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ON ASYMPTOTICALLY GENERALIZED STATISTICAL EQUIVALENT DOUBLE SEQUENCES VIA IDEALS

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Abstract. In the present paper following the line of recent work of Savaş et al. [25], we introduce the concepts of $\mathcal{I}$-statistically equivalent, $\mathcal{I}$-$[V, \lambda, \mu]$-equivalent and $\mathcal{I}$-(λ, μ)-statistically equivalent for double sequences. Moreover, we give some relations among these new notations.

1. Introduction

Fast [2] presented a generalization of the usual concept of sequential limit which their called statistical convergence. Schoenberg [23] and Salat [20] gave some basic properties of statistical convergence. There has been an effort to introduce several generalizations and variants of statistical convergence in different spaces [4, 5, 15, 21]. One such very important generalization of this notion was introduced by Kostyrko et al. [7] by using an ideal $\mathcal{I}$ of subsets of the set of natural numbers, which they called $\mathcal{I}$-convergence. More recent applications of ideals can be seen from [1, 6, 9, 24] where more references can be found. Recently, in [25], Savaş et al. introduced the notions of $\mathcal{I}$-statistical convergence, $\mathcal{I}$-λ-statistical convergence and obtained some results. On the other hand, Pobyvanets [19] introduced the concept of asymptotically regular matrix A, which preserve the asymptotic equivalence of two non-negative sequences, that is $x \sim y$ implies $Ax \sim Ay$. Subsequently, Marouf [12] and Li [11] studied the relationships between the asymptotic equivalence of two sequences in summability theory and presented some variations of asymptotic equivalence. Patterson [17] extended these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices. Later, these ideas were extended to lacunary sequences by Patterson and Savaş in [18]. More investigations in this direction and more applications can be found in [22]. In present work, we introduce some new notions $\mathcal{I}$-statistical equivalent, $\mathcal{I}$-$[V, \lambda, \mu]$-equivalent, $\mathcal{I}$-(λ, μ)-statistically equivalent for double sequences and obtain some results.

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2. Definitions and notations

First we recall some of the basic concepts, which will be used in this paper.

The notion of a statistically convergent sequence can be defined using the asymptotic density of subsets of the set of positive integers $\mathbb{N} = \{1, 2, ...\}$. For any $K \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we denote $K(n) := card K \cap \{1, 2, ..., n\}$ and we define lower and upper asymptotic density of the set K by the formulas

$$\underline{\delta}(K) := \liminf_{n\to\infty} \frac{K(n)}{n}; \ \overline{\delta}(K) := \limsup_{n\to\infty} \frac{K(n)}{n}.$$

If $\underline{\delta}(K) = \overline{\delta}(K) =: \delta(K)$, then the common value $\delta(K)$ is called the asymptotic density of the set K and

$$\delta(K) = \lim_{n\to\infty} \frac{K(n)}{n}.$$

Obviously all three densities $\underline{\delta}(K)$, $\overline{\delta}(K)$ and $\delta(K)$ (if they exist) lie in the unit interval $[0, 1]$.

$$\delta(K) = \lim_{n} \frac{1}{n} |K_n| = \lim_{n} \frac{1}{n} {}_{k=1}^{n} \chi_K(k),$$

if it exists, where χ_K is the characteristic function of the set K [3]. We say that a number sequence $x = (x_k)_{k\in\mathbb{N}}$ statistically converges to a point L if for each $\varepsilon > 0$ we have $\delta(K(\varepsilon)) = 0$, where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ and in such situation we will write $L = st\text{-}\lim x_k$.

In [14], Mursaleen introduced the idea of λ-statistical convergence for single sequences as follows:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The collection of such a sequence λ will be denoted by Δ.

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} {}_{k\in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $\lim_n t_n(x) = L$ (see [10]).

The number sequence $x = (x_k)$ is said to be λ-statistically convergent to the number L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $st_\lambda\text{-}\lim_k x_k = L$ and we denote the set of all λ-statistically convergent sequences by S_λ.

In [16], the concepts of generalized double de la Valée-Pousin mean and λ-statistical convergence of a single sequence were defined for a double sequence as follows:

Let $\lambda = (\lambda_m) \in \Delta$ and $\mu = (\mu_n) \in \Delta$. The generalized double de la Valée-Pousin mean is defined by

$$t_{m,n}(x) = \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n} x_{jk},$$

where $J_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1, n]$.

A double sequence $x = (x_{jk})$ is said to be (V, λ, μ)-summable to a number L provided that $P\text{-}\lim_{m,n} t_{m,n}(x) = L$. If $\lambda_m = m$ for all m, and $\mu_n = n$ for all n, then the space (V, λ, μ) is reduced to $(C, 1, 1)$ (see [13, 15]).

The notion of statistical convergence was further generalized in the paper [7, 8] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$, called the filter associated with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Recall the generalization of statistical convergence from [7, 8].

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)_{k \in \mathbb{N}}$ be a sequence of points in a metric space (X, ρ). We say that the sequence x is $\mathcal{I}$-convergent (or $\mathcal{I}$-converges) to a point $\xi \in X$, and we denote it by $\mathcal{I}\text{-}\lim x = \xi$, if for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, \xi) \geq \varepsilon\} \in \mathcal{I}.$$

This generalizes the notion of usual convergence, which can be obtained when we take for $\mathcal{I}$ the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. A sequence is statistically convergent if and only if it is $\mathcal{I}_\delta$-convergent, where $\mathcal{I}_\delta := \{K \subset \mathbb{N} : \delta(K) = 0\}$ is the admissible ideal of the sets of zero asymptotic density.

We also recall that the concepts of $\mathcal{I}$-statistically convergent and $\mathcal{I}$-λ-statistically convergent are studied in [25]:

A sequence $x = (x_k)$ is said to be $\mathcal{I}$-statistically convergent to $L \in X$, if for every $\varepsilon > 0$, and every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}\text{-}st\text{-}\lim_k x_k = L$ or $x_k \to L(\mathcal{I}\text{-}st)$.

A sequence $x = (x_k)$ is said to be $\mathcal{I}$-λ-statistically convergent or $\mathcal{I}$-st_λ convergent to L if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - st_\lambda\text{-}\lim x = L$ or $x_k \to L\ (\mathcal{I} - st_\lambda)$.

We can define the asymptotically equivalent of single sequences as follows (see [12]):

Definition 1 Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent of multiple L if

$$\lim_{k \to \infty} \frac{x_k}{y_k} = L$$

(denoted by $x \sim y$).

Patterson [17] presented a natural combination of the concepts of statistical convergence and asymptotically equivalent to introduce the concept of asymptotically statistically equivalent as follows.

Definition 2 Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistically equivalent of multiple L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \varepsilon \right\} \right| = 0,$$

(denoted by $x \sim^{S_L} y$) and simply asymptotically statistical equivalent if $L = 1$.

3. Main results

In this section we study the concepts of $\mathcal{I}$-statistically equivalent, $\mathcal{I}$-$[V, \lambda, \mu]$-equivalent and $\mathcal{I}$-(λ, μ)-statistically equivalent for double sequences. Moreover, we give some relations among these new notations. The results are analogues to those given by Patterson [17]. These notions generalize the notions of asymptotically statistically equivalent.

We define the following:

Definition 3 Two non-negative double sequences $x = (x_{jk})$ and $y = (y_{jk})$ are said to be asymptotically statistically equivalent of multiple L if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ j \leq m, k \leq n : \left| \left(\frac{x_{jk}}{y_{jk}} \right) - L \right| \geq \varepsilon \right\} \right| = 0,$$

(denoted by $x \sim^{S_L} y$) and simply asymptotically statistical equivalent if $L = 1$.

Following the line of Savaş et al. [25] we now introduce the following definitions for double sequences using ideals.

Definition 4 The two non-negative double sequences $x = (x_{jk})$ and $y = (y_{jk})$ is said to be asymptotically $\mathcal{I}$-statistically equivalent of multiple L if for every $\varepsilon > 0$, and every $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ j \leq m,\ k \leq n : \left| \left(\frac{x_{jk}}{y_{jk}} \right) - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I},$$

(denoted by $x \sim^{S^L(\mathcal{I})} y$) and simply asymptotically $\mathcal{I}$-statistically equivalent if $L = 1$.

Definition 5 The two non-negative double sequences $x = (x_{jk})$ and $y = (y_{jk})$ is said to be asymptotically $\mathcal{I}$-$[V, \lambda, \mu]$-equivalent of multiple L if for every $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_m \mu_n} {}_{(j,k) \in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}} \right) - L \right| \geq \delta \right\} \in \mathcal{I},$$

(denoted by $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y$) and simply asymptotically $\mathcal{I}$-$[V, \lambda, \mu]$-equivalent if $L = 1$.

Definition 6 The two non-negative double sequences $x = (x_{jk})$ and $y = (y_{jk})$ is said to be asymptotically $\mathcal{I}$-(λ, μ)-statistically equivalent of multiple L if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_m \mu_n} \left| \left\{ (j,k) \in J_m \times I_n : \left| \left(\frac{x_{jk}}{y_{jk}} \right) - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I},$$

(denoted by $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$) and simply asymptotically $\mathcal{I}$-(λ, μ)-statistically equivalent if $L = 1$.

Theorem 1 Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal and $\lambda = (\lambda_m) \in \Delta$, $\mu = (\mu_n) \in \Delta$. Then

(i) $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y \Rightarrow x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$.

(ii) If $x = (x_{jk})$, $y = (y_{jk}) \in \ell_\infty$ such that $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$, then $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y$.

(iii) $\left(x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y\right) \cap \ell_\infty = \left(x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y\right) \cap \ell_\infty$, where ℓ_∞ denotes the class of bounded sequences.

Proof. (i) Suppose $x = (x_{jk})$ and $y = (y_{jk})$ be two sequences such that $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y$. Let $\varepsilon > 0$. Since

$$\begin{aligned} {}_{(j,k)\in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| &\geq {}_{(j,k)\in J_m \times I_n \ \& \ \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \geq \varepsilon} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \\ &\geq \varepsilon . \left| \left\{ (j,k) \in J_m \times I_n : \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon \right\} \right| . \end{aligned}$$

So for a given $\delta > 0$,

$$\begin{aligned} &\frac{1}{\lambda_m \mu_n} \left| \left\{ (j,k) \in J_m \times I_n : \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon \right\} \right| \geq \delta \\ &\Rightarrow \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon . \delta . \end{aligned}$$

Therefore we have the inclusion

$$\begin{aligned} &\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_m \mu_n} \left| \left\{ (j,k) \in J_m \times I_n : \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ \subset &\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon . \delta \right\} . \end{aligned}$$

Since $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y$, so the set on the right-hand side belongs to $\mathcal{I}$ and so it follows that $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$.

(ii) If $x = (x_{jk})$, $y = (y_{jk}) \in \ell_\infty$ such that $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$. Let $\left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \leq M$ $\forall j,k$. Let $\varepsilon > 0$ be given. Now

$$\begin{aligned} \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| &= \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n \ \& \ \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \geq \varepsilon} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \\ &\quad + \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n \ \& \ \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| < \varepsilon} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \\ &\leq \frac{M}{\lambda_m \mu_n} \left| \left\{ (j,k) \in J_m \times I_n : \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon \right\} \right| + \varepsilon . \end{aligned}$$

Note that $\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_m \mu_n} \left| \left\{ (j,k) \in J_m \times I_n : \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq \varepsilon \right\} \right| \geq \frac{\varepsilon}{M} \right\} = A(\varepsilon) \in \mathcal{I}$. If $(m,n) \in (A(\varepsilon))^c$ then

$$\frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| < 2\varepsilon .$$

Hence

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_m \mu_n} {}_{(j,k)\in J_m \times I_n} \left| \left(\frac{x_{jk}}{y_{jk}}\right) - L \right| \geq 2\varepsilon \right\} \subset A(\varepsilon)$$

and so belongs to $\mathcal{I}$. This shows that $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y$.

(iii) This readily follows from (i) and (ii).

Theorem 2 Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal and $\lambda = (\lambda_m) \in \Delta$, $\mu = (\mu_n) \in \Delta$. Then

(i) $x \sim^{S^L(\mathcal{I})} y \subset x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$ if $\lim\inf_{m\to\infty} \frac{\lambda_m}{m} > 0$ and $\lim\inf_{n\to\infty} \frac{\mu_n}{n} > 0$.

(ii) If $\lim\inf_{m\to\infty} \frac{\lambda_m}{m} = 0$ and $\lim\inf_{n\to\infty} \frac{\mu_n}{n} = 0$, then $x \sim^{S^L(\mathcal{I})} y \subsetneq x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$.

Proof. (i) Suppose $x = (x_{jk})$ and $y = (y_{jk})$ be two sequences such that $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$. For given $\varepsilon > 0$,

$$\begin{aligned} & \frac{1}{mn}\left|\left\{j \le m, k \le n : \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \ge \varepsilon\right\}\right| \\ \ge & \frac{1}{mn}\left|\left\{(j,k) \in J_m \times I_n : \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \ge \varepsilon\right\}\right| \\ = & \frac{\lambda_m \mu_n}{mn}\frac{1}{\lambda_m \mu_n}\left|\left\{(j,k) \in J_m \times I_n : \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \ge \varepsilon\right\}\right|. \end{aligned}$$

If $\lim\inf_{m\to\infty} \frac{\lambda_m}{m} = a$ and $\lim\inf_{n\to\infty} \frac{\mu_n}{n} = b$ then the set $\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{\lambda_m\mu_n}{mn} < \frac{ab}{2}\right\}$ is finite. Hence for $\delta > 0$,

$$\begin{aligned} & \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_m\mu_n}\left|\left\{(j,k) \in J_m \times I_n : \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \ge \varepsilon\right\}\right| \ge \delta\right\} \\ \subset & \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\left\{j \le m, k \le n : \left|\left(\frac{x_{jk}}{y_{jk}}\right) - L\right| \ge \varepsilon\right\}\right| \ge \frac{ab}{2}.\delta\right\} \\ & \cup\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{\lambda_m\mu_n}{mn} < \frac{ab}{2}\right\}. \end{aligned}$$

Since $\mathcal{I}$ is admissible, the set on the right-hand side belongs to $\mathcal{I}$ and this completed the proof of (i).

(ii) Conversely, suppose that $x \sim^{S^L(\mathcal{I})} y$ and $\lim\inf_{m\to\infty} \frac{\lambda_m}{m} = 0$ and $\lim\inf_{n\to\infty} \frac{\mu_n}{n} = 0$. Then we can choose subsequences $(m(p))_{p=1}^{\infty}$ and $(n(q))_{q=1}^{\infty}$ such that $\frac{\lambda_{m(p)}}{m(p)} < \frac{1}{p}$ and $\frac{\mu_{n(q)}}{n(q)} < \frac{1}{q}$. Define a sequence $x = (x_{jk})$ by

$$x_{ik} = \begin{cases} 1, & \text{if } j \in J_{m(p)} \text{ and } k \in I_{n(q)}, \\ 0, & \text{otherwise.} \end{cases}$$

and $y_{jk} = 1$ for all $j,k \in \mathbb{N}$. Then $x \sim^{S^L} y$ and so by the admissibility of the ideal $x \sim^{S^L(\mathcal{I})} y$. But $x \sim^{[V,\lambda,\mu]^L(\mathcal{I})} y$ is not satisfied and therefore by Theorem 2 (ii) $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$ is not satisfied.

Theorem 3 Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal. If $\lambda, \mu \in \Delta$ be such that $\lim_m \frac{\lambda_m}{m} = 1$ and $\lim_n \frac{\mu_n}{n} = 1$, then $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y \subset x \sim^{S^L(\mathcal{I})} y$.

Proof. Suppose that $\lim_m \frac{\lambda_m}{m} = 1$ and $\lim_n \frac{\mu_n}{n} = 1$ and there exists sequences $x = (x_{jk})$ and $y = (y_{jk})$ be two sequences such that $x \sim^{S^L_{\lambda,\mu}(\mathcal{I})} y$. Let $\delta > 0$ be given. Since $\lim_m \frac{\lambda_m}{m} = 1$ and $\lim_n \frac{\mu_n}{n} = 1$, we can choose $m, n \in \mathbb{N}$, such that $\left|\frac{\lambda_m\mu_n}{mn}-1\right|<\frac{\delta}{2}$, for all $m,n\geq N_{mn}$. Now observe that, for $\varepsilon>0$

$$\begin{aligned}
& \frac{1}{mn}\left|\left\{j\leq m, k\leq n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right| \\
= & \frac{1}{mn}\left|\left\{j\leq m-\lambda_m, k\leq n-\mu_n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right| \\
& +\frac{1}{mn}\left|\left\{(j,k)\in J_m\times I_n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right| \\
\leq & \frac{mn-\lambda_m\mu_n}{mn}+\frac{1}{mn}\left|\left\{(j,k)\in J_m\times I_n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right| \\
\leq & 1-\left(1-\frac{\delta}{2}\right)+\frac{1}{mn}\left|\left\{(j,k)\in J_m\times I_n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right| \\
= & \frac{\delta}{2}+\frac{1}{mn}\left|\left\{(j,k)\in J_m\times I_n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right|,
\end{aligned}$$

for all $m,n\geq N_{mn}$. Hence

$$\begin{aligned}
& \left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\left\{j\leq m, k\leq n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\} \\
\subset & \left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\left\{(j,k)\in J_m\times I_n:\left|\left(\frac{x_{jk}}{y_{jk}}\right)-L\right|\geq\varepsilon\right\}\right|\geq\frac{\delta}{2}\right\} \\
& \cup\left\{(\mathbb{N}\times\mathbb{N}\backslash A)\cap\left(\left(\{1,2,...,i-1\}\times\mathbb{N}\right)\cup\left(\mathbb{N}\times\{1,2,...,i-1\}\right)\right)\right\}.
\end{aligned}$$

If $x\sim^{S^L_{\lambda,\mu}(\mathcal{I})}y$ then the set on the right-hand side belongs to $\mathcal{I}$ and so the set on the left-hand side also belongs to $\mathcal{I}$. This shows that $x\sim^{S^L(\mathcal{I})}y$.

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