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REVIEW: OPTION PRICING MODELS

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Abstract. Options are the important financial derivatives that control the investment risks of investors in financial market. To estimate the theoretical price of an option, or option pricing, is one of the most important issue in financial research. The objective of option pricing is to find the current fair price, for decision making, in contrast with future option price. The most effective methods for option pricing are the Black-Scholes (BS) method,the binomial Tree (BT) method, Monte Carlo (MC) simulation method, Finite difference methods and so many other approaches are used for option pricing. This article will review the important aspects of option pricing and the working of different approaches used for option pricing.

1. Introduction

An option is a security in which its owner gets the right to trade in a fixed number of shares of a specified common stock at a fixed price at any time on or before a given date. The act of making this transaction is known as exercising of the option. The fixed price is termed as the striking price and the given date as the expiration date. A call option gives the right to buy the shares and a put option gives the right to sell the shares [1].

Options are just like any other investments in many ways in which we need to understand what price we have to decide for the asset to make profit in future as compare to the market price of the option. Option pricing comes in two flavors: American and European.

American style option pricing allows its owner to exercise at any time prior to expiry date while the European style option pricing allows its owner to exercise on the expiry date. So many approaches have been developed to find the fair market value of the option, which are referred to as option pricing models.

These approaches are Black-Schols option pricing model, Binomial model, Monte-Carlo simulation method, Finite difference methods and so many other approaches are also in the list.

The history of stock options trading begins with the 1973 establishment of the Chicago board options exchange (CBOE), (CBOE is the largest business option

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exchanges in the world after that several) and the development of the Black-Scholes option pricing model [2].

Over the last few decades option pricing problem has gained a lot of attention due to the famous work of Black and schols[3]. The world of options underwent a revolutionary change in 1973 when Fischer Black, Maryon Schols and Robert C. Merton published their seminal paper on theory of option pricing.

Moreover, in the same year, Robert Merton extended the Black- Scholes (BS) option pricing model in several important ways. The BS formula has been widely used by traders to determine the price for an option.

After the Black-Scholes option pricing model in 1973, a number of other popular approaches have been developed following the BS model, in which including Cox-Ross-Rubinstein (1979) binomial tree model, Monte-Carlo Simulation method and finite difference methods to price the derivative governed by solving the underlying partial differential equation .

Over the past decade, option has developed to provide the basis for corporate hedging and for the asset/liability management of financial institutions. Options form the foundation of innovative financial instruments, which are extremely versatile securities that can be used in many different ways[4].

2. Our Approach

Our approach is based on the concept of financial derivatives. First we will discuss some definitions and mathematical tools useful in the valuation of financial derivatives and then discuss some approaches used for option pricing i.e option pricing models. Later on we will compare the models in a descriptive way.

3. Some Definitions

In recent years, a large variety of financial instruments have been created by financial institutions. The existence of financial mathematics has led to exploitation of advanced tools like martingale theory, stochastic process, markov process, brownian motion and partial differential equations in the pricing of these instruments.We concentrate on the pricing of options using these advanced mathematical tools[3].

3.1 Stochastic Process A stochastic Process $X = X(t), t$ is a collection of random variables with index set I, where t is time.A realization of X is called a sample path. A continuous time stochastic process $X(t)$ is said to have independent increments if for all $t_0 < t_1 < t_2 < .. < t_n$ the random variables

$$X(t_1) - X(t_0), X(t_2) - X(t_1), .., X(t_n) - X(t_{n-1})$$

are independent. It is said to possess stationary increments if $X(t+s)-X(t)$ has the same distribution for all t and the distribution depends only on s.

3.2 Markov Process A Markov process is a stochastic process for which everything that we know about its future is summarized by its current value.A continuous time stochastic process $X = X(t), t \geq 0$ is Markovian if

$$Prob[X(t) \leq x|X(u), 0 \leq u \leq s] \; = \; Prob[X(t) \leq x|X(s)]$$

for $s < t$.

3.3 Martingale Suppose we observe a family of random variables and let the observed process be denoted by $S_t, t \in [0,T]$ .Let us assume that time is continuous and that over an interval $[0,T]$,we can represent the various time periods as

$$0 = t_0 < t_1 <, ., < t_{k-1} < t_k = T$$

Let $I_t, t \in [0,T]$ represents a family of information sets that become continuously available to the investor as time passes. Given $S < t < T$,this family of information sets will satisfy $I_s \subseteq I_t \subseteq I_T$ ...

This set $I_t, t \in [0,T]$, is called a filtration. At some particular time t, if the value of the price process is S_t and if it is included in the information set I_t for $t \geq 0$,then it is said that $S_t, t \in [0,T]$ is adapted to $I_t, t \in [0,T]$ .This implies that the value of S_t will be known given the information set I_t .

A stochastic process $M_t, t \geq 0$ is a martingale with respect to the family of information sets I_t and with respect to the probability Q , If for all $t \geq 0$

(i) $E_Q[|M_t|] < \infty$

(ii) Whenever $0 \leq s < t$; then $E_Q[M_t \; |I_S] \; = \; M_S$

A martingale, (1) makes the expected future value conditional on its present value or on the set of information that is known. (2) It is not expected to drift upwards or downwards and thus it is a notion of a fair game and (3) is always defined with respect to some information set, and with respect to some probability measure.

3.4 Brownian Motion A random process $B_t, t \in [0,\infty]$ is a Brownian motion if

(i) B_t has both stationary and independent increments.

(ii) B_t is a continuous function of time, with $B_0 \; = \; 0$,unless otherwise stated.

(iii) For $0 \leq s \leq t, B_t - B_s$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2|t-s|$,that is $(B_t - B_S] \; N[\mu(t-s)$, $\sigma^2|t-s|$, where μ and $\sigma \neq 0$ are real numbers.

Such a process is called $(\mu\sigma)$ Brownian motion with drift μ and variance σ^2 .The $(0,1)$ Brownian motion is called the normalized Brownian motion, or again the Wiener process. A $(\mu\sigma)$ Brownian motion is also called a generalized Wiener process or the Wiener-Bachelier process.

3.5 Geometric Brownian Motion If $X(t)$ is a Brownian motion with drift rate μ and variance rate σ^2 the process $Y(t) \; = \; e^{X(t)}, t \geq 0$ is called a geometric Brownian motion, or the exponential Brownian motion,or again the lognormal diffusion.The mean and variance are given respectively by

$$E[Y(t)] = e^{(\mu + \frac{\sigma^2}{2})t}$$

$$Var[y(t)] = e^{(2\mu+\sigma^2)t} \; [e^{\sigma^2 t} - 1]$$

3.6 Arbitrage Arbitrage is a trading strategy that involves two or more securities being mispriced relative to each other to realise a profit without taking a risk.In general arbitrage opportunities are normally rare, short-lived and therefore immaterial with respect to the volume of transactions. Thus the market does not allow risk-free profits.

The main tools used to determine the fair price of a security or a derivative asset rely on the no-arbitrage principle. It is a fundamental assumption about the market. The no-arbitrage principle is that a portfolio yielding a zero return in every possible scenario must have a zero present value. Any other value would imply arbitrage opportunities, which one can realize by shorting the portfolio if its value is positive and buying it if its value is negative.

If one makes risk free profit in the market, then arbitrage opportunities exist and it implies that the economy is in an economic disequilibrium. An economic disequilibrium is a position situation in which there is mispricing in the market and investors trade. Their trading causes prices to change, moving them to new economic equilibrium. The mispricing is corrected by trading and arbitrage opportunities no longer exist. A market is Arbitrage-free if it satisfies any of the following conditions.

(i) Market Efficiency

Market efficiency is the characteristic of a market in which the prices of the instruments trading therein reflect the true economic values to investors. If the securities market is efficient, then information is widely and cheaply available to investors and all relevant and ascertainable information is already reflected in security prices. The efficient market hypothesis comes in three different forms: weak form, semi strong form, strong form.

(ii) Self Financing Strategy

It is a trading strategy in which the value change in a portfolio is as a result of a change in the value of the underlying asset and not beacause of change in the portfolio structure If we have f$_t unitsofastocks_t and ?_t unitsofabondB_t, thentheportfoliosvalueis$

$$V_t = \varphi_t \; S_t \; + \; \psi_t \; B_t$$

The strategy is self-financing if $\varphi_{t-1} \; S_t = \varphi_t \; s_t$ and $\psi_{t-1} \; B_t = \psi_t \; B_t$,That is, we have re-adjusted the portfolio while the prices have remained the same and the total value has not changed.

(iii) Risk Neutral Valuation

It is the valuation of a derivative assuming the world is risk neutral. A risk neutral world is a world where assets are valued solely in terms of their expected return. The return on all securities is the risk-free interest rate and all individuals are indifferent to risk.Thus the risk neutral valuation principle is important in option pricing. Indeed it implies that all expected returns must be zero. As a consequence, derivative prices are determined by the expected present value payoff. We assume that the world is risk neutral and the price obtained is correct not just in a risk-neutral world but also in the real world.

3.7 Risk

We can define the risk in a portfolio as the variance of the return. This definition does not take into account the distribution of the return. Example, a bank savings account or a government bond has a guaranteed return with no variance, and is thus termed as risk-less (or risk-free). A highly volatile stock with a very uncertain return has a large variance and is a risky asset. We assume the existence of risk-free investments that give a guaranteed return with no chance of default.

We have two types of risk: specific and non-specific, called market or systematic risk. Specific risk is the component of risk associated with a single asset or a sector of the market. Non-specific risk is associated with factors affecting the whole market. Diversifying away specific risk can be achieved by having a portfolio with a large number of assets from different sectors of the market. It is not possible to diversify away non-specific risk. Market risk can be eliminated from a portfolio by taking similar positions in the assets which are highly negatively correlated; as one decreases in value, the other increases.

3.8 Volatility

A measure of risk based on the standard deviation of investment fund or we can say it is the standard deviation measure of an assets potential deviating from its current price. This is the simple definition we gave for risk. For greater the volatility of the underlying, the greater the value of the option. For options, volatility is good while for other financial assets, volatility is bad.This is due to the fact that the purchaser of options enjoy only the upside potential, not downside risk. Other financial assets have both risks. Investors are usually assumed to be risk averse and they place a lower value on highly priced volatile assets. Volatility gives uncertain values and therefore risk of loss.

The price volatility in asset markets is caused mainly by information release, the process of trading, and market-making for financial instruments. The volatility estimate is a measure of the uncertainty about the returns on the asset. When pricing options, the volatility is assumed to be: (i) Time homogenous, that is, the same over the life of the asset and (ii) Constant between the pricing date and option expiry[3].

4. APPROACHES USED FOR OPTION PRICING

The basis of trading any security based on the idea of value, the value which tells us whether or not we are getting a good deal; whether or not we are buying something low or selling it high. The determination of the value of an option is based upon a complex algorithm known as The Options Pricing Model. The Option Pricing Model calculates the values of different options .All the models are connected, they are related somehow. They start at the same place setting up a probability model to predict an expected value of an option. And they all end up at a theoretical value, which is a value that an option should be worth[4].

4.1 Black Scholes Model The Black-Scholes model for calculating the premium of an option was introduced in 1973 in a paper entitled, "The Pricing of Options and Corporate Liabilities" published in the Journal of Political Economy . The Black-Scholes model is used to calculate the theoretical price of European put and call options The model assumes that the price of heavily traded assets follow a geometric Brownian motion with constant drift and volatility.

The formula, developed by three economists Fischer Black, Myron Scholes and Robert Merton is perhaps the world's most well-known options pricing model. Their dynamic hedging strategy led to a stochastic partial differential equation, now called the BlackScholes equation, which estimates the price of the option over time. Its solution is given by the BlackScholes formula [5].

The model makes certain assumptions, including: The options are European and can only be exercised at expiration. No dividends are paid out during the life of the option. Efficient markets i.e., market movements cannot be predicted. No commissions The risk-free rate and volatility of the underlying are known and constant. Follows a lognormal distribution, that is, returns on the underlying are normally distributed.

The Black-Scholes PDE describes the evolution of any derivative whose underlying asset satisfies the Black-Scholes assumptions. Equation is,

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \ S^2 \ \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

Here,
V(S,T) is the price of a derivative as a function of time and stock price.

S, be the price of the stock.

σ , is the volatility of the stock's returns.

r, is the annualized risk-free interest rate.

On Solving this partial differential equation we get an analytical formula for pricing the european style options which is known as Black-Scholes formula.The value of a call option for a non-dividend-paying unde-rlying stock in terms of the Black-Scholes parameters is:

$$C(S,T) \ = \ N(d_1)S - N(d_2)Ke^{-r(T-t)}$$

$$\mathrm{d}_1 \ = \ \tfrac{1}{\sigma\sqrt{T-t}} \ [ln\tfrac{S}{K} + (r + \tfrac{\sigma^2}{2}) \ (T-t)]$$

$$
\begin{aligned}
d_2 &= \frac{1}{\sigma\sqrt{T-t}}\ [ln\frac{S}{K} + (r - \frac{\sigma^2}{2})\ (T-t)] \\
&= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

The price of a corresponding put option based on put-call parity is:

$$
\begin{aligned}
P(S,t) &= Ke^{-r(T-t)} - S + C(S,t) \\
&= N(-d_2)\ Ke^{-r(T-t)} - N(-d_1)S
\end{aligned}
$$

Here,

N is the cumulative frequency distribution

S is the spot price of the underlying asset

K is the strike price

T-t is the time to maturity

r is the risk free rate

σ is the volatility returns of the underlying asset

The model is essentially divided into two parts, the first part, $N(d_1)S$, multiplies the price by the change in the call premium in relation to a change in the underlying price. This part of the formula shows the expected benefit of purchasing the underlying outright. The second part $N(d_2)Ke^{-r(T-t)}$ provides the current value of paying the exercise price upon expiration (remember, the Black-Scholes model applies to European options that are exercisable only on expiration day). The value of the option is calculated by taking the difference between the two parts, as shown in the equation.If we denote the current price of the underlying by S, then the payoffs at expiry, T, for a given exercise price, K, of European Calls and Puts is:

$$C(S,T) = max(S-K,0), P(S,T) = max(K-S,0) \tag{2}$$

Obtained by solving the equation for the corresponding terminal and boundary conditions.In order to price the derivative, we need to solve (1) together with some boundary conditions. This is given by (2). Also following from (2), for any strike price, $K > 0, max(0-K,0) = 0$. Thus we have,

$$C(0,t) = 0$$

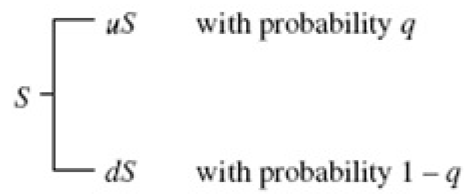
Finally, consider the case where the underlying asset increases without bound. The strike price, K becomes irrelevant and we have,

$$C(S \to \infty, t) = S$$

The European call and put analytical formulas have gained popularity in the world of finance due to the ease with which one can use the formula to value the European options [3].

4.2 Binomial Model Cox-Ross-Rubinstein presented the binomial tree model in paper Option Pricing: A Simplified Approach in 1979. The model is relatively simple and easy to understand, but it is an extremely powerful tool for pricing a wide range of option. The Binomial model is breaks down time until expiration into a series of intervals or steps. Then a tree of stock prices is produced, working forward from the present to expiration. Suppose that S is the stock price at the beginning of a given time period. At each step, its assumed that the stock price will move either up or down.The rate of return on the stock over each period can have two possible values: up to uS with probability q, or down to dS with probability $1q$, where u and d are the up and down factors with $d < 1 < u$.

files/One Step Binomial Tree.jpg



The model made the assumption that a stock will move either up or down, from one level up to the next or down to the next,continuing the pattern until you can see a two-branch type tree or binomial tree. That was the idea of the Binomial model[6].

In order to calculate the fair value of the option, first we divide the life time $[0, T]$ of the option into N time subintervals of length δt , where $\delta t = T/N$ [15]. As stock price is S for time step δt ,the expected value of S under the continuous random walk model is,

$$E(S) = \int_0^{\infty} S' \; p(S, \delta t; S', \delta t')dS' = e^{r\delta t} S$$

Where $p(S, t; S' t')$ is the probability density function, for the risk neutral random walk.The expected value of S,under the discrete binomial random walk is,

E(S) = Squ+S(1-q)d

Equating above two expected values, we get

$$Se^{r\delta t} = Squ + S(1 - q)d$$

$$e^{r\delta t} = qu + (1 - q)d \tag{3}$$

The variance of S is defined to be,

$$Var[S] = E(S^2) - [E(S)]^2 \tag{4}$$

Under the continuous random walk we have

$$E(S)^2 = \int_0^{\infty} (S')^2 \; p(S, \delta t; S', \delta t' \;)dS' = e^{(2r+\sigma^2)\delta t}(S^2)$$

Where $p(S, t; S', t' \;)$ is the probability density function. Thus the variance under the continuous process is given by

$$Var[S] = S^2 e^{2r\delta t}[e^{(\sigma^2 \delta t)} - 1] \tag{5}$$

this can be expressed as, from (4)

$$S^2 e^{2r\delta t}[e^{\sigma^2 \delta t} - 1] = qu^2 S^2 + (1 - q)d^2 S^2 - [quS + (1 - q)dS]^2$$

and this can be simplified to yield

$$e^{2r\delta t+\sigma^2\delta t} = qu^2 + (1-q)d^2 \tag{6}$$

.

If we assume $u = 1/d$, then it follows from (3) and (6) that,

$$q = \frac{e^{r\delta t} - d}{u - d} \tag{7}$$

The probability q obtained in (7) is called the risk neutral probability. It is the probability of an upward movement of the stock price that ensures that all bets are fair, that is, it ensures that there is no arbitrage[6]. The expectation of the share price can be written as

$$E[S_1] = quS + d(1-q)S \tag{8}$$

Where S_1 is the share price after one period, and using the value of q in (7) we find that $E[S_1] = Se^{r\delta t}$ which naturally follows from our assumption of the risk-neutral valuation.

We know that after one time period, the stock price can move up to uS with probability q or down to dS with probability $(1-q)$. Therefore the corresponding value of the call option at the first time movement δt is given by

$$c_u = max(uS - K, 0) : \text{ after upward movement}$$
$$c_d = max(dS - K, 0) : \text{ after downward movement}$$

We need to derive a formula to calculate the fair value of the option. The risk neutral call option price at the present time is

$$c = [qc_u + (1-q)c_d]e^{-r\delta t} \tag{9}$$

Now when we extend the model to two periods. Let c_{uu} denote the call value at time $2\delta t$ for two consecutive upward stock movement, c_{ud} for one downward and one upward movement and c_{dd} for two consecutive downward movement of the stock price. Then we have

$$c_{uu} = max(u^2S - K, 0)$$
$$c_{ud} = max(udS - K, 0)$$
$$c_{dd} = max(d^2S - K, 0) \tag{10}$$

which are illustrated in figure.2 for the three different states of the asset and call prices in the two period binomial model. Since q is the risk neutral probability, the values of call options at time, δt are

$$c_u = e^{-r\delta t}\ [qc_{uu} + (1-q)c_{ud}]$$

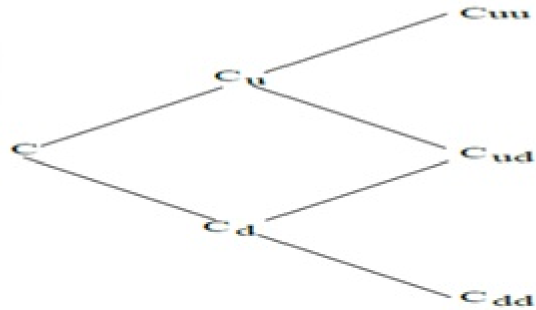
.

$$c_d = e^{-r\delta t}\ [qc_{ud} + (1-q)c_{dd}] \tag{11}$$

We substitute (11) into (9) and this gives us the current call value using time $2\delta t$ as

$$c = e^{-2r\delta t}[q^2c_{uu} + 2q(1-q)c_{ud} + (1-q)^2c_{dd}] \tag{12}$$

files/Two Step Binomial Tree.jpg



We generalize the result in (12) to value an option which expires at $T = N\delta t$ as

$$C = e^{-Nr\delta t} \textstyle\sum_{j=0}^{N} \frac{N!}{j!}(N-j)! q^j (1-q)^{N-j} \; c_u^j d^{N-j}$$

$$= e^{-Nr\delta t} \sum_{j=0}^{N} \frac{N!}{j!}(N-j)! \; q^j (1-q)^{N-j} max?(u^j d^{N-j} S - K, 0) \qquad (13)$$

Where $\frac{N!}{J!}(N-j)!$ is the binomial coefficient.

We assume that m is the smallest integer for which the options intrinsic value in (13) is greater than zero. This implies that $u^{m d N-m} S = K$. Then (13) is written as

$$C = Se^{-Nr\delta t} \sum_{j=m}^{N} \frac{N!}{J!}(N-j)! q^j (1-q)^{N-j} u^j d^{N-j} - Ke^{-Nr\delta t} \sum_{j=m}^{N} q^j (1-q)^{N-j} \qquad (14)$$

which gives us the present value of the call option. The term $e^{-Nr\delta t}$ is the discounting factor that reduces c to its present value. The first term $\frac{N!}{J!}(N-j)! q^j (1-q)^{N-j}$ is the binomial probability of j upward movements to occur after the first N trading periods and $u^j \; d^{N-j} S$ is the corresponding value of the asset after j upward move of the stock price. The second term is the present value of the options strike price[6]. Let $R = e^{r\delta t}$. We substitute R in the first term in (14) to yield

$$C = SR^{-N} \textstyle\sum_{j=m}^{N} \frac{N!}{J!}(N-j)! q^j (1-q)^{N-j} \; u^j d^{N-j} - Ke^{-Nr\delta t} \; \textstyle\sum_{j=m}^{N} \frac{N!}{J!}(N-j)! q^j (1-q)^{N-j}$$

$$= S \sum_{j=m}^{N} \frac{N!}{J!}(N-j)! [R^{-1} qu]^j [R^{-1}(1-q)d]^{N-j} - Ke^{-Nr\delta t} \sum_{j=m}^{N} \frac{N!}{J!}(N-j)! q^j (1-q)^{N-j} \qquad (15)$$

Now let $\phi(m; N, q)$ be the binomial distribution function. That is

$$\phi(m; N, q) = \sum_{j=m}^{N} \frac{N!}{J!}(N-j)! q^j (1-q)^{N-j} \qquad (16)$$

is the probability of at least m success in N independent trials, each resulting in a success with probability q and in a failure with probability $1-q$. Then , letting $q^{'} = R^{-1}\ qu$, we obtain

$$R^{-1}(1-q)d = 1-q^{'}$$

Consequently it follows from the second equality in (15) that

$$c = S\phi(m;N,q^{'}) - Ke^{-rT}\ \phi(m;N,q) \tag{17}$$

where T = Nδt.

The model in (17) was developed by Cox, Ross and Rubinstein and we will refer to it as the CRR model. The corresponding value of the European put option can be obtained using the call-put parity relationship [3].When stock price movements are governed by a multi-step binomial tree, we can treat each binomial step separately. The multi-step binomial tree can be used for the American and European style options[3].

Like the Black Scholes model, the CRR formula in (16) can only be used in the valuation of European style options and can easily be implemented in Matlab. To overcome this problem, we use multi-period binomial model for the American style options on both the dividend and non dividend paying stocks.The no-arbitrage arguments are used and no assumptions are required about the probabilities of up and down movements in the stock price at each node. We now explain the procedure for the implementation of the multi-period binomial model. At time zero, the stock price S is known. At time δt , there are two possible stock prices uS and dS. At time $2\delta t$, there are three possible stock prices $u^2\ S$, udS , $d^2\ S$ and so on. In general ,at time $i\delta t$; where $0 \le i \le N$, $(i+1)$ stock prices are considered, given by

$$Su^j\ d^{N-j} \quad \text{for} \quad j = 0,1,\ldots,N$$

Where N is the total number of movements and j is the total number of up movements. The multi-period binomial model can reflect numerous stock price outcomes if there are numerous periods. Fortunately, the binomial option pricing model is based on recombining trees, otherwise the computational burden would quickly become overwhelming as the number of moves in the tree is increased.

Options are evaluated by starting at the end of the tree at time T and working backward.We know the worth of a call and a put at time T is max $(S_T - K, O)$ and $max(K - S_T, 0)$ respectively. Because we are assuming the risk-neutral world, the value at each node at time $(T\delta t)$ can be calculated as the expected value at time T discounted at rate r for a time period δt .Similarly, the value at each node at time $(T-2\delta t)$ can be calculated as the expected value at time $(T-\delta t)$ discounted for a time period δt at rate r and so on. By working back through all the nodes, we are able to obtain the value of the option at time zero[9].

Suppose that the life of an European option on a non-dividend paying stock is divided into N subintervals of length δt . Denote the j^th node at time $i\delta t$ as the (i,j) node, where $0 \le i \le N$ and $0 \le j \le i$. Define $f_{(i,j)}$ as the value of the option at the (i,j) node. The stock price at the (i,j) node is $Su^j d^{(t-j)}$ .Then, the respective European call and put can be expressed as:

$$f_{N,j} = max(Su^j d^{N-j} - K, 0) \ \text{ for } \ j = 0, 1, ..N$$
$$f_{N,j} = max(K - Su^j d^{N-j}, 0) \ \text{ for } \ j = 0, 1..N$$

There is a probability q of moving from the (i, j) node at time $i\delta t$ to the $(i+1, j+1)$ node at time $(i+1)\delta t$, and a probability $(1-q)$ of moving from the (i, j) node at time $i\delta t$ to the $(i+1, j)$ node at time $(i+1)\delta t$ .The risk neutral valuation is

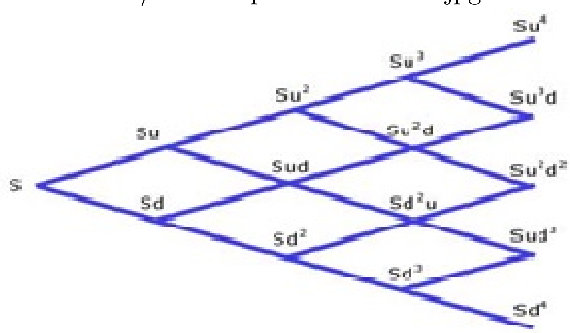
$$f_{i,j} = e^{-r\delta t} \left[q f_{i+1,j+1} + (1-q) f_{i+1,j}\right] \ \text{ and } \ 0 \le i \le N-1, \ 0 \le j \le i$$

For an American option, we check at each node to see whether early exercise is preferable to holding the option for a further time period δt . When early exercise is taken into account, this value of $f_{i,j}$) must be compared with the options intrinsic value and we have

$$f_{i,j} = max[k - Su^j d^{i-j}, e^{-r\delta t}(q f_{i+1,j+1} + (1-q) f_{i+1,j}]$$

We can represent this movement with the following diagram:

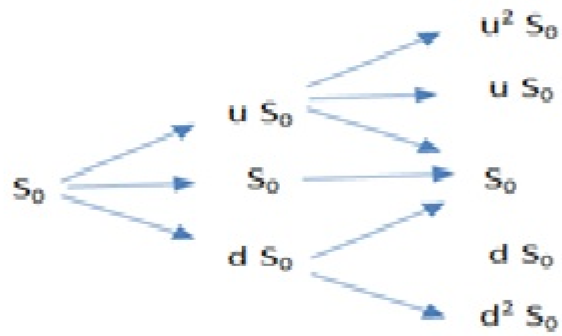
files/Multistep Binomial Tree.jpg



The model did account for early exercise and where it did have better integrity out over time it was missing something. It was missing the fact that most of the time stocks dont move. It assumed that at every level a stock either moves up or moves down to the next level. It only recognized two choices.

The response to that omission in the Binomial model was the creation of The Trinomial Model. Instead of having two choices, and a two-branch tree, the Trinomial had three choices and a three branch tree. From each step, the next step could either be up a level, down a level, or straight across sideways at the same level [4].

So, the Trinomial model was able to account for a stock not moving. The Trinomial model accepts the fact that a stock can stay still, stock does not have to move either up or down it can move sideways. Where the Binomial model improved on the Black-Scholes model by properly pricing the value of early expiration, it is failed to see the volatility smile.The Trinomial model takes into consideration the next step, that ability for the stock not to move thus accounting for the volatility smile[10].



files/Trinomial Model.jpg

4.3 Monte Carlo Simulation Method

Another model that was in existence during that early time was the Monte Carlo Simulation method. Credit for inventing the Monte Carlo Simulation method often goes to Stanislaw Ulam, a Polish born mathematician who worked for John von Neumann on the United States Manhattan Project during World War II. Ulam is primarily known for designing the hydrogen bomb with Edward Teller in 1951. He invented the Monte Carlo method in 1946 while pondering the probabilities of winning a card game of solitaire [3].

Monte-carlo simulation method also known as simulation based on the use of random numbers and probability statistics to investigate problems. Simulation is a numerical technique for conducting experiments by imitating a situation using mathematical and logical models in order to estimate the likelihood of various possible outcomes over a period of time. Monte Carlo method is an analytical technique for solving a problem by performing a large number of trial runs, called simulations, and inferring a solution from the collective results of the trial runs.It has been applied in many fields, including the pricing of financial derivatives. This method can be used in estimating option prices for derivatives that do or do not have a convenient analytical formula. It uses the risk-neutral valuation in which the expected payoff in a risk neutral world is calculated using a sampling procedure, and discounted at the risk-free interest rate. In an efficient market, the pricing of an option is equivalent to evaluating the expectation of its discounted payoff under a specified measure[12].

Main steps followed by Monte Carlo Simulation are:

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon.

- Discount the payoff corresponding to the path at the risk-free interest rate. The structure of the security in question should be adhered to.

- Repeat the procedure for a high number of simulated sample paths.

- Average the discounted cash flows over sample paths to obtain the options value.

A Monte Carlo simulation can be used as a procedure for sampling random outcomes of a process followed by the stock price

$$dS = \mu S dt + \sigma S dW_t \tag{18}$$

where dW_t is a Wiener process and S is the stock price. If S is the increase in the stock price in the next small interval of time δt then

$$\frac{\delta S}{S} = \mu \delta t + \sigma Z \sqrt{\delta t} \tag{19}$$

Where $Z \to N(0,1)$, σ is the volatility of the stock price and μ is its expected return in a risk-neutral world. So (19) is expressed as

$$S(t+\delta t) - S(t) = \mu S(t)\delta t + \sigma S(t) Z\sqrt{\delta t} \tag{20}$$

We can calculate the value of S at time $t + \delta t$ from the initial value S, then the value of S at time $t + 2\delta t$ from the value at time $t + \delta t$ and so on. We use N random samples from a normal distribution to simulate a trial for a complete path followed by S. It is more accurate to simulate lnS than S,we transform the asset price process using Itos lemma

$$dln_S = (\mu - \tfrac{\sigma^2}{2})dt + \sigma dW_t$$

So that

$$lnS(t+\delta t) - lnS(t) = (\mu - \tfrac{\sigma^2}{2})\delta t + \sigma Z\sqrt{\delta t}$$

or

$$S(t+\delta t) = S(t)exp[(\mu - \tfrac{\sigma^2}{2})\delta t + \sigma Z\sqrt{\delta t}]$$

MCS is particularly relevant when the financial derivatives pay off depends on the path followed by the underlying asset during the life of the option, that is, for path dependent options. The method can also be applied when the value of the financial derivative depends only on the final value of the underlying asset. An example is the European style option whose payoff depends on the value of S at maturity time T. The stock price process for a European option can be expressed as :

$$S_T^i = Sexp[(\mu - \tfrac{\sigma^2}{2})T + \sigma z\sqrt{T}]$$

Where $i = 1, 2, M$ and M denotes the number of trials or the different states of the world. These M simulations are the possible paths that a stock price can have at maturity date T.The estimated European call option value is

$$C = \tfrac{1}{M}\sum_{i=1}^{M} e^{-rT} max[S_T^i - K, 0]$$

This is an unbiased estimate of the derivatives price. When the number of trials M is large, the central limit theorem provides a confidence interval for the estimate, based on the sample variance of the discounted payoff.

Variance Reduction Procedures

The uncertainty about the value of the derivative is inversely proportional to the square root of the number of trials. Then, if the simulation is to give accurate results, very large number of simulated sample paths is usually necessary. This is very expensive in terms of computational time. The variance reduction technique refines and improves the efficiency of the simulation.

Antithetic Variable Technique

In this technique, a simulation trial involves calculating two values of the derivative. The first value f_1 is calculated in the usual way. The second value f_2 is calculated by changing the sign of all the random samples from the standard normal distribution. If Z is a sample used to calculate f_1 then $-Z$ is the corresponding sample used to calculate f_2 . For example, if we use (5), then we have two equations of the form

$$S_T = Sexp[(\mu - \frac{\sigma^2}{2})T + Z\sigma\sqrt{T}]$$
$$S_T = Sexp[(\mu - \frac{\sigma^2}{2})T - Z\sigma\sqrt{T}]$$

We prefer to use the random inputs obtained from the collection of antithetic pairs $(Z, -Z)$ as they are more regularly distributed than a collection of 2N independent samples. The pair is called antithetic because they exhibit negative independence. The sample mean of the antithetic pairs always equals the population mean of zero. The mean over finitely many independent samples is almost surely different from zero. We denote $\overline{f}$ as the average of f_1 and f_2 .

$$\overline{f} = \frac{f_1+f_2}{2}$$

Then,

$$Var(\overline{f}) = Var[\tfrac{1}{2}(f_1+f_2)] = \tfrac{1}{4}var[f_1] + \tfrac{1}{4}var[f_2] + \tfrac{1}{2}cov[f_1, f_2]$$

If the covariance, $Cov[f_1 f_2]$; between f_1 and f_2 is negative this will yield a smaller estimate of the variance than an independent estimate.The confidence interval is computed by estimating the standard error using the sample standard deviation of the N averaged pairs $\frac{f_1+f_2}{2}$ and not the $2N$ individual observations. Thus the antithetic variate exploits the existence of the negative correlation between two estimates[12].

Control Variate Technique

In this technique, we replace the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and a related quantity, whose expectation is known. The control variate uses a second estimate with a high positive correlation with the estimate of interest. We carry out two simulations using the same number streams and the same δt Back when models were first being auditioned for their use in option pricing, computers were simply not powerful enough to warrant the use of the Monte Carlo Simulation.There were many calculations, and it was not possible to get the all those calculations because it was uncredibly slow. Delays caused by the extended waiting periods cost traders money and that was unacceptable [14].

4.4 Finite Difference Methods Finite Difference methods are used to price options by approximating the differential equation that describes how an option price evolves over time by a set of difference equations. The discrete difference equations may then be solved iteratively to calculate a price for the option. The differential equation is converted into a set of difference equations and the difference equations are solved iteratively [3].
There are three methods
- The Explicit Euler,
- The Implicit Euler and,
- The $Crank-Nicolson$ method

to evaluate the PDE at each time step and the difference between each of the three methods is contingent on the choice of difference used for time (i.e forward , backward or central differences).The easiest scheme of the three to implement is the Explicit Euler method. Implicit Euler and Crank-Nicolson are implicit methods, which generally require a system of linear equations to be solved at each time step, which can be computationally intensive on a ?ne mesh [9].

In the formulation of a partial differential equation problem there are three components to consider:- (1) The partial differential equation. (2) The region of space-time on which the partial differential equation is required to be satisfied (3)The auxiliary boundary and initial conditions to be met.

Each finite difference method involve four step process:
- Discretize the appropriate ($continuous-time$, partial) differential equation.

- Specify a grid of potential current and future prices for the underlying asset.

- Calculate the payoff of the option at specfic boundaries of the grid of potential underlying prices.

- Iteratively determine the option price at all other grid points,including the point for the current time and underlying price (i.e the option price today). The iteration procedure is different depending on whether the explicit method, implicit method or $Crank-Nicolson$ method is being used and whether there is the possibility of early exercise of the option.

Discretizing a differential Equation

We have to discretize a partial differential equation and the boundary conditions using a forward or a backward difference approximation. The Black-Scholes PDE given by:

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}-rf=0$$

We discretize the equation with respect to time and to the underlying asset price. Divide the (S,t) plane into a sufficiently dense grid or mesh, and approximate the infinitesimal steps ΔS and Δt by some small fixed finite steps. Further, define an array of $N+1$ equally spaced grid points $t_0,t_1,,t_N$ to discretize the time derivative with

$$t_{n+1} - t_n = \Delta t \quad \text{and} \quad \Delta t = T/N$$

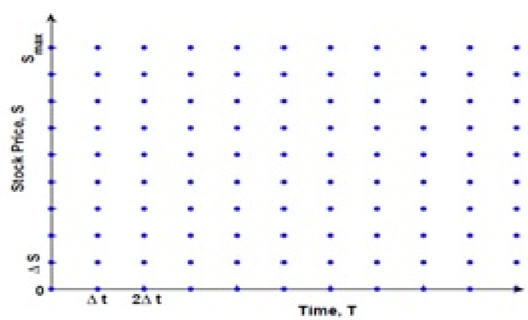
We know that the stock price cannot go below 0 and we have assumed that $S_{max} = 2S_0$ .We have $M+1$ equally spaced grid points $S_0, S_1, , S_M$ to discretize the stock price derivative with $S_{m+1} - S_m = \Delta S$ and $\Delta S = S_{max/M}$ .This gives us a rectangular region on the (S, t) plane with sides $(0, S_{max})$ and $(0, T)$.

The grid coordinates (nm) enables us to compute the solution at discrete points. The time and stock price points define a grid consisting of a total of $(M+1)(N+1)$ points. The (n,m) point on the grid is the point that corresponds to time $n\Delta t$ for $n = 0, 1, .N$, and stock price $m\Delta S$ for $m = 0, 1, ., M$ [11].

Figure 1 illustrates the discretized stock price and time derivatives into $(M+1)$ and $(N+1)$ grid points respectively. We will denote the value of the derivative at time step t_n when the underlying asset has value S_m as

$$f_{n,m} = f(n\Delta t, m\Delta S) = f(t_n, S_m) = f(t, S)$$

where n and m are the number of discrete increments in the time to maturity and stock price respectively. The discrete increments in the time to maturity and the stock price are given by Δt and ΔS , respectively.



files/image.jpg

Let $f_n = f_{n,0}, f_{n,1}, f_{n,M}$ for $n = 0, 1, N$. Then, the quantities $f_{0,m}$ and $f_{N,m}$ for $m = 0, 1, .., M$ are referred to as the boundary values which may or may not be known ahead of time but in our PDE they are known. The quantities $f_{n,m}$ for $n = 1, 2, .N-1$ and $m = 0, 1, ., M$ are referred to as interior points or values.

We classify partial differential equations as: (1) Boundary value problems, where we need to specify the full set of boundary conditions. (2) Initial value problems, where only the value of the function at one particular time needs to be specified. The majority of derivative security pricing problems, including most of the options valuation problems, are initial value problems.The idea of finite difference methods is to replace the partial derivatives occurring in the PDEs by approximations based on Taylor series expansions of functions near the point or points of interest. The derivative we seek is expressed with any desired order of accuracy.

Assuming that $f(t, s)$ is represented in the grid by $f(n, m)$,the Respective expansions of $f(t, S + \Delta S)$ and $f(t, S - \Delta S)$

$$f(t, S + \Delta S) = f(t, s) + \frac{\partial f}{\partial s}\Delta S + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\Delta S^2 + \frac{1}{6}\frac{\partial^3 f}{\partial s^3}\Delta S^3 + O(\Delta S^4) \tag{21}$$

$$f(t, S - \Delta S) = f(t, s) - \frac{\partial f}{\partial s}\Delta S + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\Delta S^2 - \frac{1}{6}\frac{\partial^3 f}{\partial s^3}\Delta S^3 + O(\Delta S^4) \tag{22}$$

Using (22), the forward difference is given by

$$\begin{aligned} \tfrac{\partial f}{\partial S}(t,S) &= \tfrac{f(t,S+\Delta S)-f(t,S)}{\Delta S} + O(\Delta S) \\ &\approx \frac{f_{n,m+1} - f_{n,m}}{\Delta S} \end{aligned} \tag{23}$$

and (23) gives the corresponding backward difference as

$$\begin{aligned} \tfrac{\partial f}{\partial s}(t,S) &= \tfrac{f(t,s)-f(t,S-\Delta S)}{\Delta S} + O(\Delta S) \\ &\approx \frac{f_{n,m} - f_{n,m-1}}{\Delta S} \end{aligned} \tag{24}$$

Subtracting (23) from (22) and taking the first order partial derivatives results in the central difference given by

$$\begin{aligned} \tfrac{\partial f}{\partial S}(t,S) &= \tfrac{f(t,S+\Delta S)-f(t,S-\Delta S)}{2\Delta S} + O(\Delta S^2) \\ &\approx \frac{f_{n,m+1} - f_{n,m-1}}{2\Delta S} \end{aligned} \tag{25}$$

The second order partial derivatives can be estimated by the symmetric central difference approximation. On adding (22) and (23) and take the second order partial derivative to have

$$\begin{aligned} \tfrac{\partial^2 f}{\partial S^2}(t,S) &= \tfrac{f(t,S+\Delta S)-2(t,S)+f(t,S-\Delta S)}{\Delta S^2} + O(\Delta S^2) \\ &\approx \frac{f_{n,m+1} - 2f_{n,m} + f_{n,m-1}}{\Delta S^2} \end{aligned} \tag{26}$$

Although there are other approximations, this approximation to $\frac{\partial^2 f}{\partial S^2}$ is preferred, as its symmetry preserves the reflectional symmetry of the second order partial derivative. It is also invariant and more accurate than other similar approximations.

We expand $f(t + \Delta t, S)$ in taylors series

$$f(t + \Delta t, S) = f(t, s) + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2 + \frac{1}{6}\frac{\partial^3 f}{\partial t^3}\Delta t^3 + O(\Delta t^4) \tag{27}$$

The forward difference for the time is given by

$$\tfrac{\partial f}{\partial t}(t,s) = \tfrac{f(t+\Delta t,S)-f(t,S}{\Delta t} + O(\Delta t)$$

$$\approx \frac{f_{n+1,m} - f_{n,m}}{\Delta t} \tag{28}$$

Replacing the first and second derivatives in the Black Scholes PDE will result in a difference equation which gives an equation that we use to approximate the solution $f(S,t)$.

A partial differential equation without the auxiliary boundary or initial conditions will either have an infinity of solutions, or have no solution. We need specify the boundary and initial conditions for the European put option whose payoff is given by $max(K - S_t, 0)$.

When the stock is worth nothing, a put is worth its strike price K.

$$f_{n,0} = K \quad \text{for n } = 0, 1, ., N$$

As the price of the underlying asset price increases, the value of the put option approaches zero.Accordingly,we choose $S_{max} = S_M$ and from this we get,

$$f_{n,M} = 0 \quad \text{for n} = 0, 1, .N$$

We know the value of the put option at time T and can impose the initial condition,

$$f_{N,m} = max(K - m\Delta S, 0) \quad \text{for m } = 0, 1., M$$

The Explicit Finite Difference Method

Given that we know the value of an option at the maturity time, it is possible to give an expression that gives us the next value $f_{m,n}$ explicitly in terms of the given values $f_{m-1,n+1}, f_{m,n+1}, f_{m+1,n+1}$.

We discretize the Black Scholes PDE above by taking the forward-difference for time discretization and the central difference for the stock price discretization. This yields

$$\frac{f_{n+1,m} - f_{n,m}}{\Delta t} + \frac{rm\Delta S}{2\Delta S}[f_{n+1,m+1} - f_{n+1,m-1}] + \frac{\sigma^2 m^2 \Delta S^2}{2\Delta S^2}[f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}] = rf_{n,m} \tag{29}$$

and re-arranging we have

$$f_{n,m} = \frac{1}{1 + r\Delta t}[\beta_{1m} f_{n+1,m-1} + \beta_{2m} f_{n+1,m} + \beta_{3m} f_{n+1,m+1}] \tag{30}$$

for n $= 0, 1, .., N - 1$ and $m = 1, 2, , M - 1$.

The forward difference for time discretization is accurate to $O(\Delta t)$ and the central difference for stock discretization to $O(\Delta S^2)$. Therefore the finite difference method is accurate to $O(\Delta t, \Delta S^2)$. The weights in (10) are given by

$$\beta_{1m} = \tfrac{1}{2}\sigma^2 m^2 \Delta t - \tfrac{1}{2} rm\Delta t \ ,$$
$$\beta_{2m} = 1 - \sigma^2 m^2 \Delta t \ ,$$

$$\beta_{3m} = \frac{1}{2} rm\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t \tag{31}$$

These weights sum to unity. They are the risk neutral probabilities of the three asset prices $S - \Delta S, S$ and $S + \Delta S$ at $t + \Delta t$ .We are assuming that the expected returns on the asset is also true in a risk neutral world. For the explicit version of the finite difference to work well, the three probabilities should be positive. The problem associated with the explicit method is that some probabilities are negative.This produces results that do not converge to the solution of the differential equation.

The condition to have non-negative probabilities is that $\sigma^2 m^2 \Delta t < 1$ and $r < \sigma^2 m$ [3].The stock price and time in the system of equations in (9) gives rise to a tridiagonal system written as $Au + \epsilon = b$ .The vector ϵ arises as a result of the boundary conditions at $m = 0$ and M for all $n > 0$. The system is represented as

$$\begin{bmatrix} \beta_{20} & \beta_{30} & 0 & \dots & 0 & 0 & 0 \\ \beta_{11} & \beta_{21} & \beta_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{1M-1} & \beta_{2M-1} & \beta_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \beta_{1M} & \beta_{2M} \end{bmatrix} \begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} = \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix} \tag{32}$$

This system of equations can be written in the form A $f_{n+1,m} = f_{n,m}$ for $m = 0, 1, , M$ and we ignore the error terms as the boundary conditions will take care of them. The vector of asset prices $f_{n+1,m}$ is known at time T from our initial condition. We can work backward by solving for $f_{n,m}(m = 0, 1, ., M)$ using the matrix A which comprises of the probabilities, $\beta_k m(k = 1, 2, 3)$ that are known. These backward iterations leads us to the value of the option obtained at time zero.The iterations in finding the solution leads to rounding errors as the difference equation is solved to give the numerical solution. If these rounding errors are not magnified at each iteration, the system is stable, otherwise it is unstable. When using finite difference grids, we encounter two kinds of problems, the stability and accuracy of the method. Our concern is to obtain an accurate solution with as few computations as possible and thats why stability and accuracy are of importance.

The Stability Issue of Explicit Method

We use the matrix A in (32) to analyze the stability of the explicit finite difference method, where the β_{km} , for $k = 1, 2, 3$ are given by (31). Matrix A is real and symmetric. If v_n is the nth eigen value of A then we have

$$\|A\|_2 = \rho(A) = max_n \ v_n \tag{33}$$

The eigen values λ_n are given by

$$\lambda_n = \beta_{2m} + 2[\beta_{1m}\beta_{3m}]^{1/2}cos?\frac{n\pi}{N} \tag{34}$$

for n $= 1, 2, , N-1$ .Further, we apply the binomial expansion on the square root part and ignore some terms. Re-arranging we get

$$\lambda_n \approx 1 - 2\sigma^2 m^2 \Delta t sin^2 ? \tfrac{n\pi}{2N}$$

Therefore the equations are stable when

$$\|A\|_2 = max|1 - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N} \leq 1$$

That is

$$1 \leq 1 - 2\sigma^2 m^2 \Delta t sin^2 \frac{n\pi}{2N} \leq 1 \tag{35}$$

for n $= 1, 2, ..N-1$ as $\Delta t \to 0$, $N \to \infty$ and $sin^2 \frac{n\pi}{2N} \to 1$.
Hence

$$0 \leq \sigma^2 m^2 \Delta t \leq 1 \tag{36}$$

In (31), the other condition is that $r < \sigma^2 m$. These conditions are necessary for the weights $\beta_{km}(k = 1, 2, 3)$ to be positive. otherwise, they will be negative. These weights areprobabilities and should always be non negative. We said that the main disadvantage of the Explicit method is that some weights are negative and thus the scheme does not converge to the solution of the differential equation.

The Implicit Finite Difference Method

We express $f_{n+1,m}$ implicitly in terms of the unknowns $f_{n,m-1}$, $f_{n,m}$ and $f_{n,m+1}$. We discretize the Black Scholes PDE above using the forward difference for time and central difference for the stock price to have

$$\frac{f_{n+1,m} - f_{n,m}}{\Delta t} + rm\Delta S[\frac{f_{n,m+1} - f_{n,m-1}}{2\Delta S}] + \frac{1}{2}\sigma^2 m^2 \Delta S^2 [\frac{f_{n,m+1} - 2f_{n,m} + f_{n,m-1}}{\Delta S^2}] \tag{37}$$

On rearranging, we get

$$f_{n+1,m} = \frac{1}{1 - r\Delta t}[\alpha_{1m} f_{n,m-1} + \alpha_{2m} f_{n,m} + \alpha_{3m} f_{n,m+1}] \tag{38}$$

for $n = 0, 1, , N-1$ and $m = 1, 2, ., M-1$.

Similar to the explicit method, the implicit method is accurate to $O(\Delta t, \Delta S^2)$ .The parameters $\alpha'_{km}s$ for $k = 1, 2, 3...$ are given as

$$\alpha_{1m} = \tfrac{1}{2} rm\Delta t - \tfrac{1}{2}\sigma^2 m^2 \Delta t$$
$$\alpha_{2m} = 1 + \sigma^2 m^2 \Delta t$$
$$\alpha_{3m} = \frac{-1}{2} rm\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t \tag{39}$$

The system of equations can be expressed as a tridiagonal system

$$\begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} = \begin{bmatrix} \alpha_{20} & \alpha_{30} & 0 & \dots & 0 & 0 & 0 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \dots & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \alpha_{1M-1} & \alpha_{2M-1} & \alpha_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_{1M} & \alpha_{2M} \end{bmatrix} \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix} \tag{40}$$

Which can be written as $Af_{n,m} = f_{n+1,m}$ for $m = 0, 1, , M$. Let $f_{n,m} = f_n$. We need to solve for f_n given matrix A and column vector f_{n+1} and this implies that $f_n = A^{-1} f_{n+1}$.

The matrix A has $\alpha_{2m} = 1 + \sigma^2 m^2 \Delta t$ in the diagonal which is positive. The product of the diagonal elements are non zero and therefore the matrix is non singular. We can solve the system by finding the inverse matrix A^{-1} .

When we apply the boundary conditions together with (38), this gives rise to some changes in the elements of matrix A with α_{20} , $\alpha_{2M} = 1$ and α_{30} , $\alpha_{1M} = 0$ .Our initial condition give values for the N^{th} time step, and we solve for f_n at t_n in terms of f_{n+1} at t_{n+1} .We set the right hand side of the system to our initial condition and solve the system to produce a solution to the equation for time step $N - 1$.

By repeatedly iterating in such a manner, we can obtain the value of f at any time step $0, 1, , N - 1$ .The implicit method allows us to use a large number of S-mesh points without having to take ridiculously small time-steps[3].

The Stability Issue of Implicit Method

We analyzed the stability of the explicit method. We apply the same principle to test for the stability of the implicit finite difference method.

The eigen values λ_n are given by

$$\lambda_n = \alpha_{2m} + 2[\alpha_{1m}\alpha_{3m}]^{1/2} cos \frac{n\pi}{N} \tag{41}$$

for $n = 1, , N - 1$ .Substituting the values of αs in (39), we have

$$\lambda_n = 1 + \sigma^2 m^2 \Delta t [1 - \frac{r^2}{\sigma^4 m^2}]^{1/2} [1 - 2 sin^2 \frac{n\pi}{2n}] \tag{42}$$

for $n = 1, 2, ., N - 1$. Furthermore, applying the binomial expansion on the square root part and re-arranging we have

$$\lambda_n \approx 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N}$$

Where there is change of sign due to the truncation of the binomial expansion .Therefore the equations are stable when

$$\|A\|_2 = max \left| 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N} \right| \leq 1$$

That is,

$$-1 \leq 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t sin^2 \frac{n\pi}{2N} \leq 1 forn = 1, 2, .., N - 1 \tag{43}$$

As $\Delta t \to 0, N \to \infty$ and $sin^2 \frac{(N-1)\pi}{2N} \to 1$,(43) reduces to $|1| \leq 1$.

$$1 + \sigma^2 m^2 \Delta t \geq 0 \text{ and } \|A\|_\infty$$

Therefore by Lax's equivalence theorem, the scheme is unconditionally stable, convergent and consistent.

Solving Systems of Linear Equations

We can apply the direct solvers or iterative solvers in solving our system of linear equations. A direct solver is one that achieves the solution within a finite number of steps. The popular direct solver is the tridiagonal solver which is the Gaussian elimination method applied to tridiagonal equations.

An iterative solver achieves a solution on the basis of satisfying an accuracy criterion. This use of accuracy as a termination criterion gives iterative solvers a dimension of flexibility and efficiency. The two main types of iterative solvers are stationary and non stationary methods.
Stationary methods use iteration schemes with parameters that remain fixed during the iterations. Examples are Jacobi, Gauss-Seidel, and Successive over-relaxation (SOR) methods.

The matrix A in the implicit method is tridiagonal and has the property that, only the diagonal, super-diagonal and Subdiagonal elements are non-zero.We can solve our system of linear equations using either the LU decomposition method or the SOR method.

The use of these techniques makes implicit method as almost as efficient as the explicit method in terms of arithmetical operations per time-step. As fewer time-steps need to be taken, the implicit finite difference method, which is unconditionally stable, is more efficient over-all than the explicit method[7].

The Crank Nicolson Method

The Crank Nicolson implicit finite difference method is the average of the implicit and explicit methods. The explicit scheme is given by (38) and the implicit by (30). We take the average of the two equations to get,

$$\frac{f_{n+1,m}-f_{n,m}}{\Delta t} + \frac{rm\Delta S}{4\Delta S}[f_{n+1,m+1} - f_{n+1,m+1} + f_{n,m+1} - f_{n,m-1}] + \frac{\sigma^2 m^2 \Delta S^2}{4\Delta S^2}[f_{n,m-1} - 2f_{n,m} + f_{n,m+1} + f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}] = \frac{1}{2}[rf_{n,m} + rf_{n+1,m}]$$

On re-arranging we get

$$[\tfrac{1}{4}rm\Delta t - \tfrac{1}{4}\sigma^2 m^2 \Delta t] \; f_{n,m-1} \; [1 + \tfrac{1}{2}r\Delta t + \tfrac{1}{2}\sigma^2 m^2 \Delta t] f_{n,m} \; + \; [\tfrac{-1}{4}\sigma^2 m^2 \Delta t - \tfrac{1}{4}rm\Delta t] f_{n,m+1}$$

$$= [\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}rm\Delta t] f_{n+1,m-1} + [1 - \frac{1}{2}r\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t] f_{n+1,m} + [\frac{1}{4}rm\Delta t + \frac{1}{4}\sigma^2 m^2 \Delta t] f_{n+1,m+1} \tag{44}$$

and we simplify to get

$$\rho_{1m} f_{n,m-1} + \rho_{2m} f_{n,m} + \rho_{3m} f_{n,m+1} = \chi_{1m} f_{n+1,m-1} + \chi_{2m} f_{n+1,m} + \chi_{3m} f_{n+1,m+1} \quad (45)$$

for $n = 0, 1, ., N-1$ and $m = 1, 2, .., M-1$. Then, the parameters ρ_{km} and χ_{km} for $k = 1, 2, 3$ are given as

$\rho_{1m} = \frac{1}{4} rm\Delta t - \frac{1}{4}\sigma^2 m^2 \Delta t$,

$\rho_{2m} = 1 + \frac{1}{2} r\Delta t + \frac{1}{2}\sigma^2 m^2 \delta t$,

$\rho_{3m} = \frac{-1}{4}\sigma^2 m^2 \delta t - \frac{1}{4} rm\delta t$,

$\chi_{1m} = \frac{1}{4}\sigma^2 m^2 \delta t - \frac{1}{4} rm\delta t$,

$\chi_{2m} = 1 - \frac{1}{2} r\delta t - \frac{1}{2}\sigma^2 m^2 \delta t$,

$$\chi_{3m} = \frac{1}{4} rm\delta t + \frac{1}{4}\sigma^2 m^2 \delta t \quad (46)$$

We express the system of equations in (45) as $Cf_n = Df_{n+1}$ .This results into a tridiagonal system given by

$$\begin{bmatrix} \rho_{20} & \rho_{30} & 0 & \dots & 0 & 0 & 0 \\ \rho_{11} & \rho_{21} & \rho_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \rho_{1M-1} & \rho_{2M-1} & \rho_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \rho_{1M} & \rho_{2M} \end{bmatrix} \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,m-1} \\ f_{n,M} \end{bmatrix} =$$

$$\begin{bmatrix} \chi_{20} & \chi_{30} & 0 & \dots & 0 & 0 & 0 \\ \chi_{11} & \chi_{21} & \chi_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \chi_{1M-1} & \chi_{2M-1} & \chi_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \chi_{1M} & \chi_{2M} \end{bmatrix} \begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix}$$

The elements of vector f_{n+1} are known at maturity time T, and we express the system as $f_n = C^{-1} D f_{n+1}$ .By repeatedly iterating from time T to time zero, we obtain the value of f as the price of the option. The diagonal entries of matrix C is $\rho_{2m} = 1 + r\Delta t/2 + \sigma^2 m^2 \ \Delta t/2$ are always positive and thus the diagonal elements are non zero. Therefore the matrix is non singular as the diagonal entries are non zero.

The boundary conditions and (45) results in some entry changes in the tridiagonal matrices C and D. For the matrix C, $\rho_{20}, \rho_{2M} = 1$ and $\rho_{30}, \rho_{1M} = 0$. For the matrix D, $\chi_{20}, \chi_{2M} = 1$ and $\chi_{30}, \chi_{1M} = 0$.

Accuracy of Crank Nicolson Method

The finite difference approximations from the Taylors series expansion leads to truncation errors and this affects the accuracy of the scheme. The Crank Nicolson method is more accurate than the Explicit and Implicit methods with an accuracy of up to $O(\Delta t^2, \Delta S^2)$. We show this accuracy by equating the central difference and the symmetric central difference at $f_{n+\frac{1}{2},m} = f(t+\frac{\Delta t}{2}, S)$. We expand $f_{n+1,m}$ in taylor series at $f_{n+1/2,m}$ to yield

$$f_{n+1,m} = f_{n+1/2,m} + \frac{1}{2}\frac{\partial f}{\partial t}\Delta t + O(\Delta t^2) \tag{47}$$

And expanding $f_{n,m}$ at $f_{n+1/2,m}$ gives

$$f_{n,m} = f_{n+1/2,m} - \frac{1}{2}\frac{\partial f}{\partial t}\Delta t + O(\Delta t^2) \tag{48}$$

Taking the average of these two equations yields

$$\tfrac{1}{2}[f_{n,m} + f_{n+1,m}] = f_{n+1/2,m} + O(\Delta t^2)$$

The subscript m was arbitrary and we can write this for subscripts $m-1, m$ and $m+1$ as follows

$$f_{n+\frac{1}{2},m-1} - 2f_{n+\frac{1}{2},m} + f_{n+\frac{1}{2},m+1} = \frac{1}{2}[f_{n,m-1} - 2f_{n,m} + f_{n,m+1}] + \frac{1}{2}[f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}] + O(\Delta t^2) \tag{49}$$

The right hand side of (49) is an average of two symmetric central differences centered at grid points n and $n+1$ Dividing by ΔS^2 we obtain the equality

$$\frac{\partial^2 f(t+\frac{1}{2}\Delta t, S)}{\partial S^2} = \frac{1}{2}[\frac{\partial^2 f(t,S)}{\partial S^2} + \frac{\partial^2 f(t+\Delta t, S)}{\partial S^2}] + O(\Delta t^2, \Delta S^2) \tag{50}$$

which is the second order partial derivative defined by the symmetric central difference approximation. The subscript m is arbitrary and we derive the central difference approximation as follows

$$f_{n+1/2,m+1} - f_{n+1/2,m-1} = \frac{1}{2}[f_{n,m+1} - f_{n,m-1}] + \frac{1}{2}[f_{n+1,m+1} - f_{n+1,m-1}] + O(\Delta t^2) \tag{51}$$

We divide the equation by $2\Delta S$ to get the equality

$$\frac{\partial f(t+\frac{1}{2}\Delta t, S)}{\partial S} = \frac{1}{2}[\frac{\partial f(t,s)}{\partial S} + \frac{\partial f(t+\Delta t, S)}{\partial S} + O(\Delta t^2, \Delta S^2) \tag{52}$$

which is the first order partial derivative defined by the symmetric central difference approximation. Now, subtract (48) from (47) to obtain the approximation of $\frac{\partial f}{\partial t}$ centered at $(t+\frac{1}{2}\Delta t, S)$.

$$\frac{\partial f(t+\frac{1}{2}\Delta t, S)}{\partial t} = \frac{f_{n+1,m} - f_{n,m}}{\Delta t} + O(\Delta t^2)$$

Hence the Black Scholes PDE centered at $(t+\frac{1}{2}\Delta t, S)$ has a finite difference approximation

$$\frac{f_{n+1,m} - f_{n,m}}{\Delta t} + \frac{rm\Delta S}{4\Delta S}[f_{n,m+1} - f_{n,m-1} + f_{n+1,m+1} - f_{n+1,m-1}] +$$
$$\frac{\sigma^2 m^2 \Delta S^2}{4\Delta S^2}[f_{n,m-1} - 2f_{n,m} + f_{n,m+1} + f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}] = rf_{n,m}$$

and re-arranging, we get an equation of the form (45) which is the exact Crank Nicolson scheme. Therefore, the scheme has a leading error of order $O(\Delta t^2 \Delta s^2)$ .The reason that Finite difference methods are a popular choice for pricing options is that all options will satisfy the Black-Scholes PDE, or appropriate variants of it.

The difference between each option contract is in determining the boundary conditions that it satisfies. Finite Difference methods can be applied to American (early exercise) Options and they can also be used for many exotic contracts [11].

5. Short Comparison of Models

In this section we are going to compare all the models which we have discussed in previous chapters in descriptive manner. Starting with the first model which is the well known and popular model among all i.e the Black-Schole Model. Black -Schole model was designed to calculate the price of European-style options and European options do not allow for early exercise. This is the main problem with the Black-Scholes model. The correctness of generated price of BS model is very depended on the accuracy of the parameters inputs Parameters like time, exercise and strike price and interest rate. Many have tested it against the option miss-pricing. It does not account for exercising early to collect interest rate. It also does not account for exercising early to collect the dividend. And further, as you go out over time, the model loses its integrity. Next model is the Binomial model, the intent here was to develop a model that was very similar to the Black-Scholes in its speed and accuracy, but adjusted for early exercise, and for better integrity out over time. And thats what the Binomial model does. To this day, there are probably more people using the Cox, Ross Rubenstein Binomial model than there are using the Black-Scholes model. But it only recognized two choices to move up and down. So Trinomial model came with three choices. It does not have to move either up or down it can move sideways. So Trinomial model was able to account for a stock not moving. Monte carlo and finite difference methods are also very effective in pricing options. Monte carlo is extremly accurate but the problem is that it is incredibly slow. So it is not possible to get all the calculations done fast to suit the needs of traders that needed instant answers and the Finite difference methods are similar to binomial and trinomial Sometimes, certain exotic options can be found to have a closed form solution. A closed form solution does not exist for other methods. In these cases, the only way a market participant will be able to obtain a price is by using an appropriate numerical method. The three finite difference methods i.e The Explicit method, Implicit method and the Crank Nicolson method has advantages and disadvantages. Crank-Nicolson exhibits the greatest accuracy of the three for a given domain discretisation. The main disadvantage to using Explicit Euler is that it is unstable for certain choices of domain discretisation. Though Implicit Euler and Crank-Nicolson involve solving linear systems of equations, they are each unconditionally stable with respect to the domain discretisation[13][16].

6. Conclusion

In this review article we explained almost all the models of option pricing and their working. It gives brief knowledge of the models of option pricing and the strategy behind each and every model. I have tried to write this review article in such a way that those students who are beginners in this field could understand without having any prior knowledge of mathematical finance, options and its models. At last this could be right to say, all models plays an eminent role in pricing options and each and every model has its advantage and disadvantage of use but its investors duty to make profit by using appropriate model at a perfect time.

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