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INCLUSION RELATIONS FOR CERTAIN CLASS OF MULTIVALENT MEROMORPHIC FUNCTIONS

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Abstract. The purpose of the present paper is to introduce new subclasses of meromorphic multivalent functions defined by using a linear operator and obtain some inclusion relationship.

1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p} \qquad (m, p \in \mathbf{N}), \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit disk

$$D = \{z \in C : 0 < |z| < 1\} = E \setminus \{0\},$$

where E is the open unit disk.
Let $P_k(\rho)$ be the class of analytic functions $p(z)$ defined in unit disc $E = D \cup \{0\}$, satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [3]. For $\rho = 0$, we obtain the class P_k defined and studied in [4], and for $\rho = 0$, $k = 2$, we get the well - known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater then ρ.

From (1.2) we can easily deduce that $p(z) \in P_k(\rho)$ if, and only if, there exist $p_1, p_2 \in P(\rho)$ such that, for E,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \tag{1.3}$$

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Let $f(z)$ is given by (1.1) and

$$g(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}b_nz^{n-p}. \tag{1.4}$$

Then the Hadamard product (or convolution) is defined by

$$(f*g)(z)=\frac{1}{z^p}+\sum_{n=m}^{\infty}a_nb_nz^{n-p}=(g*f)(z). \tag{1.5}$$

In the recent paper, Noor [3] (see also [8]) introduced the following family of integral operators defined on the meromorphic functions of the class Σ_p.

Let ${}_q\mathcal{F}_s(a_1,...,a_q;b_1,...,b_s;z)$ be a function given by

$${}_q\mathcal{F}_s(a_1,...,a_q;b_1,...,b_s;z)=\frac{1}{z^p}\ {}_qF_s(a_1,...,a_q;b_1,...,b_s;z) \tag{1.6}$$

$(q\leq s+1,\ q,s\in\mathbf{N}_0=\mathbf{N}\cup\{0\},\ z\in D,\ a_i,b_j\in C\setminus Z_0^-;\ Z_0^-=\{0,-1,...\},$
$i=1,...,q\ and\ j=1,...,s)$
where ${}_qF_s(z)$ is the well - known generalized hypergeometric function [7].

Corresponding to ${}_q\mathcal{F}_s(a_1,...,a_q;b_1,...,b_s;z)$ defined by (1.6), we introduce a function
${}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)$ by

$${}_q\mathcal{F}_s(a_1,...,a_q;b_1,...,b_s;z)*{}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)=\frac{1}{z^p(1-z)^{\lambda+p}}\quad(\lambda>-p), \tag{1.7}$$

Therefore the function ${}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)$ has the following form

$${}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)=\sum_{n=0}^{\infty}\frac{(\lambda+p)_n(b_1)_n\ ...\ (b_s)_n}{(a_1)_n\ ...\ (a_q)_n}z^{n-p}.$$

$$=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{(\lambda+p)_n(b_1)_n\ ...\ (b_s)_n}{(a_1)_n\ ...\ (a_q)_n}z^{n-p}. \tag{1.8}$$

We now define the linear operator

$${}_qI_s^{\lambda,p}(a_i;b_j):\Sigma_p\to\Sigma_p.$$

by

$$\left({}_qI_s^{\lambda,p}(a_i;b_j)f\right)(z)=\ \left({}_qI_s^{\lambda,p}(a_1,...,a_q;b_1,...,b_s)f\right)(z)=\ \left({}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)*f\right)(z) \tag{1.9}$$

$(q\leq s+1,\ q,s\in\mathbf{N}_0=\mathbf{N}\cup\{0\},\ z\in D,\ a_i,b_j\in C\setminus Z_0^-;\ Z_0^-=\{0,-1,...\},$
$i=1,...,q\ and\ j=1,...,s)$
Therefore the function ${}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)$ has the following form

$${}_q\mathcal{F}_s^{(-1)}(a_1,...,a_q;b_1,...,b_s;z)=\sum_{n=0}^{\infty}\frac{(\lambda+p)_n(b_1)_n\ ...\ (b_s)_n}{(a_1)_n\ ...\ (a_q)_n}z^{n-p}.$$

$$=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{(\lambda+p)_n(b_1)_n\ ...\ (b_s)_n}{(a_1)_n\ ...\ (a_q)_n}z^{n-p}. \tag{1.10}$$

Thus from (1.9), we have

$$\left({}_qI_s^{\lambda,p}(a_i;b_j)f\right)(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{(\lambda+p)_n(b_1)_n\,...\,(b_s)_n}{(a_1)_n\,...\,(a_q)_n}a_nz^{n-p}. \tag{1.11}$$

For convenience, we use the notation

$$\left({}_qI_s^{\lambda,p}(a_i+m;b_j+n)f\right)(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{(\lambda+p)_n(b_1)_n\,...(b_j+n)_n\,...\,(b_s)_n}{(a_1)_n\,...\,(a_i+m)_n\,..\,(a_q)_n}a_nz^{n-p}.$$

$$(i=1,...,q \text{ and } j=1,...,s)$$

Obviously the operators studied recently by Noor [3] and Yuan et al. [9] are special cases of ${}_qI_s^{\lambda,p}$ - operator defined by (1.11).

It can easily be verified that

$$z[\left({}_qI_s^{\lambda,p}(a_i+1;b_j)f\right)(z)]'=a_i\ \left({}_qI_s^{\lambda,p}(a_i;b_j)f\right)(z)-(a_i+p)\ \left({}_qI_s^{\lambda,p}(a_i+1;b_j)f\right)(z), \tag{1.12}$$

and

$$z[\left({}_qI_s^{\lambda,p}(a_i;b_j)f\right)(z)]'=(\lambda+p)\ \left({}_qI_s^{\lambda+1,p}(a_i;b_j)f\right)(z)-(\lambda+2p)\ \left({}_qI_s^{\lambda,p}(a_i;b_j)f\right)(z). \tag{1.13}$$

Definition 1.1. *Let $f\in\Sigma_p$. Then $f\in\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$ if and only if*

$$\left[(1-\beta)z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z)+\beta z^p({}_qI_s^{\lambda+1,p}(a_i;b_j)f)(z)\right]\in P_k(\rho),\quad z\in E,$$

where $\beta>0,\ k\geq 2,\ 0\leq\rho<1,\ \lambda>-p,\ p\in N$ and conditions given with (1.6) hold.

Definition 1.2. *Let $f\in\Sigma_p$. Then $f\in\ {}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$ if and only if*

$$\left[\beta z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z)+(1-\beta)z^p({}_qI_s^{\lambda,p}(a_i+1;b_j)f)(z)\right]\in P_k(\rho),\quad z\in E,$$

where $\beta>0,\ k\geq 2,\ 0\leq\rho<1,\ \lambda>-p,\ p\in N$ and conditions given with (1.6) are satisfied.

Lemma 1.1. (see [5]). *If $p(z)$ is analytic in E with $p(0)=1$ and α is a complex number satisfying $Re(\alpha)\geq 0$ $(\alpha\neq 0)$, then*

$$Re[p(z)+\alpha zp^{'}(z)]>\gamma\quad(0\leq\gamma<1)$$

implies

$$Re[p(z)]>\gamma+(1-\gamma)(2\sigma-1).$$

where σ is given by

$$\sigma=\sigma_{Re\alpha}=\int_0^1\left(1+t^{Re(\alpha)}\right)^{-1}dt.$$

Lemma 1.2. (see [6]). *Let $c>0,\ \lambda>0,\ \rho<1$ and $p(z)=1+b_1z+b_2z^2+...$ be analytic in E. let $Re[p(z)+\lambda czp^{'}(z)]>\rho$ in E, then*

$$Re[p(z)+czp^{'}(z)]\geq 2\rho-1+\left(\frac{1-\rho}{\lambda}\right)+2(1-\rho)\left(1-\frac{1}{\lambda}\right)\frac{1}{c\lambda}\int_0^1\frac{u^{\frac{1}{c\lambda}-1}}{1+u}du.$$

The result is sharp.

2. MAIN RESULTS

Theorem 1. *Let* $\beta > 0,\ \lambda > -p,\ 0 \le \rho < 1,\ p \in \boldsymbol{N}$ *and let* $f \in\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$. *Then* $z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z) \in P_k(\rho_1)$ *, where*

$$\rho_1 = \rho + (1-\rho)(2\gamma_1 - 1), \tag{2.1}$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{\frac{\beta}{(\lambda+p)}}\right)^{-1} dt. \tag{2.2}$$

with the conditions given in (1.6).

Proof . Let

$$z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z) = p(z). \tag{2.3}$$

Then $p(z)$ is analytic in E, after some calculations, we get

$$(1-\beta)z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z) + \beta z^p({}_qI_s^{\lambda+1,p}(a_i;b_j)f)(z) = p(z) + \frac{\beta}{\lambda+p}zp^{'}(z).$$

Since $f \in\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$, therefore

$$\left\{p(z) + \frac{\beta}{\lambda+p}zp^{'}(z)\right\} \in P_k(\rho) \qquad for z \in E.$$

This implies that

$$Re\left[p_i(z) + \frac{\beta}{\lambda+p}zp_i^{'}(z)\right] > \rho, \quad i = 1, 2.$$

using Lemma 1.1, we see that $Re\{p_i(z)\} > \rho_1$, where ρ_1 is givem by (2.1). Consequently $p(z) \in P_k(\rho_1)$ for $z \in E$, and proof is complete.

Similarly we have

Theorem 2. *Let* $\beta > 0,\ \lambda > -p,\ 0 \le \rho < 1,\ p \in \boldsymbol{N}$ *and let* $f \in\ {}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$. *Then* $z^p({}_qI_s^{\lambda,p}(a_i+1;b_j)f)(z) \in P_k(\rho_2)$ *, where*

$$\rho_2 = \rho + (1-\rho)(2\gamma_2 - 1), \tag{2.4}$$

and

$$\gamma_2 = \int_0^1 \left(1 + t^{\frac{\beta}{a_i}}\right)^{-1} dt. \tag{2.5}$$

with the conditions given in (1.6).

Theorem 3 . *Let* $\beta > 0,\ \lambda > -p,\ 0 \le \rho < 1,\ p \in \boldsymbol{N}$ *and let* $f \in\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$. *Then* $z^p({}_qI_s^{\lambda+1,p}(a_i;b_j)f)(z) \in P_k(\rho_3)$ *, where*

$$\rho_3 = 2\rho - 1 + \left(\frac{1-\rho}{\beta}\right) + 2(1-\rho)\left(1 - \frac{1}{\beta}\right)\left(\frac{\lambda+p}{\beta}\right)\int_0^1 \frac{u^{\frac{\lambda+p}{\beta}-1}}{1+u}du. \tag{2.6}$$

This result is sharp.

The Proof of Theorem 3 is similiar to Theorem 1. Here we use Lemma 1.2 instead of Lemma 1.1.

Similarly we have

Theorem 4. *Let* $\beta > 0,\ \lambda > -p,\ 0 \le \rho < 1,\ p \in \boldsymbol{N}$ *and let* $f \in\ {}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$. *Then* $z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z) \in P_k(\rho_4)$ *, where*

$$\rho_4 = 2\rho - 1 + \left(\frac{1-\rho}{\beta}\right) + 2(1-\rho)\left(1 - \frac{1}{\beta}\right)\left(\frac{a_i}{\beta}\right)\int_0^1 \frac{u^{\frac{a_i}{\beta}-1}}{1+u}du. \tag{2.7}$$

Next we define a function

$$F_\delta(z)=\frac{1}{\delta}\ z^{(-\frac{1}{\delta}-p)}\int_0^z t^{\frac{1}{\delta}+p-1}f(t)dt \quad (\delta>0, f(z)\in\Sigma_p) \tag{2.8}$$

Then the linear operator $({}_qI_s^{\lambda,p}(a_i;b_j)F_\delta)(z)$ satisfies the following relations.

$$z[({}_qI_s^{\lambda,p}(a_i+1;b_j)F_\delta)(z)]'=a_i\ ({}_qI_s^{\lambda,p}(a_i;b_j)F_\delta)(z)-(a_i+p)\ ({}_qI_s^{\lambda,p}(a_i+1;b_j)F_\delta)(z), \tag{2.9}$$

and

$$z[({}_qI_s^{\lambda,p}(a_i;b_j)F_\delta)(z)]'=(\lambda+p)\ ({}_qI_s^{\lambda+1,p}(a_i;b_j)F_\delta)(z)-(\lambda+2p)\ ({}_qI_s^{\lambda,p}(a_i;b_j)F_\delta)(z). \tag{2.10}$$

Theorem 5. *Let $\beta>0,\ \lambda>-p,\ 0\le\rho<1,\ p\in\boldsymbol{N}$ and let $f\in\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$. Then $F_\delta(z)\in\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho_1,(\lambda+p)\beta,a_i,b_j)$ for $z\in E$, where ρ_1 is given by (2.1) and the conditions given in (1.6) hold.*

Proof. We have

$$({}_qI_s^{\lambda,p}(a_i;b_j)F_\delta)(z)=\frac{1}{\delta}\ z^{-\frac{1}{\delta}-p}\int_0^z t^{\frac{1}{\delta}+p-1}({}_qI_s^{\lambda,p}(a_i;b_j)f)(t)dt \tag{2.11}$$

Differentiating (2.11), and using the identity (2.10), we have

$$(1-(\lambda+p)\beta)z^p({}_qI_s^{\lambda,p}(a_i;b_j)F_\delta)(z)+(\lambda+p)\delta z^p({}_qI_s^{\lambda+1,p}(a_i;b_j)F_\delta)(z)=z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z)$$

Now using Theorem 1, we obtain the required result contained in Theorem 5.
Similarly we have

Theorem 6. *Let $\beta>0,\ \lambda>-p,\ 0\le\rho<1,\ p\in\boldsymbol{N}$ and let $f\in\ {}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho,\beta,a_i,b_j)$. Then $F_\delta(z)\in\ {}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho_2,\alpha_i\ \delta,a_i,b_j)$ for $z\in E$, where ρ_2 is given by (2.4) and the conditions given in (1.6) hold.*

Theorem 7. *For $0\le\beta_2<\beta_1,\ \lambda>-p,\ 0\le\rho<1,\ p\in\boldsymbol{N},\ k\ge2$, we have*

$${}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta_1,a_i,b_j)\subset\ {}_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta_2,a_i,b_j) \tag{2.12}$$

with the conditions given in (1.6).

Proof. For $\beta_2=0$, the proof is immediate. Let $\beta_2>0$ and $f\in_q\mathcal{T}_s^{\lambda,p,k}(\rho,\beta_1,a_i,b_j)$. Then there exist two functions $h_1,h_2\in P_k(\rho)$ such that, from definition 1.1 and Theorem 1,

$$(1-\beta_1)z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z)+\beta_1z^p({}_qI_s^{\lambda+1,p}(a_i;b_j)f)(z)=h_1(z) \tag{2.13}$$

and

$$z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z)=h_2(z) \tag{2.14}$$

Hence

$$(1-\beta_2)z^p({}_qI_s^{\lambda,p}(a_i;b_j)f)(z)+\beta_2z^p({}_qI_s^{\lambda+1,p}(a_i;b_j)f)(z)=\left(\frac{\beta_2}{\beta_1}\right)h_1(z)+\left(1-\frac{\beta_2}{\beta_1}\right)h_2(z) \tag{2.15}$$

Since the class $P_k(\rho)$ is a convex set, it follows that the right-hand side of (2.15) belongs to $P_k(\rho)$ and we arrive at the result (2.12).
Similarly we have

Theorem 8. *For $0\le\beta_2<\beta_1,\ \lambda>-p,\ 0\le\rho<1,\ p\in\boldsymbol{N},\ k\ge2$ then*

$${}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho,\beta_1,a_i,b_j)\subset\ {}_q\Sigma\mathcal{S}_s^{\lambda,p,k}(\rho,\beta_2,a_i,b_j)$$

with the conditions given in (1.6).

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