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COEFFICIENT INEQUALITIES FOR STARLIKE AND CONVEX FUNCTIONS OF RECIPROCAL ORDER α

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Abstract. For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, two classes $\mathcal{S}_*(\alpha)$ and $\mathcal{K}_*(\alpha)$ of starlike functions of reciprocal order α and of convex functions of reciprocal order α, respectively, are considered. In the present paper, some interesting coefficient inequalities for $f(z)$ in the classes $\mathcal{S}_*(\alpha)$ and $\mathcal{K}_*(\alpha)$ are discussed.

1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}_*(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ satisfying

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real number α with $0\alpha < 1$. A function $f(z) \in \mathcal{S}_*(\alpha)$ is said to be starlike of reciprocal order α in $\mathbb{U}$. Alse, we define the subclass $\mathcal{K}_*(\alpha)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{1}{1+\frac{zf''(z)}{f'(z)}}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real number α with $0\alpha < 1$. A function $f(z) \in \mathcal{K}_*(\alpha)$ is also said to be convex of reciprocal order α in $\mathbb{U}$. We say that $\mathcal{S}_*(0) = \mathcal{S}_*$ and $\mathcal{K}_*(0) = \mathcal{K}_*$. We also note that $f(z) \in \mathcal{K}_*(\alpha)$ if and only if $zf'(z) \in \mathcal{S}_*(\alpha)$.
In the present paper, we consider some coefficient inequalities for $f(z)$ in the classes $\mathcal{S}_*(\alpha)$ and $\mathcal{K}_*(\alpha)$.

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Remark 1.1. Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{1.4}$$

Furthermore, the class $\mathcal{K}(\alpha)$ is defined as the set of $f(z)$ which satisfy $zf'(z) \in \mathcal{S}^*(\alpha)$. A function $f(z) \in \mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$ and $f(z) \in \mathcal{K}(\alpha)$ is called to be convex of order α in $\mathbb{U}$. For the class $\mathcal{S}^*(\alpha)$, Silverman [4] has considered the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \qquad (z \in \mathbb{U}). \tag{1.5}$$

This condition shows us that the image of $\mathbb{U}$ by $\dfrac{zf'(z)}{f(z)}$ is inside of the circle with the center at 1 and the radius $1 - \alpha$. This circle is very small. If we consider the condition

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \qquad (z \in \mathbb{U}) \tag{1.6}$$

for $0 < \alpha < 1$ by Nunokawa, Owa, Nishiwaki, Kuroki and Hayami [2], the condition (1.6) shows that

$$\mathrm{Re}\left(\frac{f(z)}{zf'(z)}\right) > \alpha \qquad (z \in \mathbb{U}),$$

which means that $f(z) \in \mathcal{S}_*(\alpha)$. This condition (1.6) gives us that the image of $\mathbb{U}$ by $\dfrac{zf'(z)}{f(z)}$ is inside of the circle with the center at $\dfrac{1}{2\alpha}$ and the radius $\dfrac{1}{2\alpha}$. Thus if $0 < \alpha\dfrac{1}{2}$, the condition (1.6) is better than (1.5). This is the motivation to discuss of the classes $\mathcal{S}_*(\alpha)$ and $\mathcal{K}_*(\alpha)$.

Remark 1.2. Let us consider the function $f(z) = ze^{(1-\alpha)z}$. Then, we have that

$$\frac{zf'(z)}{f(z)} = 2 - \alpha \qquad (z \in \mathbb{U}). \tag{1.7}$$

This shows us that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}) \tag{1.8}$$

which gives that $f(z)$ is starlike in $\mathbb{U}$ and $f(z) \in \mathcal{S}_*\left(\dfrac{1}{2-\alpha}\right)$ for $0\alpha < 1$. If we take $\alpha = \dfrac{1}{2}$, then the function $f(z) = ze^{\frac{z}{2}}$ maps $\mathbb{U}$ onto the following domain which ia starlike with respect to the origin.

Furthermore, the function

$f(z) = \dfrac{1}{1-\alpha}(e^{(1-\alpha)z} - 1)$ $(0\alpha < 1)$ belongs to the class $\mathcal{K}_*\left(\dfrac{1}{2-\alpha}\right)$ because

$zf'(z) = ze^{(1-\alpha)z} \in \mathcal{S}_*\left(\dfrac{1}{2-\alpha}\right)$.

Remark 1.3. Argument estimates for $f(z)$ in the class $\mathcal{S}_*(\alpha)$ will be found in the paper by Ravichandran and Kumar [3].

2. Coefficient inequalities

In order to consider some coefficient inequalities for $f(z)$, we have to recall here the following definition. Let $p(z)$ be analytic in $\mathbb{U}$ with

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots . \tag{2.1}$$

If $p(z)$ given by (2.1) satisfies

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}), \tag{2.2}$$

then $p(z)$ is said to be Carathéodory function in $\mathbb{U}$. We denote by $\mathcal{P}$ the class of Carathéodory functions $p(z)$ in $\mathbb{U}$. It is well known that

$$|c_n| 2 \qquad (n = 1, 2, 3, \cdots) \tag{2.3}$$

and the equality holds true for

$$p(z) = \frac{1+z}{1-z} \qquad \text{(see Carathéodory [1]).} \tag{2.4}$$

Using the coefficient inequality (2.3), we derive

Theorem 2.1. *If $f(z) \in \mathcal{S}_*(\alpha)$, then*

$$|a_n| \frac{2(1-\alpha)(1+2|a_2|)}{n-1} \prod_{k=2}^{n-2} \left(1 + \frac{2(1-\alpha)(1+k)}{k}\right) \qquad (n = 4, 5, 6, \cdots) \tag{2.5}$$

with $|a_2| 2(1-\alpha)$ and $|a_3| (1-\alpha)(1+2|a_2|)$.

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{\dfrac{f(z)}{zf'(z)} - \alpha}{1-\alpha} = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.6}$$

for $f(z) \in \mathcal{S}_*(\alpha)$, then $p(z) \in \mathcal{P}$. It follows from (2.6) that

$$f(z) = zf'(z)(\alpha + (1-\alpha)p(z)), \tag{2.7}$$

that is, that

$$z + \sum_{n=2}^{\infty} a_n z^n = (z + \sum_{n=2}^{\infty} n a_n z^n)(1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n). \tag{2.8}$$

This gives us that

$$a_n = \frac{1-\alpha}{1-n}(c_{n-1} + 2a_2 c_{n-2} + 3a_3 c_{n-3} + \cdots + (n-1)a_{n-1}c_1). \tag{2.9}$$

Noting that $|c_n| 2$ $(n = 1, 2, 3, \cdots)$, we obtain that

$$|a_n| \frac{2(1-\alpha)}{n-1}(1 + 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}|). \tag{2.10}$$

This implies that

$$|a_2| 2(1-\alpha) \tag{2.11}$$

and

$$|a_3| (1-\alpha)(1+2|a_2|). \tag{2.12}$$

For $n = 4$, we also see that

$$|a_4| \frac{2(1-\alpha)}{3}(1 + 2|a_2| + 3|a_3|) \tag{2.13}$$

$$\frac{2(1-\alpha)}{3}(1+2|a_2|+3(1-\alpha)(1+2|a_2|))$$
$$=\frac{2(1-\alpha)(1+2|a_2|)}{3}(4-3\alpha),$$

which proves that (2.5) holds true for $n=4$. We suppose that the coefficient inequality (2.5) holds true for $n=j$. Then, (2.10) shows that

$$|a_{j+1}|\frac{2(1-\alpha)}{j}(1+2|a_2|+3|a_3|+4|a_4|+\cdots+(j-1)|a_{j-1}|+j|a_j|) \tag{2.14}$$

$$\frac{2(1-\alpha)}{j}\left\{1+2|a_2|+3(1-\alpha)(1+2|a_2|)+4\frac{2(1-\alpha)}{3}(1+2|a_2|)(1+3(1-\alpha))\right.$$
$$\left.+\cdots+\frac{2(1-\alpha)}{j-1}(1+2|a_2|)\prod_{k=2}^{j-2}\left(1+\frac{2(1-\alpha)(1+k)}{k}\right)\right\}$$
$$=\frac{2(1-\alpha)(1+2|a_2|)}{j}\prod_{k=2}^{j-1}\left(1+\frac{2(1-\alpha)(1+k)}{k}\right).$$

Thus, (2.5) holds true for $n=j+1$. Therefore, applying the mathematical induction, we prove the coefficient inequality for $n=4,5,6,\cdots$. □

Remark 2.1. If we take $|a_2|=2(1-\alpha)$, then (2.5) becomes

$$|a_n|\frac{2(1-\alpha)(5-4\alpha)}{n-1}\prod_{k=2}^{n-2}\left(1+\frac{2(1-\alpha)(1+k)}{k}\right) \qquad (n=4,5,6,\cdots) \tag{2.15}$$

with $|a_2|=2(1-\alpha)$ and $|a_3|(1-\alpha)(5-4\alpha)$.

Corollary 2.1. *If $f(z)\in\mathcal{S}_*$, then*

$$|a_n|\frac{2(1+2|a_2|)}{n-1}\prod_{k=2}^{n-2}\left(\frac{3k+2}{k}\right) \qquad (n=4,5,6,\cdots) \tag{2.16}$$

with $|a_2|2$ and $|a_3|1+2|a_2|$.

For $f(z)\in\mathcal{K}_*(\alpha)$, we have

Theorem 2.2. *If $f(z)\in\mathcal{K}_*(\alpha)$, then*

$$|a_n|\frac{2(1-\alpha)(1+2|a_2|)}{n(n-1)}\prod_{k=2}^{n-2}\left(1+\frac{2(1-\alpha)(1+k)}{k}\right) \qquad (n=4,5,6,\cdots) \tag{2.17}$$

with $|a_2|1-\alpha$ and $|a_3|\frac{(1-\alpha)(1+2|a_2|)}{3}$.

Proof. We note that $f(z)\in\mathcal{K}_*(\alpha)$ if and only if $zf'(z)\in\mathcal{S}_*(\alpha)$. This shows that

$$n|a_n|\frac{2(1-\alpha)(1+2|a_2|)}{n-1}\prod_{k=2}^{n-2}\left(1+\frac{2(1-\alpha)(1+k)}{k}\right) \tag{2.18}$$

for $n=4,5,6,\cdots$, $|a_2|1-\alpha$, and $3|a_3|(1-\alpha)(1+2|a_2|)$. This completes the proof of the theorem. □

Remark 2.2. If $|a_2| = 1 - \alpha$, then (2.17) becomes

$$|a_n| \frac{2(1-\alpha)(3-2\alpha)}{n(n-1)} \prod_{k=2}^{n-2} \left(1 + \frac{2(1-\alpha)(1+k)}{k} \right) \qquad (n = 4, 5, 6, \cdots) \tag{2.19}$$

with $|a_2| = 1 - \alpha$ and $|a_3| \dfrac{(1-\alpha)(3-2\alpha)}{3}$.

Corollary 2.2. *If $f(z) \in \mathcal{K}_*$, then*

$$|a_n| \frac{2(1+2|a_2|)}{n(n-1)} \prod_{k=2}^{n-2} \left(\frac{3k+2}{k} \right) \qquad (n = 4, 5, 6, \cdots) \tag{2.20}$$

with $|a_2| 1$ *and* $|a_3| \dfrac{1+2|a_2|}{3}$.

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