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NECESSARY CONDITIONS FOR BACKWARD DOUBLY STOCHASTIC CONTROL SYSTEM

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Abstract. We consider the necessary conditions for backward doubly stochastic control system, via the second-order Taylor expansion we have obtained. All the results are got under no restriction on the convexity of control domain and the diffusion coefficient does not contain the control variable.

1. Introduction

Optimal control problem is an important subject of control science, and the study of necessary conditions for optimal control is also very hot today, such as [1], [3], [4], [8], [9], [12]. For the optimal control problems of backward doubly stochastic differential equations (BDSDEs), it is worth mentioning Pardoux and Peng [7], Peng and Wu [2], Zhang and Shi [13]. In 1994, Pardoux and Peng [7] first studied the backward doubly stochastic differential equations with the coefficients being random, and proved the existence and uniqueness result of BDSDEs. In 2010, Peng and Wu [2] discussed the optimal control problems for BDSDEs, under the assumptions that the control domain is convex. In the same year, Zhang and Shi [13] considered the maximum principle for fully coupled forward-backward doubly stochastic control system, in which the diffusion coefficient does not contain the control variable.

Motivated by the above mentioning, we are also interested in the optimal control problem of BDSDEs. In another paper (see [11]), we have considered the second-order Taylor expansion of the cost functional for backward doubly stochastic control system, in which the diffusion coefficient contain the control variable, but we could not get the necessary conditions. Via the second-order Taylor expansion, we study the necessary conditions by duality relation in the further, under no restriction on the convexity of control domain and control variable is not allowed in the diffusion coefficient. We do not assume the monotonic conditions introduced in [13], which are essential for their results. And we do not think their method can be used in our paper.

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The rest of this paper is organized as follows. Section 2 begins with a general formulation of the stochastic optimal control problem for BDSDEs. In section 3, we study the variational equation and second-order Taylor expansion for the cost functional. Some necessary conditions for an optimal control are considered in section 4.

For the sake of simplicity, we only consider the one-dimensional case.

2. Preliminaries

Let $T > 0$ and $(\Omega, \mathfrak{F}, P)$ be a completed probability space. $\{W(t) : 0 \leq t \leq T\}$ and $\{B(t) : 0 \leq t \leq T\}$ are two mutually independent standard Brownian motion processes with values both in $\mathbb{R}$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathfrak{F}$. For every $t \in [0, T]$, we define $\mathfrak{F}_t = \mathfrak{F}_t^W \vee \mathfrak{F}_{t,T}^B$, where $\mathfrak{F}_t^W = \mathcal{N} \vee \sigma\{W(r) - W(0) : 0 \leq r \leq t\}$, $\mathfrak{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B(r) - B(t) : t \leq r \leq T\}$.

Let $M^2([0, T]; \mathbb{R}^n)$ denote the set of all classes of ($dt \times dP$ a.e. equal) $\mathfrak{F}_t$-measurable stochastic processes $\{\varphi(t) : t \in [0, T]\}$ which satisfy $E \int_0^T |\varphi(t)|^2 dt < +\infty$, where E denotes the expectation on $(\Omega, \mathfrak{F}, P)$. And $S^2([0, T]; \mathbb{R}^k)$ denotes the set of all classes of $\mathfrak{F}_t$-measurable stochastic processes $\{\varphi(t) : t \in [0, T]\}$ which satisfy $E(\sup_{0 \leq t \leq T} |\varphi(t)|^2) < +\infty$.

For given $\varphi(t), \psi(t) \in M^2([0, T]; \mathbb{R}^n)$, we can define the forward Itô's integral $\int_0^{\cdot} \varphi(s) dW(s)$ and the backward Itô's integral $\int_{\cdot}^T \psi(s) dB(s)$. They are both in $M^2([0, T]; \mathbb{R}^n)$ (see [6], [10] for details).

Let U be a nonempty subset of $\mathbb{R}$, and (U, d) is a separable metric space. Define

$$U[0, T] = \left\{ u : (\omega, t) \in \Omega \times [0, T] \to U \mid u \text{ is } \mathfrak{F}_t\text{-adapted}, E \int_0^T |u(t)|^2 dt < +\infty \right\}.$$

We consider the following backward doubly stochastic control system:

$$\begin{cases} -dy(t) = f(t, y(t), z(t), u(t))dt + g(t, y(t), z(t))dB(t) - z(t)dW(t), \\ y(T) = \eta, \end{cases} \tag{1}$$

with the cost functional

$$J(u(\cdot)) = E \left\{ \int_0^T l(t, y(t), z(t), u(t))dt + \Phi(y(0)) \right\}. \tag{2}$$

We assume that the following conditions hold.

(H_1) $f, l : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are twice continuous and continuously differentiable with respect to y, z. $\Phi : \mathbb{R} \to \mathbb{R}$ is twice continuous and continuously differentiable with respect to y.

(H_2) All the derivatives are bounded.

(H_3) The maps f, g, l and Φ are measurable, and there exist constants $c > 0$, $0 < \sigma < 1$ and a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ such that for $\varphi(t, y, z, u) = f(t, y, z, u)$, $l(t, y, z, u)$, we have

$$\begin{cases} |\varphi(t, y_1, z_1, u_1) - \varphi(t, y_2, z_2, u_2)|^2 \leq c(|y_1 - y_2|^2 + |z_1 - z_2|^2) + \omega(d(u_1, u_2)), \\ |g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \leq c|y_1 - y_2|^2 + \sigma|z_1 - z_2|^2, \\ |\Phi(y_1) - \Phi(y_2)| \leq c|y_1 - y_2|, \quad \forall t \in [0, T], y_1, z_1, y_2, z_2 \in \mathbb{R}, u_1, u_2 \in U[0, T], \\ |\varphi(t, 0, 0, u)| \leq c, \quad |\Phi(0)| \leq c, \quad \forall (t, u) \in [0, T] \times U[0, T]. \end{cases} \tag{3}$$

(H_4) There exist a constant $c > 0$ and a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ such that for $\varphi(t,y,z,u) = f(t,y,z,u)$, $g(t,y,z)$, $l(t,y,z,u)$, $\Phi(y)$, we have

$$\begin{cases} |\varphi_y(t,y_1,z_1,u_1) - \varphi_y(t,y_2,z_2,u_2)|^2 \le c(|y_1-y_2|^2 + |z_1-z_2|^2) + \omega(d(u_1,u_2)), \\ |\varphi_z(t,y_1,z_1,u_1) - \varphi_z(t,y_2,z_2,u_2)|^2 \le c(|y_1-y_2|^2 + |z_1-z_2|^2) + \omega(d(u_1,u_2)), \\ |\varphi_{yy}(t,y_1,z_1,u_1) - \varphi_{yy}(t,y_2,z_2,u_2)|^2 \le c(|y_1-y_2|^2 + |z_1-z_2|^2) + \omega(d(u_1,u_2)), \\ |\varphi_{zz}(t,y_1,z_1,u_1) - \varphi_{zz}(t,y_2,z_2,u_2)|^2 \le c(|y_1-y_2|^2 + |z_1-z_2|^2) + \omega(d(u_1,u_2)), \\ \qquad \forall t \in [0,T], y_1,z_1,y_2,z_2 \in \mathbb{R}, u_1,u_2 \in U[0,T], \end{cases} \tag{4}$$

note that for function g we can see it as $d(u_1,u_2) = 0$ in (4).

Given a $u(\cdot) \in U[0,T]$, by Theorem 1.1 in [7], there exists a unique pair

$$(y(\cdot),z(\cdot)) = (y(\cdot,u(\cdot)),z(\cdot,u(\cdot))) \in S^2([0,T];\mathbb{R}) \times M^2([0,T];\mathbb{R}), \tag{5}$$

which solves the state equation (1). Thus, we can see that the cost functional $J(u(\cdot))$ is uniquely determined by the control variable $u(\cdot)$.

Then our optimal control problem can be stated as follows:

Problem (P) Find a $u^*(\cdot) \in U[0,T]$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in U[0,T]} J(u(\cdot)). \tag{6}$$

Any $u^*(\cdot) \in U[0,T]$ satisfying the above identity is called an optimal control, and the corresponding state $(y^*(\cdot),z^*(\cdot)) = (y(\cdot,u^*(\cdot)),z(\cdot,u^*(\cdot)))$ is called an optimal trajectory; $(y^*(\cdot),z^*(\cdot),u^*(\cdot))$ is called an optimal triple.

We will need the following extension of the well-known Itô's formula.

Lemma 1 (Pardoux and Peng [7]) Let $\alpha \in S^2([0,T];\mathbb{R}^k)$, $\beta \in M^2([0,T];\mathbb{R}^k)$, $\gamma \in M^2([0,T];\mathbb{R}^{k\times d})$, $\delta \in M^2([0,T];\mathbb{R}^{k\times m})$ be such that (in this lemma, $\{W(t) : 0 \le t \le T\}$ and $\{B(t) : 0 \le t \le T\}$ value respectively in $\mathbb{R}^m$ and in $\mathbb{R}^d$):

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \delta_s dW_s, \quad 0 \le t \le T.$$

Then

$$\begin{aligned} |\alpha_t|^2 \quad &= \quad |\alpha_0|^2 + 2\int_0^t (\alpha_s,\beta_s)ds + 2\int_0^t (\alpha_s,\gamma_s dB_s) + 2\int_0^t (\alpha_s,\delta_s dW_s) \\ &\quad - \int_0^t ||\gamma_s||^2 ds + \int_0^t ||\delta_s||^2 ds, \end{aligned}$$

$$E|\alpha_t|^2 = E|\alpha_0|^2 + 2E\int_0^t (\alpha_s,\beta_s)ds - E\int_0^t ||\gamma_s||^2 ds + E\int_0^t ||\delta_s||^2 ds.$$

3. Variational equation and second-order Taylor expansion

Suppose $(y^*(\cdot),z^*(\cdot),u^*(\cdot))$ is a solution to the optimal control problem (P). First, we introduce the spike variation with respect to $u^*(\cdot)$ as follows:

$$u^\varepsilon(t) = \begin{cases} v(t), & \tau \le t \le \tau+\varepsilon, \\ u^*(t), & \text{otherwise}, \end{cases} \tag{7}$$

where $\varepsilon > 0$ is sufficiently small, $v(\cdot) \in U[0,T]$ is an $\mathfrak{F}_t$-measurable random variable, and $\sup_{\omega\in\Omega} |v(\omega)| < +\infty$, $0 \le t \le T$.

Suppose $(y^\varepsilon(\cdot),z^\varepsilon(\cdot))$ is the trajectory of (1) corresponding to $u^\varepsilon(\cdot)$.

Then we have the following lemma.

Lemma 2 Let $\xi^\varepsilon(t) = y^\varepsilon(t) - y^*(t)$, $\eta^\varepsilon(t) = z^\varepsilon(t) - z^*(t)$. Then

$$E|\xi^\varepsilon(t)|^2 \le C\varepsilon, \quad E\int_t^T |\eta^\varepsilon(s)|^2 ds \le C\varepsilon, \tag{8}$$

where C is a positive constant independent of ε.

Proof. From the state equation (1), it is easy to know that

$$\begin{aligned}\xi^\varepsilon(t) &= \int_t^T [f(s, y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) - f(s, y^*(s), z^*(s), u^*(s))]ds \\ &\quad + \int_t^T [g(s, y^\varepsilon(s), z^\varepsilon(s)) - g(s, y^*(s), z^*(s))]dB(s) \\ &\quad - \int_t^T (z^\varepsilon(s) - z^*(s))dW(s).\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}|\xi^\varepsilon(t)|^2 &= |\xi^\varepsilon(T)|^2 + 2\int_t^T \xi^\varepsilon(s)[f(s, y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) - f(s, y^*(s), z^*(s), u^*(s))]ds \\ &\quad +2\int_t^T \xi^\varepsilon(s)[g(s, y^\varepsilon(s), z^\varepsilon(s)) - g(s, y^*(s), z^*(s))]dB(s) \\ &\quad -2\int_t^T \xi^\varepsilon(s)(z^\varepsilon(s) - z^*(s))dW(s) - \int_t^T |z^\varepsilon(s) - z^*(s)|^2 ds \\ &\quad + \int_t^T |g(s, y^\varepsilon(s), z^\varepsilon(s)) - g(s, y^*(s), z^*(s))|^2 ds.\end{aligned}$$

Hence from the assumptions $(H_1) - (H_3)$, it follows that

$$\begin{aligned}&E|\xi^\varepsilon(t)|^2 + E\int_t^T |\eta^\varepsilon(s)|^2 ds \\ \le\ & CE\int_t^T [|\xi^\varepsilon(s)|^2 + |\xi^\varepsilon(s)||\eta^\varepsilon(s)| + \omega(d(u^\varepsilon(s), u^*(s)))|\xi^\varepsilon(s)|]ds \\ &+\sigma E\int_t^T |\eta^\varepsilon(s)|^2 ds + E\int_t^T \omega(d(u^\varepsilon(s), u^*(s)))ds.\end{aligned}$$

From the definition and properties of a modulus of continuity (refer to pages 227 and 234 in [5]), for any $\varepsilon > 0$, there exists a positive constant K_ε such that

$$\omega(d(u^\varepsilon(s), u^*(s))) \le \varepsilon + d(u^\varepsilon(s), u^*(s))K_\varepsilon, \quad \forall \mathrm{s} \in [t, T].$$

Thus, combining with the definition of $u^\varepsilon(t)$, we have

$$\begin{aligned}&E\int_t^T \omega(d(u^\varepsilon(s), u^*(s)))ds = E\int_\tau^{\tau+\varepsilon} \omega(d(u^\varepsilon(s), u^*(s)))ds \\ \le\ & \varepsilon(\varepsilon + E\sup_{s\in[\tau,\tau+\varepsilon]} d(u^\varepsilon(s), u^*(s))K_\varepsilon).\end{aligned}$$

By Young's inequality, there exists a constant $M > 0$ such that

$$\begin{aligned}&E|\xi^\varepsilon(t)|^2 + (1 - \sigma - \frac{C_1}{M})E\int_t^T |\eta^\varepsilon(s)|^2 ds \\ \le\ & (C + C_1M + C_2)E\int_t^T |\xi^\varepsilon(s)|^2 ds \\ &+C_2E\int_t^T \omega^2(d(u^\varepsilon(s), u^*(s)))ds + \varepsilon(\varepsilon + E\sup_{s\in[\tau,\tau+\varepsilon]} d(u^\varepsilon(s), u^*(s))K_\varepsilon)\end{aligned}$$

$$\leq \quad CE\int_t^T |\xi^\varepsilon(s)|^2 ds + C_3\varepsilon.$$

We can choose some M to make sure that $(1-\sigma-\frac{C_1}{M}) > 0$. By Gronwall's inequality, we obtain (8) (where C, C_1, C_2, C_3 are positive constants, which may be different at different places throughout this paper).

Let

$$\varphi^*(\cdot) = \varphi(\cdot, y^*(\cdot), z^*(\cdot), u^*(\cdot)), \quad \varphi^\varepsilon(\cdot) = \varphi(\cdot, y^\varepsilon(\cdot), z^\varepsilon(\cdot), u^\varepsilon(\cdot)),$$

where φ denotes one of $f,\ g,\ l,\ f_y,\ f_z,\ g_y,\ g_z,\ l_y,\ l_z,\ f_{yy},\ f_{zz},\ g_{yy},\ g_{zz},\ l_{yy},\ l_{zz}$. (Note that the function g does not contain $u(\cdot)$, so the corresponding $\varphi^*(\cdot)$ and $\varphi^\varepsilon(\cdot)$ should be in the absence of $u(\cdot)$.)

Now we introduce the variational equation:

$$\begin{cases} -dx_1(t) = [f_y^*(t)x_1(t) + f_z^*(t)r_1(t) + f(u^\varepsilon(t)) - f(u^*(t))]dt \\ \qquad\qquad +[g_y^*(t)x_1(t) + g_z^*(t)r_1(t)]dB(t) - r_1(t)dW(t), \\ x_1(T) = 0, \end{cases} \tag{9}$$

where $f(u^\varepsilon(t))$ denotes $f(t, y^*(t), z^*(t), u^\varepsilon(t))$, and others are defined in the same way.

From the conditions $(H_1)-(H_3)$, it is easy to know that there exists a unique adapted solution $(x_1(t), r_1(t)) \in \mathbb{R}\times\mathbb{R}$, $0 \leq t \leq T$ satisfying (9).

Lemma 3 Let $(x_1(t),\, r_1(t))$ be the solution of the variational equation (9), then we have

$$E|x_1(t)|^2 \leq C\varepsilon, \qquad E\int_t^T |r_1(s)|^2 ds \leq C\varepsilon. \tag{10}$$

Note that $|g_z^*(t)| \leq \sigma$ for $\forall t \in [0,T]$, then by Lemma 1 and Young's inequality, (10) can be easily work out.

Lemma 4 We assume $(H_1)-(H_4)$ hold. Then we have

$$E|y^\varepsilon(t) - y^*(t) - x_1(t)|^2 \leq C\varepsilon^2, \tag{11}$$

$$E\int_t^T |z^\varepsilon(s) - z^*(s) - r_1(s)|^2 ds \leq C\varepsilon^2. \tag{12}$$

Proof. From equations (1) and (9) , it follows that

$$\begin{aligned} & y^\varepsilon(t) - y^*(t) - x_1(t) \\ = \ & \int_t^T [f(s, y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) - f(s, y^*(s), z^*(s), u^*(s)) - f_y^* x_1(s) - f_z^* r_1(s) \\ & -(f(u^\varepsilon(s)) - f(u^*(s)))]ds + \int_t^T [g(s, y^\varepsilon(s), z^\varepsilon(s)) - g(s, y^*(s), z^*(s)) \\ & -g_y^* x_1(s) - g_z^* r_1(s)]dB(s) - \int_t^T (z^\varepsilon(s) - z^*(s) - r_1(s))dW(s). \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} & E|y^\varepsilon(t) - y^*(t) - x_1(t)|^2 + E\int_t^T |z^\varepsilon(s) - z^*(s) - r_1(s)|^2 ds \\ = \ & 2E\int_t^T [f(s, y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) - f(s, y^*(s), z^*(s), u^*(s)) \\ & -f_y^* x_1(s) - f_z^* r_1(s) - (f(u^\varepsilon(s)) - f(u^*(s)))](y^\varepsilon(s) - y^*(s) - x_1(s))ds \\ & +E\int_t^T |g(s, y^\varepsilon(s), z^\varepsilon(s)) - g(s, y^*(s), z^*(s)) - g_y^* x_1(s) - g_z^* r_1(s)|^2 ds \end{aligned}$$

$$
\begin{aligned}
= \quad & 2E\int_t^T [f_y^*(y^\varepsilon(s)-y^*(s)-x_1(s))+f_z^*(z^\varepsilon(s)-z^*(s)-r_1(s)) \\
& +(f_y(s,y^*(s),z^\varepsilon(s),u^\varepsilon(s))-f_y^*)(y^\varepsilon(s)-y^*(s)) \\
& +(f_z(u^\varepsilon(s))-f_z^*)(z^\varepsilon(s)-z^*(s))](y^\varepsilon(s)-y^*(s)-x_1(s))ds \\
& +2E\int_t^T [O(|y^\varepsilon(s)-y^*(s)|^2)+O(|y^\varepsilon(s)-y^*(s)||z^\varepsilon(s)-z^*(s)|) \\
& +O(|z^\varepsilon(s)-z^*(s)|^2)](y^\varepsilon(s)-y^*(s)-x_1(s))ds \\
& +E\int_t^T |g_y^*(y^\varepsilon(s)-y^*(s)-x_1(s))+g_z^*(z^\varepsilon(s)-z^*(s)-r_1(s)) \\
& +(g_y(s,y^*(s),z^\varepsilon(s))-g_y^*)(y^\varepsilon(s)-y^*(s)) \\
& +O(|y^\varepsilon(s)-y^*(s)|^2)+O(|y^\varepsilon(s)-y^*(s)||z^\varepsilon(s)-z^*(s)|) \\
& +O(|z^\varepsilon(s)-z^*(s)|^2)|^2 ds.
\end{aligned}
$$

From Cauchy-Schwarz inequality and Lemma 2, it follows

$$
\begin{aligned}
& E\int_t^T (f_y(s,y^*(s),z^\varepsilon(s),u^\varepsilon(s))-f_y^*(s))^2(y^\varepsilon(s)-y^*(s))^2 ds \\
\leq \quad & 2E\int_t^T (f_y(s,y^*(s),z^\varepsilon(s),u^\varepsilon(s))-f_y(s,y^*(s),z^*(s),u^\varepsilon(s)))^2(y^\varepsilon(s)-y^*(s))^2 ds \\
& +2E\int_t^T (f_y(u^\varepsilon(s))-f_y(u^*(s)))^2(y^\varepsilon(s)-y^*(s))^2 ds \\
= \quad & 2E\int_t^T (f_{yz})^2(z^\varepsilon(s)-z^*(s))^2(y^\varepsilon(s)-y^*(s))^2 ds \\
& +2E\int_\tau^{\tau+\varepsilon} (f_y(u^\varepsilon(s))-f_y(u^*(s)))^2(y^\varepsilon(s)-y^*(s))^2 ds \\
\leq \quad & C[E\int_t^T (z^\varepsilon(s)-z^*(s))^4 ds]^{\frac{1}{2}}[E\int_t^T (y^\varepsilon(s)-y^*(s))^4 ds]^{\frac{1}{2}} \\
& +C\varepsilon \sup_{s\in[t,T]} E|y^\varepsilon(s)-y^*(s)|^2.
\end{aligned}
$$

From equation (1) and the above conditions, we know that processes $y^\varepsilon(t)-y^*(t)$ and $z^\varepsilon(t)-z^*(t)$ are Ornstein-Uhlenbeck process, so they are also Gauss process. Moreover, their expectation are $O(\varepsilon^{\frac{1}{2}})$ and variance are $O(\varepsilon)$, so we have

$$
E\int_t^T (y^\varepsilon(s)-y^*(s))^4 ds \leq C\varepsilon E\int_t^T (y^\varepsilon(s)-y^*(s))^2 ds \leq C\varepsilon^2,
$$

and

$$
E\int_t^T (z^\varepsilon(s)-z^*(s))^4 ds \leq C\varepsilon E\int_t^T (z^\varepsilon(s)-z^*(s))^2 ds \leq C\varepsilon^2.
$$

Thus,

$$
E\int_t^T (f_y(s,y^*(s),z^\varepsilon(s),u^\varepsilon(s))-f_y^*)^2(y^\varepsilon(s)-y^*(s))^2 ds \leq C\varepsilon^2.
$$

Using Hölder's inequality,

$$
\begin{aligned}
& E\int_t^T (f_z(u^\varepsilon(s))-f_z^*(s))^2(z^\varepsilon(s)-z^*(s))^2 ds \\
= \quad & E\int_\tau^{\tau+\varepsilon} (f_z(u^\varepsilon(s))-f_z^*(s))^2(z^\varepsilon(s)-z^*(s))^2 ds
\end{aligned}
$$

$$\begin{aligned} &\leq \quad CE\int_\tau^{\tau+\varepsilon}(z^\varepsilon(s)-z^*(s))^2ds \\ &\leq \quad C[E\int_\tau^{\tau+\varepsilon}1^p ds]^{\frac{1}{p}}[E\int_\tau^{\tau+\varepsilon}(z^\varepsilon(s)-z^*(s))^{2q}ds]^{\frac{1}{q}}, \end{aligned}$$

where $q \in N_+$, $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Similarly, we can obtain

$$E\int_\tau^{\tau+\varepsilon}(z^\varepsilon(s)-z^*(s))^{2q}ds \leq C\varepsilon^q.$$

So we get

$$E\int_t^T (f_z(u^\varepsilon(s))-f_z^*(s))^2(z^\varepsilon(s)-z^*(s))^2ds \leq C\varepsilon^{1+\frac{1}{p}},$$

with

$$E\int_t^T (g_y(s,y^*(s),z^\varepsilon(s))-g_y^*(s))^2(y^\varepsilon(s)-y^*(s))^2ds \leq C\varepsilon^2.$$

Let $p \to 1$, we can get $\varepsilon^{1+\frac{1}{p}} \to \varepsilon^2$. Although the order could not reach 2, it is enough for us to get the second-order Taylor expansion. We can choose some p small enough, and we will give some analysis later. For convenience, here we denote it by 2.
By Young's inequality and Lemma 2, we deduce

$$\begin{aligned} &E|y^\varepsilon(t)-y^*(t)-x_1(t)|^2+\alpha E\int_t^T|z^\varepsilon(s)-z^*(s)-r_1(s)|^2ds \\ \leq \quad &CE\int_t^T|y^\varepsilon(s)-y^*(s)-x_1(s)|^2ds+C\varepsilon^2. \end{aligned}$$

We can make sure that $\alpha > 0$, then by Gronwall's inequality, we obtain the results (Note that the real order should be $1+\frac{1}{p}$).

Let $(x_2(t), r_2(t))$ be the solution of the following stochastic differential equation:

$$\begin{cases} -dx_2(t) = [f_y^*x_2(t)+f_z^*r_2(t)+\frac{1}{2}(f_{yy}^*x_1^2(t)+2f_{yz}^*x_1(t)r_1(t)+f_{zz}^*r_1^2(t))]dt \\ \qquad\qquad +[g_y^*x_2(t)+g_z^*r_2(t)+\frac{1}{2}(g_{yy}^*x_1^2(t)+2g_{yz}^*x_1(t)r_1(t)+g_{zz}^*r_1^2(t))]dB(t) \\ \qquad\qquad -r_2(t)dW(t), \\ x_2(T)=0. \end{cases} \tag{13}$$

Lemma 5 We assume $(H_1)-(H_4)$ hold. Then we have

$$E|x_2(t)|^2 \leq C\varepsilon^2, \quad E\int_t^T|r_2(s)|^2ds \leq C\varepsilon^2. \tag{14}$$

Lemma 6 We assume $(H_1)-(H_4)$ hold. Then we have

$$E|y^\varepsilon(t)-y^*(t)-x_1(t)-x_2(t)|^2 \leq C\varepsilon^3, \tag{15}$$

$$E\int_t^T|z^\varepsilon(s)-z^*(s)-r_1(s)-r_2(s)|^2ds \leq C\varepsilon^3. \tag{16}$$

The above two Lemmas can be similarly proved. Now we give an elementary lemma which will be used below, and its proof is very simple and straightforward.

Lemma 7 Let $g \in C^2(\mathbb{R}^n)$. Then for any $x, \bar{x} \in \mathbb{R}^n$,

$$g(x) = g(\bar{x}) + \langle g_x(\bar{x}), x - \bar{x}\rangle + \int_0^1 \langle \theta g_{xx}(\theta\bar{x} + (1-\theta)x)(x - \bar{x}, x - \bar{x})\rangle d\theta.$$

Now we give the second-order Taylor expansion of the cost functional along the optimal control $u^*(\cdot)$.

Theorem 8 Let $(H_1) - (H_4)$ hold. Then the following Taylor expansion holds for the cost functional (2):

$$\begin{aligned} J(u^\varepsilon(\cdot)) \quad = \quad & J(u^*(\cdot)) + E\Phi_y(y^*(0))(x_1(0) + x_2(0)) + \frac{1}{2}E\Phi_{yy}(y^*(0))x_1^2(0) \\ & + E\int_0^T \{l(u^\varepsilon(t)) - l(u^*(t)) + l_y^*(t)(x_1(t) + x_2(t)) + l_z^*(t)(r_1(t) + r_2(t)) \\ & + \frac{1}{2}l_{yy}^*(t)x_1^2(t) + l_{yz}^*(t)x_1(t)r_1(t) + \frac{1}{2}l_{zz}^*(t)r_1^2(t)\}dt + o(\varepsilon). \end{aligned} \tag{17}$$

Proof. By the definition of $J(u(\cdot))$ and Lemma 7, we have

$$\begin{aligned} & J(u^\varepsilon(\cdot)) - J(u^*(\cdot)) \\ = \quad & E(\Phi(y^\varepsilon(0)) - \Phi(y^*(0))) + E\int_0^T (l^\varepsilon(t) - l^*(t))dt \\ = \quad & E\Phi_y(y^*(0))(y^\varepsilon(0) - y^*(0)) \\ & + E\int_0^1 \theta\Phi_{yy}(\theta y^*(0) + (1-\theta)y^\varepsilon(0))(y^\varepsilon(0) - y^*(0))^2 d\theta \\ & + E\int_0^T \{l(u^\varepsilon(t)) - l(u^*(t)) + l_y(t, y^*(t), z^\varepsilon(t), u^\varepsilon(t))(y^\varepsilon(t) - y^*(t)) \\ & + \int_0^1 \theta l_{yy}(t, \theta y^*(t) + (1-\theta)y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t))(y^\varepsilon(t) - y^*(t))^2 d\theta \\ & + l_z(t, y^*(t), z^*(t), u^\varepsilon(t))(z^\varepsilon(t) - z^*(t)) \\ & + \int_0^1 \theta l_{zz}(t, y^*(t), \theta z^*(t) + (1-\theta)z^\varepsilon(t), u^\varepsilon(t))(z^\varepsilon(t) - z^*(t))^2 d\theta\}dt. \end{aligned}$$

Let

$$\begin{aligned} I_1 \quad = \quad & E\int_0^T \{l(u^\varepsilon(t)) - l(u^*(t)) + l_y(t, y^*(t), z^\varepsilon(t), u^\varepsilon(t))(y^\varepsilon(t) - y^*(t)) \\ & + \int_0^1 \theta l_{yy}(t, \theta y^*(t) + (1-\theta)y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t))(y^\varepsilon(t) - y^*(t))^2 d\theta\}dt, \end{aligned}$$

$$\begin{aligned} I_2 \quad = \quad & E\int_0^T \{l_z(t, y^*(t), z^*(t), u^\varepsilon(t))(z^\varepsilon(t) - z^*(t)) \\ & + \int_0^1 \theta l_{zz}(t, y^*(t), \theta z^*(t) + (1-\theta)z^\varepsilon(t), u^\varepsilon(t))(z^\varepsilon(t) - z^*(t))^2 d\theta\}dt, \end{aligned}$$

$$\begin{aligned} I_3 \quad = \quad & E\Phi_y(y^*(0))(y^\varepsilon(0) - y^*(0)) \\ & + E\int_0^1 \theta\Phi_{yy}(\theta y^*(0) + (1-\theta)y^\varepsilon(0))(y^\varepsilon(0) - y^*(0))^2 d\theta. \end{aligned}$$

In the further, via some equivalent transformations, we obtain

$$\begin{aligned} I_1 \quad = \quad & E\int_0^T \{l(u^\varepsilon(t)) - l(u^*(t)) + l_y(t, y^*(t), z^\varepsilon(t), u^*(t))(x_1(t) + x_2(t)) \\ & + l_y(t, y^*(t), z^\varepsilon(t), u^*(t))(y^\varepsilon(t) - y^*(t) - x_1(t) - x_2(t)) \end{aligned}$$

$$
\begin{aligned}
&+(l_y(t,y^*(t),z^\varepsilon(t),u^\varepsilon(t))-l_y(t,y^*(t),z^\varepsilon(t),u^*(t)))(y^\varepsilon(t)-y^*(t))\\
&+\int_0^1\theta[l_{yy}(t,\theta y^*(t)+(1-\theta)y^\varepsilon(t),z^\varepsilon(t),u^\varepsilon(t))\\
&-l_{yy}(t,y^*(t),z^\varepsilon(t),u^\varepsilon(t))](y^\varepsilon(t)-y^*(t))^2d\theta+\frac{1}{2}l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t))x_1^2(t)\\
&+\frac{1}{2}l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t))(y^\varepsilon(t)-y^*(t)-x_1(t))(y^\varepsilon(t)-y^*(t)+x_1(t))\\
&+\frac{1}{2}(l_{yy}(t,y^*(t),z^\varepsilon(t),u^\varepsilon(t))-l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t)))(y^\varepsilon(t)-y^*(t))^2\}dt,
\end{aligned}
$$

$$
\begin{aligned}
I_2 \;=\;& E\int_0^T\{l_z(t,y^*(t),z^*(t),u^*(t))(r_1(t)+r_2(t))\\
&+l_z(t,y^*(t),z^*(t),u^*(t))(z^\varepsilon(t)-z^*(t)-r_1(t)-r_2(t))\\
&+(l_z(t,y^*(t),z^*(t),u^\varepsilon(t))-l_z(t,y^*(t),z^*(t),u^*(t)))(z^\varepsilon(t)-z^*(t))\\
&+\int_0^1\theta[l_{zz}(t,y^*(t),\theta z^*(t)+(1-\theta)z^\varepsilon(t),u^\varepsilon(t))\\
&-l_{zz}(t,y^*(t),z^*(t),u^\varepsilon(t))](z^\varepsilon(t)-z^*(t))^2d\theta+\frac{1}{2}l_{zz}(t,y^*(t),z^*(t),u^*(t))r_1^2(t)\\
&+\frac{1}{2}l_{zz}(t,y^*(t),z^*(t),u^*(t))(z^\varepsilon(t)-z^*(t)-r_1(t))(z^\varepsilon(t)-z^*(t)+r_1(t))\\
&+\frac{1}{2}(l_{zz}(t,y^*(t),z^*(t),u^\varepsilon(t))-l_{zz}(t,y^*(t),z^*,u^*(t)))(z^\varepsilon(t)-z^*(t))^2\}dt,
\end{aligned}
$$

$$
\begin{aligned}
I_3 \;=\;& E\Phi_y(y^*(0))(x_1(0)+x_2(0))+E\Phi_y(y^*(0))(y^\varepsilon(0)-y^*(0)-x_1(0)-x_2(0))\\
&+E\int_0^1\theta[\Phi_{yy}(\theta y^*(0)+(1-\theta)y^\varepsilon(0))-\Phi_{yy}(y^*(0))](y^\varepsilon(0)-y^*(0))^2d\theta\\
&+\frac{1}{2}E\Phi_{yy}(y^*(0))(y^\varepsilon(0)-y^*(0)-x_1(0))(y^\varepsilon(0)-y^*(0)+x_1(0))\\
&+\frac{1}{2}E\Phi_{yy}(y^*(0))x_1^2(0).
\end{aligned}
$$

From the conditions $(H_1)-(H_4)$ and all the above lemmas , we can show that

$$
\begin{aligned}
I_1 \;=\;& E\int_0^T\{l(u^\varepsilon(t))-l(u^*(t))+l_y(t,y^*(t),z^\varepsilon(t),u^*(t))(x_1(t)+x_2(t))\\
&+\frac{1}{2}l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t))x_1^2(t)\}dt+O(\varepsilon^{\frac{3}{2}})\\
&+O(E|z^\varepsilon(t)-z^*(t)-r_1(t)||z^\varepsilon(t)-z^*(t)+r_1(t)|)\\
&+\varepsilon O(E\int_0^T|z^\varepsilon(t)-z^*(t)|dt)+C(1-\theta)O(E|y^\varepsilon(t)-y^*(t)|^3)\\
&+\varepsilon O(E|y^\varepsilon(t)-y^*(t)|^2)\\
=\;& E\int_0^T\{l(u^\varepsilon(t))-l(u^*(t))+l_y(t,y^*(t),z^\varepsilon(t),u^*(t))(x_1(t)+x_2(t))\\
&+\frac{1}{2}l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t))x_1^2(t)\}dt+o(\varepsilon)\\
=\;& E\int_0^T\{l(u^\varepsilon(t))-l(u^*(t))+l_y^*(t)(x_1(t)+x_2(t))+\frac{1}{2}l_{yy}^*x_1^2(t)\\
&+(l_y(t,y^*(t),z^\varepsilon(t),u^*(t))-l_y^*(t))(x_1(t)+x_2(t))\\
&+\frac{1}{2}(l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t))-l_{yy}^*)x_1^2(t)\}dt+o(\varepsilon).
\end{aligned}
$$

By Lemma 2-5, it is easy to get

$$\begin{aligned}
& E\int_0^T (l_y(t,y^*(t),z^\varepsilon(t),u^*(t)) - l_y^*(t))(x_1(t)+x_2(t))dt \\
= \quad & E\int_0^T l_{yz}^*(t)(z^\varepsilon(t)-z^*(t))(x_1(t)+x_2(t))dt + o(\varepsilon) \\
= \quad & E\int_0^T l_{yz}^*(t)(z^\varepsilon(t)-z^*(t)-r_1(t))(x_1(t)+x_2(t)) \\
& + l_{yz}^*(t)x_1(t)r_1(t) + l_{yz}^*(t)x_2(t)r_1(t)dt + o(\varepsilon) \\
= \quad & E\int_0^T l_{yz}^*(t)x_1(t)r_1(t)dt + o(\varepsilon),
\end{aligned}$$

and by condition (H_4), Lemma 2 and Lemma 3, we have

$$E\int_0^T \frac{1}{2}(l_{yy}(t,y^*(t),z^\varepsilon(t),u^*(t)) - l_{yy}^*)x_1^2(t)dt = o(\varepsilon).$$

Combining the above three identities, it follows that

$$I_1 = E\int_0^T [l(u^\varepsilon(t)) - l(u^*(t)) + l_y^*(t)(x_1(t)+x_2(t)) + l_{yz}^*(t)x_1(t)r_1(t) + \frac{1}{2}l_{yy}^* x_1^2(t)]dt + o(\varepsilon).$$

Similarly,

$$I_2 = E\int_0^T [l_z^*(t)(r_1(t)+r_2(t)) + \frac{1}{2}l_{zz}^*(t)r_1^2(t)]dt + o(\varepsilon),$$

$$I_3 = E\Phi_y(y^*(0))(x_1(0)+x_2(0)) + \frac{1}{2}E\Phi_{yy}(y^*(0))x_1^2(0) + o(\varepsilon).$$

Hence, our conclusion follows.

Remark 9 Recall Lemma 4, the real order is only $(1+\frac{1}{p})$. From the above proof, we know that the real order in Taylor expansion is a little smaller then $\frac{3}{2}$. So we can choose some p to make sure that the order of ε in (17) is $o(\varepsilon)$.

4. Some necessary conditions

In this section, we want to derive the necessary conditions by duality relation.

From Theorem 8, we can conclude that a necessary condition for a given optimal triple $(y^*(\cdot), z^*(\cdot), u^*(\cdot))$ is the following inequality:

$$\begin{aligned}
0 \quad \leq \quad & J(u^\varepsilon(\cdot)) - J(u^*(\cdot)) \\
= \quad & E\Phi_y(y^*(0))(x_1(0)+x_2(0)) + \frac{1}{2}E\Phi_{yy}(y^*(0))x_1^2(0) \\
& + E\int_0^T \{l(u^\varepsilon(t)) - l(u^*(t)) + l_y^*(t)(x_1(t)+x_2(t)) + l_z^*(t)\cdot \\
& (r_1(t)+r_2(t)) + \frac{1}{2}l_{yy}^*(t)x_1^2(t) + l_{yz}^*(t)x_1(t)r_1(t) \\
& + \frac{1}{2}l_{zz}^*(t)r_1^2(t)\}dt + o(\varepsilon), \forall v(\cdot) \in U[0,T], \forall \varepsilon > 0.
\end{aligned} \tag{18}$$

As the usual method for Pontryagin's maximum principle, we have to get rid of $x_1(\cdot), r_1(\cdot)$, $x_2(\cdot), r_2(\cdot)$, and then pass to the limit. To this end, we need some duality analysis.

First, we introduce the following adjoint equation with respect to the variational equation (9) using the dual technique:

$$\begin{cases} dp(t) = (f_y^* p(t) + g_y^* q(t) + l_y^*(t))dt \\ \qquad\qquad +(f_z^* p(t) + g_z^* q(t) + l_z^*(t))dW(t) - q(t)dB(t), \\ p(0) = \Phi_y(y^*(0)), \end{cases} \tag{19}$$

where $(p(t), q(t)) \in \mathbb{R} \times \mathbb{R}$. From the conditions $(H_1) - (H_3)$ and the Theorem 1.1 in [7], we conclude that (19) has a unique solution $(p(t), q(t))$, $0 \le t \le T$.

Using Itô's formula to $p(t)x_1(t)$, it follows that

$$\begin{aligned} & E\int_0^T d(p(t)x_1(t)) \\ = \; & E(p(T)x_1(T)) - E(p(0)x_1(0)) \\ = \; & E\int_0^T [(f_y^* p(t) + g_y^* q(t) + l_y^*(t))x_1(t) \\ & -(f_y^* x_1(t) + f_z^* r_1(t) + f(u^\varepsilon(t)) - f(u^*(t)))p(t) \\ & +(f_z^* p(t) + g_z^* q(t) + l_z^*(t))r_1(t) - (g_y^* x_1(t) + g_z^* r_1(t))q(t)]dt \\ = \; & E\int_0^T [l_y^* x_1(t) + l_z^* r_1(t) - (f(u^\varepsilon(t)) - f(u^*(t)))p(t)]dt \\ = \; & -E\Phi_y(y^*(0))x_1(0). \end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned} & E\int_0^T d(p(t)x_2(t)) = E(p(T)x_2(T)) - E(p(0)x_2(0)) \\ = \; & E\int_0^T [(f_y^* p(t) + g_y^* q(t) + l_y^*)x_2(t) - (f_y^* x_2(t) + f_z^* r_2(t) \\ & +\frac{1}{2}(f_{yy}^* x_1^2(t) + 2f_{yz}^* x_1(t)r_1(t) + f_{zz}^* r_1^2(t)))p(t) \\ & +(f_z^* p(t) + g_z^* q(t) + l_z(t, y^*(t), z^*(t), u^*(t)))r_2(t) \\ & -(g_y^* x_2(t) + g_z^* r_2(t) + \frac{1}{2}(g_{yy}^* x_1^2(t) + 2g_{yz}^* x_1(t)r_1(t) + g_{zz}^* r_1^2(t)))q(t)]dt \\ = \; & E\int_0^T [l_y^* x_2(t) + l_z^* r_2(t) - \frac{1}{2}(f_{yy}^* x_1^2(t) + 2f_{yz}^* x_1(t)r_1(t) + f_{zz}^* r_1^2(t))p(t) \\ & -\frac{1}{2}(g_{yy}^* x_1^2(t) + 2g_{yz}^* x_1(t)r_1(t) + g_{zz}^* r_1^2(t))q(t)]dt \\ = \; & -E\Phi_y(y^*(0))x_2(0)). \end{aligned}$$

Thus,

$$\begin{aligned} & E\Phi_y(y^*(0))(x_1(0) + x_2(0)) \\ = \; & -E\int_0^T [l_y^*(t)(x_1(t) + x_2(t)) + l_z^*(t)(r_1(t) + r_2(t)) \\ & -(f(u^\varepsilon(t)) - f(u^*(t)))p(t) \\ & -\frac{1}{2}(f_{yy}^* x_1^2(t) + 2f_{yz}^* x_1(t)r_1(t) + f_{zz}^* r_1^2(t))p(t) \\ & -\frac{1}{2}(g_{yy}^* x_1^2(t) + 2g_{yz}^* x_1(t)r_1(t) + g_{zz}^* r_1^2(t))q(t)]dt. \end{aligned}$$

We define the Hamilton function as

$$H(t, y, z, u, p, q) = f(t, y, z, u)p + g(t, y, z)q + l(t, y, z, u), \tag{21}$$

where $H:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. On one hand, let $(y,z,u)=(y^*,z^*,u^*)$ in (21),

$$\begin{aligned}
&\frac{1}{2}(H^*_{yy}x_1^2(t)+2H^*_{yz}x_1(t)r_1(t)+H^*_{zz}r_1^2(t))\\
=\ &\frac{1}{2}(f^*_{yy}p(t)x_1^2(t)+g^*_{yy}q(t)x_1^2(t)+l^*_{yy}(t)x_1^2(t))\\
&+(f^*_{yz}p(t)x_1(t)r_1(t)+g^*_{yz}q(t)x_1(t)r_1(t)+l^*_{yz}(t)x_1(t)r_1(t))\\
&+\frac{1}{2}(f^*_{zz}p(t)r_1^2(t))+g^*_{zz}q(t)r_1^2(t)+l^*_{zz}(t)r_1^2(t)).
\end{aligned}$$

On the other hand, from (20) we can get

$$\begin{aligned}
&E\int_0^T[H(u^\varepsilon(t))-H(u^*(t))]dt\\
=\ &E\int_0^T[l^*_yx_1(t)+l^*_zr_1(t)+l(u^\varepsilon(t))-l(u^*(t))]dt+E\Phi_y(y^*(0))x_1(0).
\end{aligned}$$

So (18) can be rewritten as

$$\begin{aligned}
0\ \le\ &\frac{1}{2}E\Phi_{yy}(y^*(0))x_1^2(0)+E\int_0^T[H(u^\varepsilon(t))-H(u^*(t))\\
&+\frac{1}{2}(H_{yy}x_1^2(t)+2H_{yz}x_1(t)r_1(t)+H_{zz}r_1^2(t))]dt+o(\varepsilon). \qquad (22)
\end{aligned}$$

In the further, from the variational equation (9), we have

$$\begin{cases}
dx_1^2(t)=[-2f^*_y(t)x_1^2(t)-2f^*_z(t)x_1(t)r_1(t)-2(f(u^\varepsilon(t))-f(u^*(t)))x_1(t)\\
\qquad\qquad +r_1^2(t)-(g^*_y(t)x_1(t)+g^*_z(t)r_1(t))^2]dt\\
\qquad\qquad -[2g^*_y(t)x_1^2(t)+2g^*_z(t)x_1(t)r_1(t)]dB(t)+2x_1(t)r_1(t)dW(t),\\
x_1^2(T)=0.
\end{cases} \qquad (23)$$

And we also attempt to derive the adjoint equation corresponding to (23), we get

$$\begin{cases}
dP(t)=[2f^*_y(t)P(t)+g^{*2}_y(t)P(t)-2g^*_y(t)Q(t)+H_{yy}(t)]dt\\
\qquad\qquad +[f^*_z(t)P(t)+g^*_yg^*_zP(t)-g^*_zQ(t)+H_{yz}(t)]dW(t)+Q(t)dB(t),\\
P(0)=\Phi_{yy}(y^*(0)).
\end{cases} \qquad (24)$$

Now using Itô's formula to $P(t)x_1^2(t)$, it follows

$$\begin{aligned}
&E\int_0^T d(P(t)x_1^2(t))=-E\Phi_{yy}(y^*(0))x_1^2(0)\\
=\ &E\int_0^T[(-2f^*_y(t)x_1^2(t)-2f^*_z(t)x_1(t)r_1(t)-2(f(u^\varepsilon(t))-f(u^*(t)))x_1(t)\\
&+r_1^2(t)-(g^*_y(t)x_1(t)+g^*_z(t)r_1(t))^2)P(t)+(2f^*_y(t)P(t)+g^{*2}_y(t)P(t)\\
&-2g^*_y(t)Q(t)+H_{yy}(t))x_1^2(t)+2(f^*_z(t)P(t)+g^*_y(t)g^*_z(t)P(t)-g^*_z(t)Q(t)\\
&+H_{yz}(t))x_1(t)r_1(t)+(2g^*_y(t)x_1^2(t)+2g^*_z(t)x_1(t)r_1(t))Q(t)]dt\\
=\ &E\int_0^T[(1-g^{*2}_z(t))P(t)r_1^2(t)+H_{yy}(t)x_1^2(t)+2H_{yz}(t)x_1(t)r_1(t)]dt+o(\varepsilon). \qquad (25)
\end{aligned}$$

Thus, (22) can be changed into

$$0\le E\int_0^T[H(u^\varepsilon(t))-H(u^*(t))+\frac{1}{2}(H_{zz}-(1-g^{*2}_z(t))P(t))r_1^2(t)]dt+o(\varepsilon). \qquad (26)$$

So we have obtained three necessary conditions for an optimal control, i.e. (18), (22) and (26). (18) is too complicated and not convenient for using. By duality relation, we turn it into (22). Now, (22) is simpler, but still there is one term outside the integral. We could not pass to limit, so we change it to (26). We find that we can not get rid of $r_1(t)$ completely. So the necessary conditions (22) or (26) is not very perfect. Compared to the forward stochastic differential equation, a solution to a backward stochastic differential equation is a pair such as $(y(\cdot), z(\cdot))$, not only a stochastic process $y(\cdot)$. This is the reason that $r_1(\cdot)$ and $r_2(\cdot)$ appear in the above. And this causes some difficulty to our problem. We may overcome it and try to derive the Pontryagin's maximum principle in future.

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