Electronic Journal of Mathematical Analysis and Applications,
Vol. 3(1) Jan. 2015, pp. 180-187.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

ASYMPTOTIC DISTRIBUTIONS OF ORDER STATISTICS AND RECORD VALUES ARISING FROM SOME FAMILIES OF EXTENDED DISTRIBUTION FUNCTIONS

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Abstract. In this review article the asymptotic behavior of the order statistics and record values based on Marshall and Olkin, beta, gamma and Kumaraswamy -generated-distributions families is studied. In each case, the relation between the weak convergence of the base distribution and the generated family is revealed. Moreover the relations between the limit types of the base distribution and its generated family are found.

1. Introduction

Adding parameters to a well-established distribution (a base distribution function (df)) is an effective way to enlarge the behavior range of this distribution and to obtain more flexible family of distributions to model various types of data. This technique, has been tackled by many authors, among them are [14], [12], [13], [1] and [11]. We now discuss some of these known extended distributions. In the sequel we consider $F_X(x) = P(X \leq x)$ as a base df, with probability density function (pdf) $f_x(x)$.

Marshall and Olkin [14] introduced a new way to expand df's and applied it to yield a two-parameter extension of the exponential df, which can serve as a competitor to such commonly-used two-parameter distributions as the Weibull, gamma and lognormal distributions. Marshal-Olkin's way is summed in introducing a parametrization operation for adding a parameter $\alpha > 0$ to any base df F_X. The df of the extended family is defined by

$$\mathrm{M}_{F_X}(x;\alpha) = \frac{F_X(x)}{\alpha + (1-\alpha)F_X(x)}. \tag{1}$$

The pdf of the family 1 is given by

$$M_{F_X}(x;\alpha) = \frac{\alpha f_X(x)}{[1-(1-\alpha)(1-F_X(x))]^2}.$$

2010 *Mathematics Subject Classification.* 62F20, 60E05, 62G30.
Key words and phrases. Asymptotic distributions, order statistics, record values, Marshall and Olkin family, beta-generated-distributions family, Kumaraswamy-generated-distributions family, gamma-generated-distributions family.
Submitted Jun 11, 2014.

Clearly, the Marshall and Olkin distribution includes the base distribution F_X, as a special case, when $\alpha = 1$. Moreover, it is stable in the sense that

$$\mathrm{M}_{\mathrm{M}_{F_X}(.;\alpha)}(x;\beta) = \mathrm{M}_{F_X}(x;\alpha\beta),$$

or in other words, if the operation is applied twice, nothing new will be obtained the second time around. It should be noted that not all the known operators for adding a parameter satisfy the stability property, e.g., the exponentiation operator $\mathrm{P}_\alpha(F) = F^\alpha$ satisfies this property, while the Azzalini family, introduced in [5], with pdf $A_{F_X}(x;\lambda) = 2f_X(x)F_X(\lambda x)$, where F_X is symmetric and λ is any real number, does not satisfy this property. Although, the family $\mathrm{M}_{F_X}(x;\alpha)$ does not involve any special function, but it has a somewhat complicated form. To our knowledge, the family $\mathrm{M}_{\mathrm{N}}(x;\alpha)$, where N is the standard normal distribution, has not yet studied.

Eugene et al. [12] defined the beta normal distribution. Following the work of [12], Jones [13] proposed a new family of distributions motivated by order statistics. Let $I_u(a,b),\ a,b \geq 0,$ be the incomplete beta ratio function (beta df). Then the proposed new family of continuous df's is given by

$$\mathrm{B}_{F_X}(x;a,b) = I_{F_X(x)}(a,b) = \frac{1}{\beta(a,b)}\int_0^{F_X(x)} t^{a-1}(1-t)^{b-1}dt, \tag{2}$$

where $\beta(.,.)$ is the beta function. The pdf of this family is given by

$$B_{F_X}(x;a,b) = \frac{1}{\beta(a,b)}F_X^{a-1}(x)(1-F_X(x))^{b-1}f_X(x).$$

Of course $B_{F_X}(x;1,1) = f_X(x)$. The basic exemplar of the family B_{F_X} is the beta distribution itself which arises immediately if F_X is taken to be the uniform distribution. The family B_{F_X} will be most tractable when the base df F_X has a simple analytic form. Clearly, $\mathrm{B}_{F_X}(x;r,n-r+1)$, where r is an integer such that $1 \leq r \leq n$, is the df of the rth order statistic $X_{r:n}$ of size n from the df F_X. An alternative motivation for this family comes through the inverse probability integral transformation $\eta = F_X^{-1}(Y)$, where $Y \sim I_x(a,b)$ (beta df) which is immediately seen to yield $\eta \sim \mathrm{B}_{F_X}$. Thus, $\mathrm{E}(\eta)$ can be obtained using $\mathrm{E}(\eta) = \mathrm{E}(F_X^{-1}(Y))$. We may look at Jones's family as a parametrization operation for adding two parameters $a, b > 0$ to any base df F_X. Clearly, this operation, like all well-known parametrization operations for adding two parameters, does not satisfy the stability property. To our knowledge, the problem of finding stable parametrization operation for adding two parameters, is still unsolved.

Alzaatreh [1] and Alzaatreh et al. [2] suggested and studied a new family of distributions motivated by the upper record values. Let $\Gamma_x(\alpha,\beta) = (\beta^\alpha\Gamma(\alpha))^{-1}$ $\int_0^x t^{\alpha-1}e^{-t/\beta}dt$ be the incomplete gamma ratio function (the gamma df). Then the proposed new family of continuous df's, with base df F_X, is given by

$$G_{F_X}(x;\alpha,\beta) = \Gamma_{-\log(1-F_X(x))}(\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^{-\log(1-F_X(x))} t^{\alpha-1}e^{-t/\beta}dt. \tag{3}$$

The pdf of this family is given by

$$g_{F_X}(x;\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)}(-\log(1-F_X(x)))^{\alpha-1}(1-F_X(x))^{\frac{1}{\beta}-1}f_X(x).$$

Clearly, $G_{F_X}(x;1,1) = F_X(x)$. When $\alpha = n$ and $\beta = 1$, the gamma family 3 is the df of the nth upper record value arising from a sequence $\{X_i\}$ of identically independent random variables (rv's) with the pdf $f_X(x)$ and df $F_X(x)$ (see [4]). Moreover, if $Y \sim \Gamma_x(\alpha,\beta), x \geq 0$, and $\eta \sim G_{F_X}(x;\alpha,\beta)$, then $\eta = F_X^{-1}(1-e^{-Y})$. Thus $\mathrm{E}(\eta)$ can be obtained by using the relation $\mathrm{E}(\eta) = \mathrm{E}(F_X^{-1}(1-e^{-Y}))$.

In 2011, Cordeiro and de Castro [11] have created a family of generalized distributions derived from the distribution initially proposed by Kumaraswamy. Honoring this author, Cordeiro and de Castro [11] called this family of kum. For any base F_X df, the df of Kum family is defined by

$$K_{F_X}(x;a,b) = 1 - \left(1 - F_X^a(x)\right)^b, \ a,b > 0. \tag{4}$$

The pdf of the expanded family 4 has a simple form

$$k_{F_X}(x;a,b) = abF_X^{a-1}\left(1 - F_X^a(x)\right)^{b-1} f_X(x).$$

The new family 4 has an advantage over the class of generalized beta distributions 2, since it does not involve any special function. Clearly, $K_{F_X}(x;1,1) = F_X(x)$ and with $a = 1$, the Kum family coincides with the beta family 2 generated by the $I_u(1,b)$ distribution. Furthermore, for $b = 1$ and a being an integer, 4 is the distribution of the maximum of a random sample of size a from F_X. It is worth mentioning that the family 4 is relevant to the order statistics in an interesting way. Namely, if $X_{11},...,X_{1m};X_{21},...,X_{2m};...;X_{n1},...,X_{nm}$ are i.i.d rv's from the base df F_X, then $\eta = \min_{1\leq i\leq n}\max_{1\leq j\leq n}\{X_{ij}\} \sim K_{F_X}(x;n,m)$.

Actually, there are many reasons call us to expand a family of df's, including for example survival analysis (in this case we focus on the resulted survival and hazard rate functions, etc) and data modeling (in this case we focus on obtaining a wide range of the indices of skewness and kurtosis). Whatever the purpose for which the base distribution was extended to a more flexible family, it is of a great benefit to have mathematical relationships between the family and its base, which enable us to deduce the different statistical properties for this family from the corresponding properties of its base. Clearly, some of the most beneficial and important of those statistical properties are the asymptotic behavior of the df's of the different order statistics (extreme, intermediate and central order statistics) and record values. Actually, the knowledge of these asymptotic behaviors facilitate to use these flexible families to build statistical models for many important random phenomena.

In this review and expository article, which is based on the works of [8], [7] and [9], we study the weak convergence of general order statistics (extreme, intermediate and central order statistics) as well as lower and upper record values arising from a given base df F_X comparing with the weak convergence of those corresponding statistics arising from the families $\mathrm{M}_{F_X}(x;\alpha)$, $\mathrm{B}_{F_X}(x;a,b), G_{F_X}(x;\alpha,\beta)$ and $K_{F_X}(x;a,b)$.

2. Asymptotic distribution of extreme order statistics

A df $F(x)$ is said to belong to the domain of maximal (minimal) attraction of a non degenerate df $H(x)$ $(G(x))$ denoted by $F(x) \in D_{max}(H(x))$ $(F(x) \in D_{min}(G(x))$ if there exist normalizing constants $a_n > 0$ and b_n ($c_n > 0$ and d_n) such that $P(X_{n:n} \leq a_nx + b_n) \longrightarrow H(x)$ $(P(X_{1:n} \leq c_nx + d_n) \longrightarrow G(x))$ for all continuity points of $H(x)$ $(G(x))$. Sometimes, we use the notation $F(a_nx+b_n) \in D_{max}(H(x))$ $(F(c_nx+d_n) \in D_{min}(G(x)))$ when our attention is focused on some

specific normalizing constants $a_n > 0$ and b_n ($c_n > 0$ and d_n). It is well known, (see [3], Pages 210-213), that $H(x)$ is only one of the types:
(i) $H_1(x;\alpha) = e^{-x^{-\alpha}},\ x, \alpha > 0.$
(ii) $H_2(x;\alpha) = e^{-(-x)^{\alpha}},\ x \le 0, \alpha > 0.$
(iii) $H_3(x) = e^{-e^{-x}},\ -\infty < x < \infty.$
Moreover, $G(x)$ is related to $H(x)$ by $G(x) = 1 - H(-x)$.
Lemma 1. (See [3], Page 218).
(i) $F(a_nx + b_n) \in D_{max}(H(x))$ if and only if $n(1 - F(a_nx + b_n)) \longrightarrow -\log H(x)$, as $n \to \infty$.
(ii) $F(c_nx + d_n) \in D_{min}(G(x))$ if and only if $nF(c_nx + d_n) \longrightarrow -\log(1 - G(x))$, as $n \to \infty$.
Theorem 1. For any base df F and suitable normalizing constants $a_n, c_n > 0$, b_n, d_n, we have:
Part I. $\mathrm{M}_F(.;\alpha),\ K_F(.;a,b), \mathrm{B}_F(.;a,b) \in D_{max}(H)\ (D_{min}(G))$ if and only if $F \in D_{max}(H)\ (D_{min}(G))$.
Part II. $F(a_nx+b_n) \in D_{max}(H)$ (or $F(c_nx+d_n) \in D_{min}(G)$), implies $G_F(a_nx+b_n;a) \notin D_{max}(H')$ (or $G_F(c_nx+d_n;a) \notin D_{max}(G')$), for any non-degenerate limit H' (or G').
Remark 1. Theorem 1, Part I, shows that the asymptotic behavior of the extreme order statistics based on Kumaraswamy and beta-generated-distributions families are the same.
Example 1. If F is an exponential(σ) df, it can be shown that $F(a_nx + b_n) \in D_{max}(H_3(x))$ and $F(c_nx) \in D_{min}(G_2(x;1))$, where $(a_n, c_n) = (\frac{1}{\sigma}, \frac{1}{n\sigma})$ and $b_n = \frac{1}{\sigma}\log n$. An application of Theorem 1 thus yields $K_F(a_{\varphi(n;b)}x + b_{\varphi(n;b)}; a, b) \in D_{max}(H_3((bx - b\log a)))$ and $K_F(c_{\varphi(n;a)}x; a, b) \in D_{min}(G_2(b^{\frac{1}{a}}x; a))$. Note that (cf., Example 2.1 of [7]) $\mathrm{B}_F(a_{\varphi(n;b)}x + b_{\varphi(n;b)}; a, b) \in D_{max}(H_3((bx + \log b\beta(a,b))))$ and $\mathrm{B}_F(c_{\varphi(n;a)}x; a, b) \in D_{min}(G_2((a\beta(a,b))^{-\frac{1}{a}}x; a))$. Moreover, $\mathrm{M}_F(a_nx+b_n;\alpha) \in D_{max}(H_3(x - \log\alpha))$.

3. Asymptotic distribution of intermediate and central order statistics

The limit theory of the order statistic $X_{r:n}$, with variable rank (i.e., $\min(r, n - r) \to \infty$, as $n \to \infty$) was studied by many authors, such as [15], [10] and [16]. When $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \to 0$, as $n \to \infty$, $0 < \lambda < 1$, a df F is said to belong to the domain of normal $\lambda-$attraction of a non degenerate df Φ, denoted by $F \in D_\lambda(\Phi)$, if there exist normalizing constants $a_n > 0$ and b_n such that $P(X_{r:n} \le a_nx + b_n) \longrightarrow \Phi(x)$ for all continuity points of $\Phi(x)$ (when we have specific normalizing constants $a_n > 0$ and b_n, the notation $F(a_nx+b_n) \in D_\lambda(\Phi)$ may be used). Smirnov [15] showed that $F(a_nx + b_n) \in D_\lambda(\Phi)$ if and only if

$$\sqrt{n}\,\frac{F(a_nx + b_n) - \lambda}{C_\lambda} \to \mathrm{N}^{-1}(\Phi(x)),\ \text{as } n \to \infty,$$

where $C_\lambda = \sqrt{\lambda(1-\lambda)}$ and N is the standard df. Moreover, the df Φ has only one of the types
(i) $\Phi_1(x;\alpha) = \mathrm{N}(cx^\alpha),\ x, c, \alpha > 0.$
(ii) $\Phi_2(x;\alpha) = \mathrm{N}(-c(-x)^\alpha),\ x \le 0, c, \alpha > 0.$
(iii) $\Phi_3(x;\alpha) = \mathrm{N}(-c_1(-x)^\alpha),\ x \le 0,\ \ \Phi_3(x;\alpha) = \mathrm{N}(c_2x^\alpha),\ x > 0$, where

$c_1, c_2, \alpha > 0$.
(iv) $\Phi_4(x) = \frac{1}{2},\ -1 \le x \le 1$.

When the variable rank r is such that $\frac{r}{n} \to 0$, as $n \to \infty$, a df F is said to belong to the domain of attraction of a possible non degenerate lower intermediate limit df Ψ, denoted by $F \in D_r(\Psi)$, if there exist normalizing constants $a_n > 0$ and b_n such that $P(X_{r:n} \le a_n x + b_n) \longrightarrow \Psi(x)$ for all continuity points of $\Psi(x)$ (again, we use the notation $F(a_n x + b_n) \in D_r(\Psi)$, when our attention is focused on some specific normalizing constants $a_n > 0$ and b_n). An intermediate rank $r = r_n$ is said to satisfy Chibisov's condition if $\lim_{n\to\infty}(\sqrt{r_{n+z_n(\nu)}} - \sqrt{r_n}) = \frac{\theta\nu\ell}{2}$, for any sequence of integer values $\{z_n(\nu)\}$, for which $\frac{z_n(\nu)}{n^{1-\frac{\theta}{2}}} \to \nu$, as $n \to \infty$, where $0 < \theta < 1$, $\ell > 0$ and ν is any real number. It is easy to show that Chibisov's condition implies the condition $\frac{r_n}{n^\theta} \to \ell^2$. Moreover, the latter condition implies Chibisov's condition, see [6], which means that the class of intermediate rank sequences, which satisfies the Chibisov condition is a very wide class. Chibisov [10] has proved that $F(a_n x + b_n) \in D_r(\Psi)$ if and only if

$$\sqrt{n}\,\frac{F(a_n x + b_n) - R}{\sqrt{R}} = \frac{nF(a_n x + b_n) - r}{\sqrt{r}} \longrightarrow \mathrm{N}^{-1}(\Psi(x)),\ \text{as } n \to \infty,$$

where $R = \frac{r}{n}$. Moreover, the df Ψ has only one of the types

$$\left.\begin{array}{l}(i)\ \Psi_1(x;\alpha) = \mathrm{N}(\alpha \log x),\ x, \alpha > 0.\\ (ii)\ \Psi_2(x;\alpha) = \mathrm{N}(-\alpha \log(-x)),\ x \le 0,\ \alpha > 0.\\ (iii)\ \Psi_3(x) = \mathrm{N}(x).\end{array}\right\} \tag{5}$$

Theorem 2. Let $Y_{r:n}$ and $X_{r:n}$ be the rth order statistics based on F and $\mathrm{M}_F(x;\alpha)$, respectively, where $\frac{r}{n} \to \lambda \in [0,1]$ and $\min(r, n-r) \to \infty$, as $n \to \infty$. Then the weak convergence of the df $P(Y_{r:n} \le a_n x + b_n)$ to a non-degenerate distribution implies that the df $P(X_{r:n} \le a_n x + b_n)$, for all $0 < \alpha \ne 1$, does not converge to any non-degenerate limit df and vice versa.

Theorem 3. For any df F, and $0 < \lambda < 1$, we have:

Part I. $\mathrm{B}_F \in D_{\lambda^*}(\Phi)$ if and only if $F \in D_\lambda(\Phi)$, where $\lambda^* = I_\lambda(a,b)$. More specifically, let $A = \frac{\lambda^{a-\frac{1}{2}}(1-\lambda)^{b-\frac{1}{2}}}{\beta(a,b)C_{\lambda^*}}$. Then, we have

(1) $F \in D_\lambda(\Phi_i(x;\alpha))$ if and only if $\mathrm{B}_F \in D_{\lambda^*}(\Phi_i(A^{\frac{1}{\alpha}}x;\alpha)),\ i = 1,2,3$.
(2) $F \in D_\lambda(\Phi_4(x))$ if and only if $\mathrm{B}_F \in D_{\lambda^*}(\Phi_4(x))$.

In the above four cases the normalizing constants for the base df F and the family B_F are the same.

Part II. $G_F \in D_{\lambda^*}(\Phi)$ if and only if $F \in D_\lambda(\Phi)$, where $\lambda^* = \Gamma_{\hat{\lambda}}(a)$ and $\hat{\lambda} = -\log(1-\lambda)$. More specifically, let $\zeta = \frac{C_\lambda \hat{\lambda}^{a-1}}{C^*_{\hat{\lambda}}\Gamma(a)}$. Then, we have

(1) $F \in D_\lambda(\Phi_i(x;\alpha))$ if and only if $G_F \in D_{\lambda^*}(\Phi_i(\zeta^{\frac{1}{\alpha}}x;\alpha)),\ i = 1,2,3$.
(2) $F \in D_\lambda(\Phi_4(x))$ if and only if $G_F \in D_{\lambda^*}(\Phi_4(x))$.

Part III. $K_F \in D_{\tilde{\lambda}}(\Phi)$ if and only if $F \in D_\lambda(\Phi)$, where $\tilde{\lambda} = 1 - (1-\lambda^a)^b$. More specifically, let $\eta = \frac{abC_\lambda \lambda^{a-1}(1-\lambda^a)^{b-1}}{C_{\tilde{\lambda}}}$. Then, we have

(1) $F \in D_\lambda(\Phi_i(x;\alpha))$ if and only if $K_F \in D_{\tilde{\lambda}}(\Phi_i(\eta^{\frac{1}{\alpha}}x;\alpha)),\ i = 1,2,3$.
(2) $F \in D_\lambda(\Phi_4(x))$ if and only if $K_F \in D_{\tilde{\lambda}}(\Phi_4(x))$.

In all the cases of Parts I-III, the normalizing constants for the base df F and the families B_F, G_F and K_F are the same.

Example 2. While, the family $\mathrm{B}_F(x;a,b)$ is difficult to deal with except for these well behaved choices for F, but Theorem 3 enables us to reveal the asymptotic behavior of the sample quantiles arising from $\mathrm{B}_F(x;a,b)$ even if F has not a simple analytic form such as the normal df. For example, consider the sample median and set $\frac{1}{2}=I_\lambda(a,b)$. It is well known (see, [15]) that $\mathrm{N}(a_nx)\in D_\lambda(\mathrm{N}(x))$, where $a_n=\sqrt{\frac{2\pi\lambda(1-\lambda)}{n}}$. Theretofore, Theorem 3 implies that $\mathrm{B}_{\mathrm{N}}(a_nx;a,b)\in D_{\frac{1}{2}}(\mathrm{N}(Ax))$, where $A=\frac{2\lambda^{a-\frac{1}{2}}(1-\lambda)^{b-\frac{1}{2}}}{\beta(a,b)}$. As special case when $a=b=2$, we get $\lambda=\frac{1}{2}$, $a_n=\sqrt{\frac{\pi}{2n}}$ and $A=0.75$. Moreover, for every $0<\bar{\lambda}<1$, Theorem 3 implies that

(1) $G_{\mathrm{N}}(A_nx;a)\in D_{\bar{\lambda}}(\mathrm{N}(x))$, where $A_n=\frac{C_{\bar{\lambda}}\Gamma(a)}{(-\log(1-\lambda))^{a-1}}\sqrt{\frac{2\pi}{n}}$ and λ is determined from the relation $\bar{\lambda}=\Gamma_\lambda(a)$.
(2) $K_{\mathrm{N}}(B_nx;a,b)\in D_{\bar{\lambda}}(\mathrm{N}(x)$, where $B_n=\frac{C_{\bar{\lambda}}}{ab[1-(1-\bar{\lambda}^{\frac{1}{b}})]^{\frac{a-1}{a}}(1-\bar{\lambda})^{\frac{b-1}{b}}}\sqrt{\frac{2\pi}{n}}$.

Theorem 4. Let $r\sim\ell^2n^\theta,\ 0<\theta<1$, be a Chibisov rank sequence.
Part I. Let $0<a<(1-\theta)^{-1}$. Furthermore, let $r^*=nR^*$ be another Chibisov rank sequence, where $R^*=I_R(a,b)$ and $R=\frac{r}{n}$. Then with the same normalizing constants $F\in D_r(\Psi)$ implies $\mathrm{B}_F\in D_{r^*}(\Psi)$, only if $a=1$. In this case $R^*=1-(1-R)^b\sim bR$, i.e., $r^*\sim br$. More specifically, we have

(1) $\mathrm{B}_F(x;1,b)\in D_{r^*}(\Psi_i(x;\alpha\sqrt{b}))$ if $F\in D_r(\Psi_i(x;\alpha)), i=1,2$.
(2) $\mathrm{B}_F(x;1,b)\in D_{r^*}(\Psi_3(\sqrt{b}x))$ if $F\in D_r(\Psi_3(x))$.

Let $R=\frac{r}{n}$ and for suitable normalizing constants $a_n>0$ and b_n, let $F(a_nx+b_n)\in D_r(\Psi)$. Then
Part II. $G_F(a_nx+b_n;a)\notin D_{\acute{r}}(\Psi(x))$, for any Chibisov rank sequence $\acute{r}$.
Part III. $K_F(a_nx+b_n;a,b)\in D_{r^*}(\Psi(x))$, where $r^*=nR^*$, $R^*=1-(1-R^a)^b$ and $0<a<(1-\theta)^{-1}$, only if $a=1$ (in this case we have $R^*=1-(1-R^a)^b\sim bR$). More specifically,

(1) $K_F(a_nx+b_n;1,b)\in D_{r^*}(\Psi_i(x;\alpha\sqrt{b}))$ if $F(a_nx+b_n)\in D_r(\Psi_i(x;\alpha)), i=1,2$.
(2) $K_F(a_nx+b_n;1,b)\in D_{r^*}(\Psi_3(\sqrt{b}x))$ if $F(a_nx+b_n)\in D_r(\Psi_3(x))$.

Example 3. Consider the Chibisov rank sequence $r=[\sqrt{n}]$, where $\ell^2=1$ and $\theta=\frac{1}{2}$. It is well known (see,[10]) that $\mathrm{N}\in D_r(\mathrm{N})$. Theretofore, an application of Theorem 4 yields $\mathrm{B}_{\mathrm{N}}(x;1,b)=1-\mathrm{N}^b(-x)\in D_{r^*}(\mathrm{N})$, for every Chibisov rank sequence $r^*\sim\kappa^2n^\rho$, where $\kappa=\sqrt{b}$ and $\rho=\frac{1}{2}$. On the other hand, we have $R^*=1-(1-R)^b\sim bR$, where $R=\frac{r}{n}$ and $R^*=\frac{r^*}{n}$, then r^* is a Chibisov rank sequence if and only if r is a Chibisov rank sequence. Thus, an application of Theorem 4 yields that $K_{\mathrm{N}}(x;1,b)=1-\mathrm{N}^b(-x)\in D_{r^*}(\mathrm{N})$, for any Chibisov rank sequence r^*.

4. Asymptotic distribution of record values

An observation X_j will be called an upper record value if $X_j>X_i$ for every $i<j$. An analogous definition deals with lower record values. By convention X_1 is an upper as well as lower record value. The upper and lower record value sequences $\{R_n\}$ and $\{L_n\}$ can be defined by $R_n=X_{N_n}$ and $L_n=X_{M_n}$, respectively, where $N_n=\min\{j:j>N_{n-1},X_j>X_{N_{n-1}},\ n>1\}$, and $M_n=\min\{j:j>M_{n-1},X_j<X_{M_{n-1}},\ n>1\}$ (note that $N_1=M_1=1$) are the upper and lower record time sequences, respectively. A df F is said to belong to the domain of upper (lower) record value attraction of a non degenerate df Ψ (Ψ^*) and write $F \in D_{urec}(\Psi)$ ($F \in D_{lrec}(\Psi^*)$) if there exist normalizing constants $a_n > 0$ and b_n ($c_n > 0$ and d_n) such that $P(R_n \leq a_n x + b_n) \longrightarrow \Psi(x)$ ($P(L_n \leq c_n x + d_n) \longrightarrow \Psi^*(x)$), for all continuity points of $\Psi(x)$ ($\Psi^*(x)$) (again, when our attention is focused on some specific normalizing constants $a_n > 0$ and b_n ($c_n > 0$ and d_n) we use the notation $F(a_n x+b_n) \in D_{urec}(\Psi)$ ($F(c_n x+d_n) \in D_{lrec}(\Psi^*)$). It is well known that $\Psi(x)$ has only one of the types 5. Moreover, $\Psi^*(x) = 1 - \Psi(-x)$, see [3].

Lemma 2. (c.f. [3]). Let $U_{n:F}(x) = -\log(1 - F(x))$ and $V_{n:F}(x) = -\log F(x)$. Then

(1) $F(a_n x + b_n) \in D_{urec}(\Psi(x))$ if and only if $\frac{U_{n:F}(a_n x+b_n)-n}{\sqrt{n}} \longrightarrow \mathrm{N}^{-1}(\Psi(x))$.

(2) $F(c_n x + d_n) \in D_{lrec}(\Psi^*(x))$ if and only if $\frac{V_{n:F}(c_n x+d_n)-n}{\sqrt{n}} \longrightarrow \mathrm{N}^{-1}(\Psi^*(x))$.

Theorem 5. For any df F, we have:

(1) $\mathrm{M}_F(a_n x+b_n;\alpha) \in D_{urec}(\Psi)$ ($\mathrm{M}_F(c_n x+d_n;\alpha) \in D_{lrec}(\Psi^\star)$), for all $\alpha > 0$, if and only if $F(a_n x + b_n) \in D_{urec}(\Psi)$ ($F(c_n x + d_n) \in D_{lrec}(\Psi^\star)$).

(2) $F(a_n x+b_n) \in D_{urec}(\Psi_i(x))$ implies $\mathrm{B}_F(a_n x+b_n;a,b) \in D_{urec}(\Psi_i(x))$, $i = 1,2,3$, only if $b = 1$
and
$F(c_n x+d_n) \in D_{lrec}(\Psi_i^*(x))$ implies $\mathrm{B}_F(c_n x+d_n;a,b) \in D_{lrec}(\Psi_i^*(x))$, $i = 1,2,3$, only if $a = 1$.

(3) $F(a_n x + b_n) \in D_{urec}(\Psi(x))$ implies $G_F(a_n x + b_n;a) \in D_{urec}(\Psi(x))$.

(4) $F(c_n x + d_n) \in D_{lrec}(\Psi^*(x))$ implies $G_F((c_{[\frac{n}{a}]}x + d_{[\frac{n}{a}]})) \in D_{lrec}(\Psi^*(x))$. More specifically, $F(c_n x+d_n) \in D_{lrec}(\Psi_i^*(x;\alpha))$ implies $G_F((c_{[\frac{n}{a}]}x+d_{[\frac{n}{a}]})) \in D_{lrec}(\Psi_i^*(x;\sqrt{a}\alpha))$, $i = 1,2$, and $F(c_n x + d_n) \in D_{lrec}(\Psi_3^*(x))$ implies $G_F((c_{[\frac{n}{a}]}x + d_{[\frac{n}{a}]})) \in D_{lrec}(\Psi_3^*(\sqrt{a}x))$.

(5) $F(a_n x + b_n) \in D_{urec}(\Psi(x))$ implies $K_F(a_n x + b_n;a,b) \in D_{urec}(\Psi(x))$, only if $b = 1$.

(6) $F(c_n x+d_n) \in D_{lrec}(\Psi^*(x))$ implies $K_F(c_n x+d_n;a,b) \in D_{lrec}(\Psi^*(x))$, only if $a = 1$.

Example 4.

(1) It is well known that (see, [4]) the Weibull df $W(x) = (1 - e^{-x^c}) \in D_{urec}(\Psi_3)$, $c,x > 0$, with $a_n = (n + \sqrt{n})^{\frac{1}{c}} - n^{\frac{1}{c}}$ and $b_n = n^{\frac{1}{c}}$ (note that $\Psi_3 = \mathrm{N}$). Therefore, in view of Theorem 5, it follows that $\mathrm{M}_W(a_n x + b_n;\alpha)$ (i.e., the Marshall-Olkin extended Weibull distribution), $G_W(a_n x + b_n;a), W^a(a_n x + b_n) \in D_{urec}(\mathrm{N})$.

(2) It is well known that (see, [4]), the Logistic df $L(x) = (\frac{e^x}{1+e^x}) \in D_{urec}(\mathrm{N})$, with $a_n = \log(e^{n+\sqrt{n}} - 1) - \log(e^n - 1)$ and $b_n = \log(e^n - 1)$. Therefore, in view of Theorem 5, we get $G_L(a_n x + b_n;a), L^a(a_n x + b_n) \in D_{urec}(\mathrm{N})$.

(3) It is well known that (see, [4]) the standard normal df $\mathrm{N} \in D_{urec}(\mathrm{N})$, with $a_n = \frac{(n+\sqrt{n})^2}{2} + \log(n + \sqrt{n}) - \frac{n^2}{2} - \log n$ and $b_n = \frac{n^2}{2}\log n$. Therefore, in view of Theorem 5, we get $G_{\mathrm{N}}(a_n x + b_n;a), \mathrm{N}^a(a_n x + b_n) \in D_{urec}(\mathrm{N})$.

(4) It is well known that (see, [4]) the df $\mathrm{F}(x;\alpha) = (1 - e^{-\frac{\alpha^2}{4}(\log x)^2}) \in D_{urec}(\Psi_1(x;\alpha))$, $\alpha > 0, 1 < x < \infty$, with $a_n = e^{\frac{2}{\alpha}\sqrt{n}}$ and $b_n = 0$. Therefore, in view of Theorem 5, we get $G_{\mathrm{F}}(a_n x + b_n;a,\alpha), \mathrm{F}^a(a_n x + b_n;\alpha) \in D_{urec}(\Psi_1(x;\alpha))$.

(5) It is well known that (see, [4]) the df $\mathrm{L}(x;\alpha,\delta) = (1 - e^{-\frac{\alpha^2}{4}(\log(\delta - x))^2}) \in D_{urec}(\Psi_2(x;\alpha))$, $\delta - 1 \leq x < \delta < \infty$, with $a_n = e^{\frac{2}{\alpha}\sqrt{n}}$ and $b_n = \delta$. Therefore, in view of Theorem 5, we get $G_{\mathrm{L}}(a_n x + b_n; a), \mathrm{L}^a(a_n x + b_n; \alpha, \delta) \in D_{urec}(\Psi_2(x;\alpha))$.

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