.$$

Hence,

$$
\begin{aligned}
\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\left|\sin\left(4k+\frac{1}{2}\right)\left(s+\frac{\pi}{2}\right)\right|}{\sin\frac{s+\frac{\pi}{2}}{2}}ds &\geq \frac{\sqrt{2}}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\left|\sin\left(4k+\frac{1}{2}\right)s+\cos\left(4k+\frac{1}{2}\right)s\right|}{\frac{s}{2}+\frac{\pi}{4}}ds \\
&= \frac{\sqrt{2}}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\left|\sin\left(8k+1\right)\frac{s}{2}+\cos\left(8k+1\right)\frac{s}{2}\right|}{\frac{s}{2}+\frac{\pi}{4}}ds \\
&\underset{t=\frac{s}{2}}{=} \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{4}} \frac{\left|\sin\left(8k+1\right)t+\cos\left(8k+1\right)t\right|}{t+\frac{\pi}{4}}dt \\
&\geq \frac{\sqrt{2}}{\pi}\sum_{j=1}^{8k}\int_{\frac{j\pi}{4(8k+1)}}^{\frac{(j+1)\pi}{4(8k+1)}} \frac{\left|\sin\left(8k+1\right)t+\cos\left(8k+1\right)t\right|}{t+\frac{\pi}{4}}dt \\
&\geq \frac{\sqrt{2}}{\pi}\sum_{j=1}^{8k}\frac{1}{\frac{(j+1)\pi}{4(8k+1)}+\frac{\pi}{4}}\int_{\frac{j\pi}{4(8k+1)}}^{\frac{(j+1)\pi}{4(8k+1)}} \left|\sin\left(8k+1\right)t+\cos\left(8k+1\right)t\right|dt \\
&\geq \frac{\sqrt{2}}{\pi}\sum_{j=1}^{8k}\frac{1}{\frac{(j+1)\pi}{4(8k+1)}+\frac{\pi}{4}}\left|\int_{\frac{j\pi}{4(8k+1)}}^{\frac{(j+1)\pi}{4(8k+1)}} \left(\sin\left(8k+1\right)t+\cos\left(8k+1\right)t\right)dt\right|.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_{\frac{j\pi}{4(8k+1)}}^{\frac{(j+1)\pi}{4(8k+1)}} \sin\left(8k+1\right)t\,dt &= -\frac{1}{8k+1}\left\{\cos\left[\left(8k+1\right)\frac{\left(j+1\right)\pi}{4\left(8k+1\right)}\right]-\cos\left[\left(8k+1\right)\frac{j\pi}{4\left(8k+1\right)}\right]\right\} \\
&= -\frac{1}{8k+1}\left[\cos\left(j+1\right)\frac{\pi}{4}-\cos j\frac{\pi}{4}\right] \\
&= -\frac{1}{8k+1}\left(\cos j\frac{\pi}{4}\cos\frac{\pi}{4}-\sin j\frac{\pi}{4}\sin\frac{\pi}{4}-\cos j\frac{\pi}{4}\right) \\
&= \frac{\sqrt{2}}{2\left(8k+1\right)}\left(-\cos j\frac{\pi}{4}+\sin j\frac{\pi}{4}\right)+\frac{1}{8k+1}\cos j\frac{\pi}{4},
\end{aligned}
$$

while

$$
\begin{aligned}
\int_{\frac{j\pi}{4(8k+1)}}^{\frac{(j+1)\pi}{4(8k+1)}} \cos\left(8k+1\right)t\,dt &= \frac{1}{8k+1}\left\{\sin\left[\left(8k+1\right)\frac{\left(j+1\right)\pi}{4\left(8k+1\right)}\right]-\sin\left[\left(8k+1\right)\frac{j\pi}{4\left(8k+1\right)}\right]\right\} \\
&= \frac{1}{8k+1}\left(\sin\left(j+1\right)\frac{\pi}{4}-\sin j\frac{\pi}{4}\right) \\
&= \frac{1}{8k+1}\left(\sin j\frac{\pi}{4}\cos\frac{\pi}{4}+\cos j\frac{\pi}{4}\sin\frac{\pi}{4}-\sin j\frac{\pi}{4}\right) \\
&= \frac{\sqrt{2}}{2\left(8k+1\right)}\left(\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}\right)-\frac{1}{8k+1}\sin j\frac{\pi}{4}.
\end{aligned}
$$

Then,

$$\frac{\sqrt{2}}{\pi}\sum_{j=1}^{8k}\frac{1}{\frac{(j+1)\pi}{4(8k+1)}+\frac{\pi}{4}}\left|\int_{\frac{j\pi}{4(8k+1)}}^{\frac{(j+1)\pi}{4(8k+1)}}\left(\sin\left(8k+1\right)t+\cos\left(8k+1\right)t\right)dt\right|$$
$$=\frac{\sqrt{2}}{\pi}\frac{1}{(8k+1)}\sum_{j=1}^{8k}\frac{1}{\frac{(j+1)\pi}{4(8k+1)}+\frac{\pi}{4}}\left|\frac{\sqrt{2}}{2}\left(-\cos j\frac{\pi}{4}+\sin j\frac{\pi}{4}\right)+\cos j\frac{\pi}{4}\right.$$
$$\left.+\frac{\sqrt{2}}{2}\left(\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}\right)-\sin j\frac{\pi}{4}\right|$$
$$=\frac{\sqrt{2}}{\pi}\frac{1}{(8k+1)}\sum_{j=1}^{8k}\frac{1}{\frac{(j+1)\pi}{4(8k+1)}+\frac{\pi}{4}}\left|\sqrt{2}\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}-\sin j\frac{\pi}{4}\right|$$
$$=\frac{\sqrt{2}}{\pi}\sum_{j=1}^{8k}\frac{1}{(j+1)\frac{\pi}{4}+\frac{\pi}{4}(8k+1)}\left|\left(\sqrt{2}-1\right)\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}\right|.$$

Let us observe that

$$\left(\sqrt{2}-1\right)\sin\left(j+8l\right)\frac{\pi}{4}+\cos\left(j+8l\right)\frac{\pi}{4}=\left(\sqrt{2}-1\right)\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}$$

for any $l=0,1,2,....$

On the other hand, here are the values of $f(j)=\left(\sqrt{2}-1\right)\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}$, for $j=1,2,...,8$:

j	$f(j)$	j	$f(j)$
1	1	5	-1
2	$\sqrt{2}-1$	6	$1-\sqrt{2}$
3	$1-\sqrt{2}$	7	$\sqrt{2}-1$
4	-1	8	1

.

Therefore, the values of $f(j)$ for $j=9,10,11,...,16$, are the values for $j=1,2,3,...,8$, respectively, and so on. As a consequence of this observation,

$$\left|\left(\sqrt{2}-1\right)\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}\right|\geq\sqrt{2}-1$$

for every j.

Then,

$$\frac{\sqrt{2}}{\pi}\sum_{j=1}^{8k}\frac{1}{(j+1)\frac{\pi}{4}+\frac{\pi}{4}(8k+1)}\left|\left(\sqrt{2}-1\right)\sin j\frac{\pi}{4}+\cos j\frac{\pi}{4}\right|$$
$$\geq\frac{\sqrt{2}}{\pi}\left(\sqrt{2}-1\right)\sum_{j=1}^{8k}\frac{1}{(j+1)\frac{\pi}{4}+\frac{\pi}{4}(8k+1)}$$
$$\geq\frac{\sqrt{2}}{\pi}\left(\sqrt{2}-1\right)\frac{8k}{2\left(8k+1\right)\frac{\pi}{4}}=\frac{2}{\pi^2}\left(2-\sqrt{2}\right)\frac{8k}{8k+1}\underset{k\to\infty}{\to}\frac{2}{\pi^2}\left(2-\sqrt{2}\right).$$

Hence,

$$\int_{\delta\leq|t|\leq\pi}\left|\mathcal{D}_{4k}\left(t\right)\right|dt$$

does not converge to zero as $k \to \infty$ and, as a consequence,

$$\int_{\delta \le |t| \le \pi} |\mathcal{D}_n(t)|\, dt$$

does not converge to zero as $n \to \infty$.

This completes the proof of the theorem. $\square$

In page 86 of [8], Zygmund introduces the following uniform pointwise version of condition 3) in Definition 1.

For each $0 < \delta < \pi$, there is

$$\lim_{n \to \infty} \sup_{\delta \le |t| \le \pi} |\mathcal{K}_n(t)| = 0. \tag{14}$$

Zygmund observes that the Dirichlet kernel does not satisfy (14) and the claim is repeated in page 370 of [3]. To be sure, $\mathcal{D}_n$ cannot satisfy (14), otherwise it would satisfy condition 3). However, since we do not know of any source where the claim is proved directly, we conclude by giving our own proof.

Proposition 9. *For each $0 < \delta \le \frac{\pi}{2}$, the expression*

$$\sup_{\delta \le |t| \le \pi} |\mathcal{D}_n(t)|$$

does not go to zero as $n \to \infty$.

Proof. Let us fix $0 < \delta \le \frac{\pi}{2}$. Then,

$$\sup_{\delta \le |t| \le \pi} |\mathcal{D}_n(t)| \ge \sup_{\frac{\pi}{2} \le |t| \le \pi} |\mathcal{D}_n(t)|.$$

Therefore, it is sufficient to assume that $\delta = \frac{\pi}{2}$.

$$\begin{aligned}\sup_{\frac{\pi}{2} \le |t| \le \pi} |\mathcal{D}_n(t)| &= \frac{1}{2\pi} \sup_{\frac{\pi}{2} \le t \le \pi} \frac{\left|\sin\left(n + \frac{1}{2}\right) t\right|}{\sin \frac{t}{2}} \underset{s=\frac{t}{2}}{=} \frac{1}{2\pi} \sup_{\frac{\pi}{4} \le s \le \frac{\pi}{2}} \frac{|\sin(2n+1)s|}{\sin s} \\ &\ge \frac{1}{2\pi} \sup_{\frac{\pi}{4} \le s \le \frac{\pi}{2}} |\sin(2n+1)s|.\end{aligned}$$

When $\frac{\pi}{4} \le s \le \frac{\pi}{2}$, we have

$$\frac{\pi}{4}(2n+1) \le (2n+1)s \le \frac{\pi}{2}(2n+1)$$

or

$$\frac{\pi}{4}(2n+1) \le (2n+1)s \le 2\frac{\pi}{4}(2n+1).$$

That is to say, the argument of the function $|\sin(2n+1)s|$ takes values in the interval

$$\left[\frac{\pi}{4} + 2n\frac{\pi}{4}, 2\left(\frac{\pi}{4} + 2n\frac{\pi}{4}\right)\right].$$

If $n = 2k$ for $k = 0, 1, 2, ...$,

$$\frac{\pi}{4} + 2n\frac{\pi}{4} = \frac{\pi}{4} + 4k\frac{\pi}{4} = \frac{\pi}{4} + k\pi,$$

while if $n = 2k+1$ for $k = 0, 1, 2, ...$,

$$\frac{\pi}{4} + 2n\frac{\pi}{4} = \frac{\pi}{4} + 2(2k+1)\frac{\pi}{4} = \frac{3\pi}{4} + k\pi.$$

Therefore, by periodicity and symmetry,

$$\frac{1}{2\pi}\sup_{\frac{\pi}{4}\leq s\leq\frac{\pi}{2}}\left|\sin\left(2n+1\right)s\right|\geq\frac{1}{2\pi}\sin\frac{\pi}{4}=\frac{\sqrt{2}}{4\pi}.$$

This completes the proof of the proposition. □

Remark 6. *With minor adjustments, (13) can replace condition 3) in the proof of Theorem 4. We refer to ([5], p. 10, Lemma) for the details.*

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