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SOLVABILITY FOR NONLOCAL PROBLEM OF SECOND-ORDER DIFFERENTIAL EQUATION

EMAN M. A. HAMD-ALLAH

Abstract. Here, we study the existence of a positive nondecreasing solution for the nonlocal problem of the differential equation

$$x^{''}(t) \ = \ f(t, x(t)), \ t \in (0,1) \tag{1}$$

with the nonlocal condition

$$x(0) = \sum_{k=1}^{n-2} a_k \ x(\tau_k), \ x'(0) = \sum_{j=1}^{m-2} b_j \ x'(\eta_j) \tag{2}$$

where:
$\tau_k, \eta_j \in (0,1), \ 0 < \tau_1 < \tau_2 < .... < \tau_{n-2} < 1$ and $0 < \eta_1 < \eta_2 < .... < \eta_{m-2} < 1$.
As an application, we prove that the existence of the maximal and minimal positive solutions for the nonlocal problem of the differential equation (1)with the nonlocal condition

$$x(0) = \alpha \ x(b), \ x'(0) = \beta \ x'(c). \tag{3}$$

where $b \in [\tau_1, \ \tau_{n-2}], \ c \in [\eta_1, \ \eta_{m-2}], \ \alpha = \sum_{k=1}^{n-2} a_k$ and $\beta = \sum_{j=1}^{m-2} b_j$.

1. Introduction

The study of initial value problems with nonlocal conditions is of significance, since they have applications in problems in physics and other areas of applied mathematics ([18],[19]).
Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred examples, to ([1]-[5]), ([8]-[13])and ([27]-[30]) and references therein.

2. Integral equation representation

Consider the nonlocal problem (1) and (2). Assume the following assumptions

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(i) $f : [0,1] \times I\!R^+ \to I\!R^+$ is measurable in $t \in [0,1]$ for all $x \in I\!R^+$ and continuous in $x \in I\!R^+$ for almost all $t \in [0,1]$.
(ii) There exists an integrable function $m \in L^1[0,1]$ such that $f(t,x) \leq m(t)$.
(iii) $\int_0^1 m(s)\ ds \leq M$.
(iv) $\sum_{k=1}^{n-2} a_k < 1$, $\sum_{j=1}^{m-2} b_j < 1.$

Lemma 1. The solution of the nonlocal problem (1)-(2) can be expressed by the integral equation

$$\begin{aligned} x(t) &= AB\left(\sum_{k=1}^{n-2} a_k \tau_k\right)\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ A\left(\sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s,x(s))ds\right) \\ &+ B\ t\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ \int_0^t (t-s) f(s,x(s))ds, \end{aligned}$$

where $A = \left(1 - \sum_{k=1}^{n-2} a_k\right)^{-1}$ and $B = \left(1 - \sum_{j=1}^{m-2} b_j\right)^{-1}$.

Proof. Integrating equation (1), we obtain

$$x'(t) = x'(0) + \int_0^t f(s,x(s))ds. \tag{4}$$

Let $t = \eta_j$ in (4), we get

$$x'(\eta_j) = x'(0) + \int_0^{\eta_j} f(s,x(s))ds,$$

$$\sum_{j=1}^{m-2} b_j x'(\eta_j) = x'(0)\sum_{j=1}^{m-2} b_j + \sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds,$$

and

$$x'(0) = x'(0)\sum_{j=1}^{m-2} b_j + \sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds,$$

$$\left(1 - \sum_{j=1}^{m-2} b_j\right) x'(0) = \sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds,$$

$$x'(0) = B\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds, \tag{5}$$

where $B = \left(1 - \sum_{j=1}^{m-2} b_j\right)^{-1}$.

Integrating equation (4), we obtain

$$x(t) \ = \ x(0) \ + \ x'(0) \ t \ + \ \int_0^t (t-s) f(s, x(s)) ds. \tag{6}$$

Let $t = \tau_k$ in (6), we get

$$x(\tau_k) \ = \ x(0) \ + x'(0) \ \tau_k \ + \ \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds,$$

$$\sum_{k=1}^{n-2} a_k \ x(\tau_k) \ = \ x(0) \sum_{k=1}^{n-2} a_k \ + x'(0) \sum_{k=1}^{n-2} a_k \ \tau_k \ + \ \sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds,$$

and

$$x(0) \ = \ x(0) \sum_{k=1}^{n-2} a_k \ + x'(0) \sum_{k=1}^{n-2} a_k \ \tau_k \ + \ \sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds,$$

$$\left(1 \ - \ \sum_{k=1}^{n-2} a_k\right) \ x(0) \ = \ x'(0) \sum_{k=1}^{n-2} a_k \ \tau_k \ + \ \sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds,$$

$$x(0) \ = \ A x'(0) \ \sum_{k=1}^{n-2} a_k \ \tau_k \ + \ A \ \sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds \tag{7}$$

where $A \ = \ \left(1 \ - \ \sum_{k=1}^{n-2} a_k\right)^{-1}$.
Substitute from (5) into (7), we deduce that

$$\begin{aligned} x(0) \quad &= \quad AB \left(\sum_{k=1}^{n-2} a_k \tau_k \right) \left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s, x(s)) ds \right) \\ &+ \quad A \left(\sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds \right). \end{aligned} \tag{8}$$

Substitute from (5) and (8) into (6), we get

$$\begin{aligned} x(t) \quad &= \quad AB \left(\sum_{k=1}^{n-2} a_k \tau_k \right) \left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s, x(s)) ds \right) \\ &+ \quad A \left(\sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(s)) ds \right) \\ &+ \quad B \ t \left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s, x(s)) ds \right) \\ &+ \quad \int_0^t (t-s) f(s, x(s)) ds, \end{aligned} \tag{9}$$

which proves that the solution of the nonlocal problem (1)-(2) can be expressed by the integral equation (9).

3. Existence of solution

Now, we study the existence of a solution of the nonlocal problem (1)-(2).

Theorem 1. Let the assumptions (i)-(iv) be satisfied. Then the nonlocal problem (1)-(2) has at least one solution $x \in C[0,1]$.

proof. Define the subset $Q_r \subset C[0,1]$ by $Q_r = \{x \in I\!R : |x(t)| \leq r \}$.
Clearly the set Q_r , is nonempty, closed and convex.
Let H be an operator defined by

$$\begin{aligned}
(Hx)(t) &= AB\left(\sum_{k=1}^{n-2} a_k \tau_k\right)\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\
&+ A\left(\sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s) f(s,x(s))ds\right) \\
&+ B\,t\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\
&+ \int_0^t (t-s) f(s,x(s))ds.
\end{aligned}$$

Now, let $x \in Q_r$ then

$$\begin{aligned}
|(Hx)(t)| &\leq AB\left(\sum_{k=1}^{n-2} a_k \tau_k\right)\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} |f(s,x(s))|ds\right) \\
&+ A\left(\sum_{k=1}^{n-2} a_k \int_0^{\tau_k} (\tau_k - s)|f(s,x(s))|ds\right) + B\,t\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} |f(s,x(s))|ds\right) \\
&+ \int_0^t (t-s)|f(s,x(s))|ds. \\
&\leq AB\left(\sum_{k=1}^{n-2} a_k \tau_k\right)\left(\sum_{j=1}^{m-2} b_j \int_0^1 m(s)ds\right) + A\left(\sum_{k=1}^{n-2} a_k \int_0^1 m(s)ds\right) \\
&+ B\left(\sum_{j=1}^{m-2} b_j \int_0^1 m(s)ds\right) + \int_0^1 m(s)ds, \\
&\leq AB\left(\sum_{k=1}^{n-2} a_k\right)\left(\sum_{j=1}^{m-2} b_j\right) M + A\left(\sum_{k=1}^{n-2} a_k\right) M + B\left(\sum_{j=1}^{m-2} b_j\right) M + M \\
&= r,
\end{aligned}$$

where $r = (ABCD + AC + BD + 1)\,M$, $C = \sum_{k=1}^{n-2} a_k$ and $D = \sum_{j=1}^{m-2} b_j$.
Then $\{Hx(t)\}$ is uniformly bounded in Q_r .

Also, for $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
(Hx)(t_2) - (Hx)(t_1) &= B(t_2 - t_1)\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\
&+ \int_0^{t_2} (t_2 - s) f(s,x(s))ds - \int_0^{t_1} (t_1 - s) f(s,x(s))ds, \\
&= B(t_2 - t_1)\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\
&+ \int_0^{t_1} (t_2 - t_1) f(s,x(s))ds + \int_{t_1}^{t_2} (t_2 - s) f(s,x(s))ds.
\end{aligned}$$

Then

$$\begin{aligned}
|(Hx)(t_2) - (Hx)(t_1)| &\leq B|t_2 - t_1|\left(\sum_{j=1}^{m-2} b_j \int_0^{1} m(s)ds\right) \\
&+ |t_2 - t_1| \int_0^1 m(s)ds + \int_{t_1}^{t_2} (t_2 - s) m(s)ds \\
&= BD|t_2 - t_1|M + |t_2 - t_1|M + \int_{t_1}^{t_2} (t_2 - s) m(s)ds.
\end{aligned}$$

Therefore $\{Hx(t)\}$ is equi-continuous. By Arzela-Ascolis Theorem $\{Hx(t)\}$ is relatively compact.
Since all conditions of Schauder fixed point theorem are hold, then H has a fixed point in Q_r which proves that the existence of at least one solution $x \in C[0,1]$ of the integral equation (9).
To complete the proof, we prove that the integral equation (9) satisfies the nonlocal problem (1)-(2).
Differentiating (9), we get

$$x'(t) = B\left(\sum_{j=1}^{m-2} b_j \int_0^{\eta_j} f(s,x(s))ds\right) + \int_0^t f(s,x(s))ds \tag{10}$$

and

$$x''(t) = f(t,x(t)).$$

Let $t = \tau_k$ in (9), we get

$$\begin{aligned} x(\tau_k) &= AB\left(\sum_{k=1}^{n-2} a_k\tau_k\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ A\left(\sum_{k=1}^{n-2} a_k\int_0^{\tau_k}(\tau_k - s)f(s,x(s))ds\right) \\ &+ B\ \tau_k\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ \int_0^t (\tau_k - s)f(s,x(s))ds. \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{n-2} a_k\ x(\tau_k) &= AB\left(\sum_{k=1}^{n-2} a_k\right)\left(\sum_{k=1}^{n-2} a_k\tau_k\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ A\ \left(\sum_{k=1}^{n-2} a_k\right)\left(\sum_{k=1}^{n-2} a_k\int_0^{\tau_k}(\tau_k - s)f(s,x(s))ds\right) \\ &+ B\ \left(\sum_{k=1}^{n-2} a_k\tau_k\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ \sum_{k=1}^{n-2} a_k\ \int_0^t (\tau_k - s)f(s,x(s))ds. \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{n-2} a_k\ x(\tau_k) &= AB\left(\sum_{k=1}^{n-2} a_k\tau_k\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right)\left(\sum_{k=1}^{n-2} a_k + \frac{1}{A}\right) \\ &+ A\ \left(\sum_{k=1}^{n-2} a_k\int_0^{\tau_k}(\tau_k - s)f(s,x(s))ds\right)\left(\sum_{k=1}^{n-2} a_k + \frac{1}{A}\right), \end{aligned}$$

but $\left(\sum_{k=1}^{n-2} a_k + \frac{1}{A}\right) = 1$. Then

$$\begin{aligned} \sum_{k=1}^{n-2} a_k\ x(\tau_k) &= AB\left(\sum_{k=1}^{n-2} a_k\tau_k\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right) \\ &+ A\ \left(\sum_{k=1}^{n-2} a_k\int_0^{\tau_k}(\tau_k - s)f(s,x(s))ds\right) \\ &= x(0). \end{aligned}$$

Let $t = \eta_j$ in (10), we obtain

$$x'(\eta_j) = B\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right) + \int_0^{\eta_j} f(s,x(s))ds$$

$$\begin{aligned}
\sum_{j=1}^{m-2} b_j\ x'(\eta_j) &= B\left(\sum_{j=1}^{m-2} b_j\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right)\\
&+ \sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\\
&= B\left(\sum_{j=1}^{m-2} b_j+\frac{1}{B}\right)\left(\sum_{j=1}^{m-2} b_j\int_0^{\eta_j} f(s,x(s))ds\right)\\
&= x'(0).
\end{aligned}$$

This completes the proof.

Corollary 1. The solution $x(t)$ of the nonlocal problem (1)-(2) is positive and nondecreasing.

As a particular case of Theorem 1, we have the following corollary.

Corollary 2. The nonlocal problem

$$x^{''}(t) = f(t,x(t)),\ t\in(0,1) \tag{11}$$

with the nonlocal condition

$$x(0) = \alpha x(b)\ ,\ x^{'}(0) = \beta x^{'}(c) \tag{12}$$

has at least one positive nondecreasing solution in the form

$$\begin{aligned}
x(t) &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,x(s))ds + \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,x(s))ds\\
&+ \frac{\beta\ t}{(1-\beta)}\int_0^c f(s,x(s))ds+\int_0^t (t-s)f(s,x(s))ds. \qquad (13)
\end{aligned}$$

4. Maximal and minimal solutions

Definition.
let $q(t)$ be a solution of (13). Then q is said to be a maximal solution of (13) if every solution $x(t)$ of (13) satisfies the inequality $x(t) < q(t)$.
A minimal solution s(t) can be defined by similar way by reversing the above inequality i.e. $x(t) > s(t)$.

The following lemma will be used later.
Lemma 2.
Let x, y are continuous functions on $[0,1]$, satisfying

$$\begin{aligned}
x(t) &\leq \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,x(s))ds + \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,x(s))ds\\
&+ \frac{\beta\ t}{(1-\beta)}\int_0^c f(s,x(s))ds+\int_0^t (t-s)f(s,x(s))ds,
\end{aligned}$$

$$
\begin{aligned}
y(t) \quad &\geq \quad \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,y(s))ds \ + \ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,y(s))ds \\
&+ \quad \frac{\beta\ t}{(1-\beta)}\int_0^c f(s,y(s))ds + \int_0^t (t-s)f(s,y(s))ds
\end{aligned}
$$

and one of them is strict. If $f(t,x)$ is monotonic nondecreasing in x , then

$$x(t) < y(t),\ t > 0 \tag{14}$$

proof. Let the conclusion (14) be false, then there exists t_1 such that

$$x(t_1) \ = \ y(t_1),\ t_1 > 0$$

and

$$x(t) < y(t),\ 0 < t \leq t_1.$$

From the monotonicity of f in x , we get

$$
\begin{aligned}
x(t_1) \quad &\leq \quad \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,x(s))ds \ + \ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,x(s))ds \\
&+ \quad \frac{\beta\ t_1}{(1-\beta)}\int_0^c f(s,x(s))ds + \int_0^{t_1} (t_1-s)f(s,x(s))ds \\
&< \quad \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,y(s))ds \ + \ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,y(s))ds \\
&+ \quad \frac{\beta\ t_1}{(1-\beta)}\int_0^c f(s,y(s))ds + \int_0^{t_1} (t_1-s)f(s,y(s))ds \\
&< \quad y(t_1),
\end{aligned}
$$

which contradicts the fact that $x(t_1) = y(t_1)$, then $x(t) < y(t)$.

For the existence of the maximal and minimal solutions we have the following theorem,

Theorem 2.

Let the assumptions of Theorem 1 be satisfied. If f is a nondecreasing in x on $[0,1]$. Then there exist maximal and minimal solutions of the integral equation (13).

proof. Firstly we shall prove the existence of the maximal solution of (13). Let $\epsilon > 0$ be given and consider the integral equation

$$
\begin{aligned}
x_\epsilon(t) \quad &= \quad \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f_\epsilon(s,x_\epsilon(s))ds \ + \ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f_\epsilon(s,x_\epsilon(s))ds \\
&+ \quad \frac{\beta\ t}{(1-\beta)}\int_0^c f_\epsilon(s,x_\epsilon(s))ds + \int_0^t (t-s)f_\epsilon(s,x_\epsilon(s))ds,\ t\in[0,1] \qquad (15)
\end{aligned}
$$

where $f_\epsilon(t,x_\epsilon(t)) \ = \ f(t,x_\epsilon(t)) \ + \ \epsilon$.

Clearly the function $f_\epsilon(t,x_\epsilon(t))$ satisfies assumptions (i)-(ii) of Theorem 1 and therefore equation (15) has at least a positive solution $x_\epsilon(t) \in C[0,1]$.

let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$. Then

$$\begin{aligned} x_{\epsilon_2}(t) &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)} \int_0^c f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds \ + \ \frac{\alpha}{(1-\alpha)} \int_0^b (b-s) f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds \\ &+ \frac{\beta\ t}{(1-\beta)} \int_0^c f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds + \int_0^t (t-s) f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds \\ &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)} \int_0^c (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds \ + \ \frac{\alpha}{(1-\alpha)} \int_0^b (b-s)(f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds \\ &+ \frac{\beta\ t}{(1-\beta)} \int_0^c (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds + \int_0^t (t-s)(f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds, \qquad (16) \end{aligned}$$

$$\begin{aligned} x_{\epsilon_1}(t) &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)} \int_0^c (f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds \ + \ \frac{\alpha}{(1-\alpha)} \int_0^b (b-s)(f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds \\ &+ \frac{\beta\ t}{(1-\beta)} \int_0^c (f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds + \int_0^t (t-s)(f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds, \end{aligned}$$

$$\begin{aligned} x_{\epsilon_1}(t) &> \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)} \int_0^c (f(s, x_{\epsilon_1}(s)) + \epsilon_2)ds \ + \ \frac{\alpha}{(1-\alpha)} \int_0^b (b-s)(f(s, x_{\epsilon_1}(s)) + \epsilon_2)ds \\ &+ \frac{\beta\ t}{(1-\beta)} \int_0^c (f(s, x_{\epsilon_1}(s)) + \epsilon_2)ds + \int_0^t (t-s)(f(s, x_{\epsilon_1}(s)) + \epsilon_2)ds. \qquad (17) \end{aligned}$$

Applying Lemma 2 on (16) and (17), we have

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t)\ for\ t \in [0, 1].$$

As shown before the family of functions $x_\epsilon(t)$ is equi-continuous and uniformly bounded. Hence by Arzela-Ascolis Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, 1]$. Denote this limit by q , then from the continuity of the function $f_\epsilon(t, x_\epsilon)$ in the second argument, we get

$$\begin{aligned} q(t) \ = \ \lim_{n\to\infty} x_{\epsilon_n}(t) &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)} \int_0^c f(s, q(s))ds \ + \ \frac{\alpha}{(1-\alpha)} \int_0^b (b-s) f(s, q(s))ds \\ &+ \frac{\beta\ t}{(1-\beta)} \int_0^c f(s, q(s))ds + \int_0^t (t-s) f(s, q(s))ds, \qquad (18) \end{aligned}$$

which implies that q is a solution of (13).

Finally, we shall show that q is the maximal solution of (13). To do that, let x_ϵ be any solution of (13). Then

$$\begin{aligned} x_\epsilon(t) &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f_\epsilon(s,x_\epsilon(s))ds\ +\ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f_\epsilon(s,x_\epsilon(s))ds \\ &+ \frac{\beta\ t}{(1-\beta)}\int_0^c f_\epsilon(s,x_\epsilon(s))ds + \int_0^t (t-s)f_\epsilon(s,x_\epsilon(s))ds \\ &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c (f(s,x_\epsilon(s))+\epsilon)ds\ +\ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)(f(s,x_\epsilon(s))+\epsilon)ds \\ &+ \frac{\beta\ t}{(1-\beta)}\int_0^c (f(s,x_\epsilon(s))+\epsilon)ds + \int_0^t (t-s)(f(s,x_\epsilon(s))+\epsilon)ds \\ &> \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,x_\epsilon(s))ds\ +\ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,x_\epsilon(s))ds \\ &+ \frac{\beta\ t}{(1-\beta)}\int_0^c f(s,x_\epsilon(s))ds + \int_0^t (t-s)f(s,x_\epsilon(s))ds. \end{aligned} \tag{19}$$

And for any solution $x(t)$ of (13), we have

$$\begin{aligned} x(t) &= \frac{\alpha\ \beta\ b}{(1-\alpha)(1-\beta)}\int_0^c f(s,x(s))ds\ +\ \frac{\alpha}{(1-\alpha)}\int_0^b (b-s)f(s,x(s))ds \\ &+ \frac{\beta\ t}{(1-\beta)}\int_0^c f(s,x(s))ds + \int_0^t (t-s)f(s,x(s))ds. \end{aligned} \tag{20}$$

Applying Lemma 2,we get

$$x(t) < x_\epsilon(t)\ for\ t\in[0,1],$$

from the uniqueness of the maximal solution, it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t\in[0,1]$ as $\epsilon_n\to 0$.
By similar way as done above we can prove the existence of the minimal solution of (13).

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Eman M. A. Hamd-Allah

Faculty of Science, Alexandria University, Alexandria, Egypt

E-mail address: eman_hr2003@yahoo.com