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PERIODIC SOLUTIONS FOR NONLINEAR NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE DELAY

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Abstract. The nonlinear neutral difference equation with variable delay

$$x(n+1) = a(n)x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n), x(n-\tau(n))),$$

is considered in this work. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, we establish some criteria for the existence and uniqueness of periodic solutions to the neutral difference equation.

1. Introduction

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness of solutions for delay difference equations, see the references in this article and references therein.

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of delay difference equations. Motivated by the papers [1], [2], [4], [5] and the references therein, we concentrate on the existence and uniqueness of periodic solutions for the nonlinear neutral difference equation with variable delay

$$x(n+1) = a(n)x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n), x(n-\tau(n))), \quad (1)$$

where

$$g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}, \ f : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(n) = x(n+1) - x(n)$ for any sequence $\{x(n), \ n \in \mathbb{Z}\}$. Also, we define the operator E by $Ex(n) = x(n+1)$. For more on the calculus of difference equations, we refer the reader to [3].

The purpose of this paper is to use Krasnoselskii's fixed point theorem to show the existence of periodic solutions for equation (1). To apply Krasnoselaskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is completely continuous. We also use the contraction mapping principle to show the existence of a unique periodic solution of (1). It is important to note that, in our consideration, the neutral term $\triangle g(n, x(n-\tau(n)))$ of (1) produces nonlinearity

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in the neutral term $\triangle x(n-\tau(n))$. While, the neutral term $\triangle x(n-\tau(n))$ in [4] enters linearly. As a consequence, we have performed an appropriate analysis which is different from that used in [4] to construct the mappings in order to employ fixed point theorems.

The organization of this paper is as follows. In Section 2, we present the inversion of difference equation (1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [6]. In Section 3, we present our main results on existence and uniqueness of periodic solutions of (1).

2. Preliminaries

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z},\mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z},\mathbb{R})$ is the space of all real valued functions. Then $(P_T, \|.\|)$ is a Banach space with the maximum norm

$$\|x\| = \max_{n\in[0,T-1]\cap\mathbb{Z}} |x(n)|.$$

Since we are searching for the existence of periodic solutions for equation (1), it is natural to assume that

$$a(n+T) = a(n),\ \tau(n+T) = \tau(n), \tag{2}$$

with τ being scalar sequence and $\tau(n) \geq \tau^* > 0$. Also, we assume

$$\prod_{s=n-T}^{n-1} a(s) \neq 1. \tag{3}$$

Throughout this paper we assume $a(n) \neq 0$ for all $n \in [0,T-1]\cap\mathbb{Z}$. Since we are searching for periodic solutions, it is natural to ask that the functions $g(n,x)$ and $f(n,x,y)$ are periodic in n with period T and Lipschitz continuous in x and in x and y, respectively. That is

$$g(n+T,x) = g(n,x),\ f(n+T,x,y) = f(n,x,y), \tag{4}$$

and there are positive constants L_1, L_2, L_3 such that

$$|g(n,x) - g(n,y)| \leq L_1 \|x-y\|, \tag{5}$$

and

$$|f(n,x,y) - f(n,z,w)| \leq L_2 \|x-z\| + L_3 \|y-w\|. \tag{6}$$

The following lemma is fundamental to our results.

Lemma 2.1. *Suppose (2)–(4) hold. If $x \in P_T$, then x is a solution of equation (1) if and only if*

$$\begin{aligned} x(n) &= g(n,x(n-\tau(n))) \\ &\quad + \left(1 - \prod_{s=n-T}^{n-1} a(s)\right)^{-1} \sum_{u=n-T}^{n-1} [f(u,x(u),x(u-\tau(u))) \\ &\quad + (a(u)-1)\, g(u,x(u-\tau(u)))] \prod_{s=u+1}^{n-1} a(s). \end{aligned} \tag{7}$$

Proof. We consider two cases, $n \geq 1$ and $n \leq 0$. Let $x \in P_T$ be a solution of (1). For $n \geq 1$ equation (1) is equivalent to

$$\begin{aligned} &\triangle\left[x(n)\prod_{s=0}^{n-1}a^{-1}(s)\right] \\ &=\left[\triangle g(n,x(n-\tau(n)))+f(n,x(n),x(n-\tau(n)))\right]\prod_{s=0}^{n}a^{-1}(s). \end{aligned} \tag{8}$$

By summing (8) from $n-T$ to $n-1$, we obtain

$$\begin{aligned} &\sum_{u=n-T}^{n-1}\triangle\left[x(u)\prod_{s=0}^{u-1}a^{-1}(s)\right] \\ &=\sum_{u=n-T}^{n-1}\left[\triangle g(u,x(u-\tau(u)))+f(u,x(u),x(u-\tau(u)))\right]\prod_{s=0}^{u}a^{-1}(s). \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &x(n)\prod_{s=0}^{n-1}a^{-1}(s)-x(n-T)\prod_{s=0}^{n-T-1}a^{-1}(s) \\ &=\sum_{u=n-T}^{n-1}\left[\triangle g(u,x(u-\tau(u)))+f(u,x(u),x(u-\tau(u)))\right]\prod_{s=0}^{u}a^{-1}(s). \end{aligned}$$

Since $x(n-T)=x(n)$, we obtain

$$\begin{aligned} &x(n)\left[\prod_{s=0}^{n-1}a^{-1}(s)-\prod_{s=0}^{n-T-1}a^{-1}(s)\right] \\ &=\sum_{u=n-T}^{n-1}\left[\triangle g(u,x(u-\tau(u)))+f(u,x(u),x(u-\tau(u)))\right]\prod_{s=0}^{u}a^{-1}(s). \end{aligned} \tag{9}$$

Rewrite

$$\begin{aligned} &\sum_{u=n-T}^{n-1}\triangle g(u,x(u-\tau(u)))\prod_{s=0}^{u}a^{-1}(s) \\ &=\sum_{u=n-T}^{n-1}E\left[\prod_{s=0}^{u-1}a^{-1}(s)\right]\triangle g(u,x(u-\tau(u))). \end{aligned}$$

Performing a summation by parts on the on the above equation, we get

$$\begin{aligned}
&\sum_{u=n-T}^{n-1} \triangle g\left(u, x\left(u-\tau\left(u\right)\right)\right) \prod_{s=0}^{u} a^{-1}\left(s\right) \\
&= g\left(n, x\left(n-\tau\left(n\right)\right)\right)\left[\prod_{s=0}^{n-1} a^{-1}\left(s\right) - \prod_{s=0}^{n+T-1} a^{-1}\left(s\right)\right] \\
&- \sum_{u=n-T}^{n-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right) \triangle\left[\prod_{s=0}^{u-1} a^{-1}\left(s\right)\right] \\
&= g\left(n, x\left(n-\tau\left(n\right)\right)\right)\left[\prod_{s=0}^{n-1} a^{-1}\left(s\right) - \prod_{s=0}^{n+T-1} a^{-1}\left(s\right)\right] \\
&- \sum_{u=n-T}^{n-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right)\left[1-a\left(u\right)\right] \prod_{s=0}^{u} a^{-1}\left(s\right).
\end{aligned} \tag{10}$$

Substituting (10) into (9), we obtain

$$\begin{aligned}
&x\left(n\right)\left[\prod_{s=0}^{n-1} a^{-1}\left(s\right) - \prod_{s=0}^{n+T-1} a^{-1}\left(s\right)\right] \\
&= g\left(n, x\left(n-\tau\left(n\right)\right)\right)\left[\prod_{s=0}^{n-1} a^{-1}\left(s\right) - \prod_{s=0}^{n+T-1} a^{-1}\left(s\right)\right] \\
&- \sum_{u=n-T}^{n-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right)\left[1-a\left(u\right)\right] \prod_{s=0}^{u} a^{-1}\left(s\right) \\
&+ \sum_{u=n-T}^{n-1} f\left(u, x\left(u\right), x\left(u-\tau\left(u\right)\right)\right) \prod_{s=0}^{u} a^{-1}\left(s\right).
\end{aligned}$$

Dividing both sides of the above equation by $\prod_{s=0}^{n-1} a^{-1}\left(s\right) - \prod_{s=0}^{n+T-1} a^{-1}\left(s\right)$, we obtain (7).

Now for $n \leq 0$, equation (1) is equivalent to

$$\begin{aligned}
&\triangle\left[x\left(n\right) \prod_{s=n}^{0} a^{-1}\left(s\right)\right] \\
&= \left[\triangle g\left(n, x\left(n-\tau\left(n\right)\right)\right) + f\left(n, x\left(n\right), x\left(n-\tau\left(n\right)\right)\right)\right] \prod_{s=n+1}^{0} a^{-1}\left(s\right).
\end{aligned}$$

Summing the above expression from $n-T$ to $n-1$, we obtain (7) by a similar argument. This completes the proof. □

Using (7) we define the mapping $H : P_T \to P_T$ by

$$\begin{aligned}(H\varphi)(n) &= g(n, \varphi(n - \tau(n))) \\ &\quad + \left(1 - \prod_{s=n-T}^{n-1} a(s)\right)^{-1} \sum_{u=n-T}^{n-1} [f(u, \varphi(u), \varphi(u - \tau(u))) \\ &\quad + (a(u) - 1) g(u, \varphi(u - \tau(u)))] \prod_{s=u+1}^{n-1} a(s). \qquad (11)\end{aligned}$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1). For its proof we refer the reader to [6].

Theorem 2.2 (Krasnoselskii)**.** *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*
(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(ii) $\mathcal{A}$ is completely continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. Existence of periodic solutions

To apply Theorem 2.2, we need to construct two mappings, one is a contraction and the other is compact. Therefore, we express equation (11) as

$$(H\varphi)(n) = (\mathcal{B}\varphi)(n) + (\mathcal{A}\varphi)(n), \qquad (12)$$

where $\mathcal{A}, \mathcal{B} : P_T \to P_T$ are defined by

$$(\mathcal{B}\varphi)(n) = g(n, \varphi(n - \tau(n))), \qquad (13)$$

and

$$\begin{aligned}(\mathcal{A}\varphi)(n) &= \left(1 - \prod_{s=n-T}^{n-1} a(s)\right)^{-1} \sum_{u=n-T}^{n-1} [f(u, \varphi(u), \varphi(u - \tau(u))) \\ &\quad + (a(u) - 1) g(u, \varphi(u - \tau(u)))] \prod_{s=u+1}^{n-1} a(s). \qquad (14)\end{aligned}$$

To simplify notations, we introduce the following constants.

$$\eta = \max_{n \in [0, T-1] \cap \mathbb{Z}} \left| \left(1 - \prod_{s=n-T}^{n-1} a(s)\right)^{-1} \right|, \ \rho = \max_{u \in [0, T-1] \cap \mathbb{Z}} |a(u) - 1|,$$

$$\gamma = \max_{u \in [n-T, n-1] \cap \mathbb{Z}} \prod_{s=u+1}^{n-1} a(s).$$

Lemma 3.1. *Suppose that the conditions (2)–(6) hold. Then $\mathcal{A} : P_T \to P_T$ is completely continuous.*

Proof. We first show that $(\mathcal{A}\varphi)(n + T) = (\mathcal{A}\varphi)(n)$.

Let $\varphi \in P_T$. Then using (14) we arrive at

$$\begin{aligned}(\mathcal{A}\varphi)(n+T) &= \left(1-\prod_{s=n}^{n+T-1} a(s)\right)^{-1} \sum_{u=n}^{n+T-1} \left[f\left(u, \varphi(u), \varphi(u-\tau(u))\right)\right. \\ &\quad \left. + (a(u)-1)\, g\left(u, \varphi(u-\tau(u))\right)\right] \prod_{s=u+1}^{n+T-1} a(s).\end{aligned}$$

Let $j = u - T$, then

$$\begin{aligned}&(\mathcal{A}\varphi)(n+T) \\ &= \left(1-\prod_{s=n}^{n+T-1} a(s)\right)^{-1} \sum_{j=n-T}^{n-1} \left[f\left(j+T, \varphi(j+T), \varphi(j+T-\tau(j+T))\right)\right. \\ &\left. + (a(j+T)-1)\, g\left(j+T, \varphi(j+T-\tau(j+T))\right)\right] \prod_{s=j+T+1}^{n+T-1} a(s) \\ &= \left(1-\prod_{s=n}^{n+T-1} a(s)\right)^{-1} \sum_{j=n-T}^{n-1} \left[f\left(j, \varphi(j), \varphi(j-\tau(j))\right)\right. \\ &\left. + (a(j)-1)\, g\left(j, \varphi(j-\tau(j))\right)\right] \prod_{s=j+T+1}^{n+T-1} a(s).\end{aligned}$$

Now let $k = s - T$, then

$$\begin{aligned}(\mathcal{A}\varphi)(n+T) &= \left(1-\prod_{k=n-T}^{n-1} a(k)\right)^{-1} \sum_{j=n-T}^{n-1} \left[f\left(j, \varphi(j), \varphi(j-\tau(j))\right)\right. \\ &\quad \left. + (a(j)-1)\, g\left(j, \varphi(j-\tau(j))\right)\right] \prod_{k=j+1}^{n-1} a(k) \\ &= (\mathcal{A}\varphi)(n).\end{aligned}$$

To see that $\mathcal{A}$ is continuous, we let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\delta = \epsilon / M$ with $M = \eta\gamma T\left(L_2 + L_3 + \rho L_1\right)$, where L_1, L_2 and L_3 are given by (5) and (6).

Now, for $\|\varphi-\psi\|<\delta$, we obtain

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(n)-(\mathcal{A}\psi)(n)\right| \\
&=\Bigg|\left(1-\prod_{s=n-T}^{n-1}a(s)\right)^{-1} \\
&\times\sum_{u=n-T}^{n-1}\left[\left(f\left(u,\varphi(u),\varphi(u-\tau(u))\right)-f\left(u,\psi(u),\psi(u-\tau(u))\right)\right)\right. \\
&\left.+(a(u)-1)\left(g\left(u,\varphi(u-\tau(u))\right)-g\left(u,\psi(u-\tau(u))\right)\right)\right]\prod_{s=u+1}^{n-1}a(s)\Bigg| \\
&\leq\eta\sum_{u=n-T}^{n-1}\left[L_2\|\varphi-\psi\|+L_3\|\varphi-\psi\|+\rho L_1\|\varphi-\psi\|\right]\gamma \\
&\leq\eta\gamma T\left(L_2+L_3+\rho L_1\right)\|\varphi-\psi\| \\
&=M\|\varphi-\psi\|<M\delta=\epsilon.
\end{aligned}$$

Then $\|\mathcal{A}\varphi-\mathcal{A}\psi\|<\epsilon$. This proves $\mathcal{A}$ is continuous.

Next, we show that $\mathcal{A}$ maps bounded subsets into compact sets. Let J be given, $S=\{\varphi\in P_T:\|\varphi\|\leq J\}$ and $Q=\{\mathcal{A}\varphi:\varphi\in S\}$, then S is a subset of $\mathbb{R}^T$ which is closed and bounded thus compact. As $\mathcal{A}$ is continuous it maps compact sets into compact sets. Then $Q=\mathcal{A}(S)$ is compact. Therefore $\mathcal{A}$ is completely continuous. This completes the proof. □

Lemma 3.2. *Suppose that (5) holds. If $\mathcal{B}$ is given by (13) with*

$$L_1<1, \tag{15}$$

then $\mathcal{B}:P_T\to P_T$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (13). Obviously, $(\mathcal{B}\varphi)(n+T)=(\mathcal{B}\varphi)(n)$. So, for any $\varphi,\psi\in P_T$, we have

$$\begin{aligned}
\left|(\mathcal{B}\varphi)(n)-(\mathcal{B}\psi)(n)\right|&\leq\left|g\left(n,\varphi(n-\tau(n))\right)-g\left(n,\psi(n-\tau(n))\right)\right| \\
&\leq L_1\|\varphi-\psi\|.
\end{aligned}$$

Then $\|\mathcal{B}\varphi-\mathcal{B}\psi\|\leq L_1\|\varphi-\psi\|$. Thus $\mathcal{B}:P_T\to P_T$ is a contraction by (15). □

Observe that in view of (5) and (6) we have

$$\begin{aligned}
|g(n,x)|&=|g(n,x)-g(n,0)+g(n,0)| \\
&\leq|g(n,x)-g(n,0)|+|g(n,0)| \\
&\leq L_1\|x\|+\alpha,
\end{aligned}$$

and

$$\begin{aligned}
|f(n,x,y)|&=|f(n,x,y)-f(n,0,0)+f(n,0,0)| \\
&\leq|f(n,x,y)-f(n,0,0)|+|f(n,0,0)| \\
&\leq L_2\|x\|+L_3\|y\|+\beta,
\end{aligned}$$

where

$$\alpha=\max_{n\in[0,T-1]\cap\mathbb{Z}}|g(n,0)|\text{ and }\beta=\max_{n\in[0,T-1]\cap\mathbb{Z}}|f(n,0,0)|.$$

Theorem 3.3. *Suppose (2)–(6) and (15) hold. Let J be a positive constant satisfying the inequality*

$$L_1 J+\alpha+\eta\gamma T\left[\left(L_2+L_3\right) J+\beta+\rho\left(L_1 J+\alpha\right)\right] \leq J.$$

Let $\mathbb{D}=\{\varphi \in P_T:\|\varphi\| \leq J\}$. Then equation (1) has a T-periodic solution x in the subset $\mathbb{D}$.

Proof. By Lemma 3.1, the operator $\mathcal{A}: \mathbb{D} \rightarrow P_T$ is completely continuous. Also, from Lemma 3.2, the operator $\mathcal{B}: \mathbb{D} \rightarrow P_T$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}
&|(\mathcal{B}\psi)(n)+(\mathcal{A}\varphi)(n)| \\
&=\Bigg| g(n, \psi(n-\tau(n)))+\left(1-\prod_{s=n-T}^{n-1} a(s)\right)^{-1} \\
&\times \sum_{u=n-T}^{n-1}[f(u, \varphi(u), \varphi(u-\tau(u)))+(a(u)-1) g(u, \varphi(u-\tau(u)))] \prod_{s=u+1}^{n-1} a(s)\Bigg| \\
&\leq L_1\|\psi\|+\alpha+\eta\gamma \sum_{u=n-T}^{n-1}\left[\left(L_2+L_3\right)\|\varphi\|+\beta+\rho\left(L_1\|\varphi\|+\alpha\right)\right] \\
&\leq L_1 J+\alpha+\eta\gamma T\left[\left(L_2+L_3\right) J+\beta+\rho\left(L_1 J+\alpha\right)\right] \leq J.
\end{aligned}$$

Then $\|\mathcal{B}\psi+\mathcal{A}\varphi\| \leq J$. This shows that $\mathcal{B}\psi+\mathcal{A}\varphi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x=\mathcal{B}x+\mathcal{A}x$. By Lemma 2.1 this fixed point is a solution of (1) and the proof is complete. □

Remark 3.4. The constant J of Theorem 3.3 serves as a priori bound on all possible T-periodic solutions of equation (1).

Theorem 3.5. *Suppose (2)–(6) and (15) hold. If*

$$L_1+\eta\gamma T\left(L_2+L_3+\rho L_1\right) \leq \nu<1,$$

then equation (1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (12). For $\varphi, \psi \in P_T$, in view of (12), we have

$$\begin{aligned}
\|H\varphi-H\psi\| &=\|\mathcal{B}\varphi+\mathcal{A}\varphi-\mathcal{B}\psi-\mathcal{A}\psi\| \\
&\leq\|\mathcal{B}\varphi-\mathcal{B}\psi\|+\|\mathcal{A}\varphi-\mathcal{A}\psi\| \\
&\leq L_1\|\varphi-\psi\|+\eta\gamma T\left(L_2\|\varphi-\psi\|+L_3\|\varphi-\psi\|+\rho L_1\|\varphi-\psi\|\right) \\
&\leq\left[L_1+\eta\gamma T\left(L_2+L_3+\rho L_1\right)\right]\|\varphi-\psi\| \\
&\leq \nu\|\varphi-\psi\| .
\end{aligned}$$

This completes the proof by invoking the contraction mapping principle. □

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