*Electronic Journal of Mathematical Analysis and Applications*
Vol. 1(2) July 2013, pp. 178-187.
ISSN: 2090-792X (online)
http://ejmaa.6te.net/

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# DENSITY PROBLEMS IN $L^p_\Delta(\mathbb{T},\mathbb{R})$ SPACE

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Abstract. In this paper, we present a generalization of the density $C_{rd}(\mathbb{T},\mathbb{R})$ in $L^p_\Delta(\mathbb{T},\mathbb{R})$, where $\mathbb{T}$ is a limited time scale. We use the notions of equi-integrability $L^p_\lambda([a,b],\mathbb{R})$ and convergence in measure.

## 1. Introduction

The study dynamic on time scales, which has been created in order to unify differential and difference equations, is an area of mathematics that has been subject of interest. Moreover, a lot of results in this area have been deeply studied in the PhD thesis of Hilger [4] and the monographs the Bohner and Peterson, [1, 2] Lakshmikantham et al [5] and the references therein.

Recently, the Lebesgue $\Delta$-integral has been introduced by Bohner and Guseinov in ([2], Chapter 5). For the fundamental relationship between Riemann and Lebesgue $\Delta$-integrals see A. Cabada, D. Vivero [3]. The goal of this paper is to investigate the density propriety between a space of rd-continuous function and Lebesgue $\Delta$-integral. This is the first step in this direction. In the sequel could be interesting to deepen this study in Morrey spaces $L^{p,\lambda}$; (see e. g.[6] ).

## 2. Preliminaries

We will briefly recall some basic definitions and facts from time scale calculus that we will use in the sequel.

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}\qquad\text{and}\qquad\rho(t):=\sup\{s\in\mathbb{T}:s<t\}.$$

(supplemented by $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$) are well defined. The point $t\in\mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t)=t$, $\rho(t)<t$, $\sigma(t)=t$, $\sigma(t)>t$, respectively. If $\mathbb{T}$ has a right-scattered minimum $m$, define $\mathbb{T}_k:=\mathbb{T}-\{m\}$; otherwise, set $\mathbb{T}_k=\mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum $M$, define $\mathbb{T}^k:=\mathbb{T}-\{M\}$; otherwise, set $\mathbb{T}^k=\mathbb{T}$.

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2000 *Mathematics Subject Classification.* 05C38, 15A15, 05A15,15A18.
*Key words and phrases.* Time scale, Density, Measure.
Submitted Dec. 6, 2012.

**Lemma 2.1.** *The set of all right-scattered points of* $\mathbb{T}$ *is at most countable, that is, there are* $J \subset N$ *and* $\{t_j\}_{j\in J} \subset \mathbb{T}$ *such that*

$$\mathcal{R} := \{t \in \mathbb{T}, \sigma(t) > t\} = \{t_j\}_{j\in J}.$$

**Definition 2.2.** *The function* $\varphi : \mathbb{T} \to \mathbb{R}$ *will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write* $\varphi \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T},\mathbb{R})$.

**Definition 2.3.** *Assume* $\varphi : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^k$. *Then we define* $f^\Delta$ *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$, *there is a neighbourhood* $U$ *of* $t$ *(i.e.,*$U = (t-\delta, t+\delta)\cap\mathbb{T}$*) for some* $\delta > 0$ *such that*

$$|\varphi(\sigma(t)) - \phi(s) - \phi^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

*We call* $f^\Delta$ *the delta (or Hilger) derivative of* $f$ *at* $t$.

$C([a,b]_\mathbb{T},\mathbb{R})$ is the Banach space of all continuous functions from $[a,b]_\mathbb{T}$ into $\mathbb{R}$ where $[a,b]_\mathbb{T} = [a,b]\cap\mathbb{T}$ with the norm

$$\|x\|_\infty := \sup_{t\in[a,b]_\mathbb{T}} |x(t)|.$$

**Remark 2.4.** *It is known that if* $\varphi$ *is continuous, then it is rd−continuous. Moreover, if* $\varphi$ *is delta differentiable at* $t$, *then it is continuous there.*

**Definition 2.5.** *The function* $F$ *is an antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided*

$$F^\Delta(t) = f(t) \quad \text{for each } t \in \mathbb{T}^k.$$

**Definition 2.6.** *The function* $p : \mathbb{T} \to \mathbb{R}$ *is said to be regressive if*

$$1 + \mu(t)p(t) \ne 0 \quad \text{for all } t \in \mathbb{T}^k,$$

*where* $\mu(t) := \sigma(t) - t$ *is called the graininess function. The set of all regressive and rd-continuous functions* $f : \mathbb{T} \to \mathbb{R}$. *We denote by:*

$$\mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T},\mathbb{R}).$$

In this section, bearing in mind the criterion for 1-measurability of sets, we compare Lebesgue $\Delta$-measurable functions with Lebesgue measurable functions. In order to do this, given a function $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$, we need an auxiliary function which extends $\varphi$ to the interval $[a,b]$ defined as

$$\widetilde{\varphi}(t) := \begin{cases} \varphi(t), & \text{if } t \in \mathbb{T}, \\ \varphi(t_j), & \text{if } t \in (t_j, \sigma(t_j)), \text{ for all } j \in J. \end{cases} \tag{1}$$

Let $E \subset \mathbb{T}$, we define

$$J_E := \{j \in J : t_j \in E \cap \mathcal{R}\},$$

and

$$\widetilde{E} := E \cup \bigcup_{j\in J_E} (t_j, \sigma(t_j)). \tag{2}$$

**Proposition 2.7.** ([3]) *Let* $A \subset \mathbb{T}$. *Then* $A$ *is a* $\Delta-$*measurable if and only if,* $A$ *is Lebesgue measurable.*

*In this case the following properties hold for every* $\Delta-$*measurable set* $A$ :

*1. If* $b \notin A$, *then*

$$\mu_\Delta(A) = \mu_L(A) + \sum_{j\in J_A} \mu(t_j).$$

*2.* $\mu_\Delta(A) = \mu_L(A)$ *if and only if* $b \notin A$ *and* $A$ *has no right-scattered point.*

**Definition 2.8.** ([3]) *We say that $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}} = [-\infty, +\infty]$ is $\Delta-$measurable if for every $\alpha \in \mathbb{R}$, the set*

$$\varphi^{-1}\left([-\infty, \alpha)\right) = \{t \in \mathbb{T} : \varphi(t) < \alpha\}$$

*is $\Delta-$measurable.*

**Theorem 2.9.** ([3]) *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable such that $b \notin E$, let $\widetilde{E}$ be the set defined in* (2)*, let $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$ be a $\Delta-$measurable function and $\widetilde{\varphi} : [a, b] \longrightarrow \overline{\mathbb{R}}$ be the extension of $\varphi$ to $[a, b]$. Then, $\varphi$ is Lebesgue $\Delta-$integrable on $E$ if and only if $\widetilde{\varphi}$ is Lebesgue integrable on $\widetilde{E}$ and we have*

$$\int_E \varphi(t)\, \Delta t = \int_{\widetilde{E}} \widetilde{\varphi}(t)\, dt = \int_E \varphi(t)\, dt + \sum_{j \in J_E} \mu(t_j)\, \varphi(t_j). \tag{3}$$

**Definition 2.10.** *Let $p \in [1, +\infty)$ and $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$ be a $\Delta-$measurable function. We said that $\varphi$ belongs to $L^p_\Delta(\mathbb{T}, \mathbb{R})$ provided that*

$$\int_{[a,b)\cap\mathbb{T}} |\varphi(s)|^p \Delta s < +\infty.$$

Let us define a second type of extension for a function $\varphi$ on $[a, b]$. We introduce the following function

$$\overline{\varphi}(t) := \begin{cases} \varphi(t), & \text{if } t \in \mathbb{T}, \\ \dfrac{\varphi(\sigma(t_j)) - \varphi(t_j)}{\mu(t_j)}(t - t_j) + \varphi(t_j), & \text{if } t \in (t_j, \sigma(t_j)), \text{ for all } j \in J. \end{cases} \tag{4}$$

## 3. Main results

We will need the following auxiliary result in order to prove our main density theorem.

**Proposition 3.1.** *Let $p \in [1, +\infty)$, $L^p_\Delta(\mathbb{T}, \mathbb{R})$ is a Banach space equipped with the norm*

$$\|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})} := \left(\int_{[a,b)\cap\mathbb{T}} |\varphi(t)|^p \Delta t\right)^{\frac{1}{p}}.$$

*Proof.* It is clear that $L^p_\Delta(\mathbb{T}, \mathbb{R})$ is a normed space. Now we show that $L^p_\Delta(\mathbb{T}, \mathbb{R})$ is a Banach space. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p_\Delta(\mathbb{T}, \mathbb{R})$, then for all $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$, such that, for all $n, m \geq n_0$, we have

$$\|\varphi_n - \varphi_m\|_{L^p_\Delta(\mathbb{T},\mathbb{R})} < \varepsilon.$$

By (3) we obtain that $(\widetilde{\varphi}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p_\lambda([a, b], \mathbb{R})$. It follows that, since $L^p_\lambda$ is a Banach space, then there exists $\psi \in L^p_\lambda([a, b], \mathbb{R})$ such that

$$\widetilde{\varphi}_n \to \psi \text{ as } n \to \infty.$$

Since

$$\int_{t_j}^{\sigma(t_j)} |\widetilde{\varphi}_n(t) - \widetilde{\varphi}_m(t)|^p\, dt \leq \int_{[a,b[} |\widetilde{\varphi}_n(t) - \widetilde{\varphi}_m(t)|^p\, dt, \text{ for all } j \in J, \tag{5}$$

and

$$\int_{t_j}^{\sigma(t_j)} |\widetilde{\varphi}_n(t) - \widetilde{\varphi}_m(t)|^p\, dt = \mu(t_j)\, |\varphi_n(t_j) - \varphi_m(t_j)|^p, \text{ for all } j \in J, \tag{6}$$

for every $j\in J$, $(\varphi_n(t_j))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, then there exists $x_j\in\mathbb{R}$, such that

$$\varphi_n(t_j)\to x_j \text{ as } n\to\infty.$$

We show that $\psi=x_j$ on $(t_j,\sigma(t_j))$, for all $j\in J$. Also

$$\int_{t_j}^{\sigma(t_j)}|\widetilde{\varphi}_n(t)-\psi(t)|^p\,dt\le\int_{[a,b[}|\widetilde{\varphi}_n(t)-\psi(t)|^p\,dt,\text{ for all } j\in J,\tag{7}$$

and

$$\int_{t_j}^{\sigma(t_j)}|\widetilde{\varphi}_n(t)-x_j|^p\,dt=\mu(t_j)\,|\varphi_n(t_j)-x_j|^p,\text{ for all } j\in J.\tag{8}$$

Thus

$$\widetilde{\varphi}_n\to\psi \text{ as } n\to\infty \text{ and } x_j\in L^p_\lambda((t_j,\sigma(t_j)),\mathbb{R}),$$

this implies that

$$\psi(t)=x_j,\quad\text{for each } t\in(t_j,\sigma(t_j)).$$

Let

$$\varphi(t):=\begin{cases}\psi(t), & \text{if } t\in\mathbb{T}\setminus\mathcal{R},\\ x_j, & \text{if } t=t_j, \text{ for all } j\in J.\end{cases}\tag{9}$$

Hence

$$\widetilde{\varphi}(t)=\begin{cases}\psi(t), & \text{if } t\in\mathbb{T}\setminus\mathcal{R},\\ x_j, & \text{if } t\in[t_j,\sigma(t_j)), \text{ for all } j\in J.\end{cases}\tag{10}$$

From (10), $\widetilde{\varphi}=\psi$ $\lambda$ a.e on $[a,b]$, then $(\varphi_n)_{n\in\mathbb{N}}$ converges to $\varphi$ in $L^p_\Delta(\mathbb{T},\ \mathbb{R})$. □

Set

$$\mathcal{U}_\varepsilon(t):=(t-\varepsilon,t+\varepsilon)\cap\mathbb{T},\text{ for all }\varepsilon>0,\ t\in\mathbb{T}.$$

The following remarks will be useful in the sequel.

- All the right-dense or left-dense points are accumulation points.
- For all $t$ right-dense or left-dense point, we have

$$\lim_{s\longrightarrow t}\sigma(s)=t.\tag{11}$$

Let

$$I:=\{j\in J:\rho(t_j)=t_j\}.$$

For all $i\in I$,

$$\alpha_i:=\inf\{\lambda_i:[\lambda_i,t_i]\subseteq\mathbb{T}\}.$$

Let

$$\begin{aligned}\mathcal{F}&:=\mathbb{T}\setminus\bigcup_{i\in I}[\alpha_i,t_i]\cup\mathcal{R},\\ I_1&:=\{i\in I:\alpha_i\neq t_i\},\\ I_2&:=\{i\in I:\alpha_i=t_i\}.\end{aligned}$$

**Theorem 3.2.** *Let $p\in[1,\infty)$, then $C_{rd}(\mathbb{T},\mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T},\ \mathbb{R})$.*

*Proof.* Let $\varphi\in L^p_\Delta(\mathbb{T},\mathbb{R})$, then $\widetilde{\varphi}\in L^p_\lambda([a,b],\mathbb{R})$. Since $C^1([a,b],\mathbb{R})$ is dense in $L^p_\lambda([a,b],\mathbb{R})$, then there exists a sequence $(\psi_n)_{n\in\mathbb{N}}\in C^1([a,b],\mathbb{R})$ that converges to $\widetilde{\varphi}$ in $L^p_\lambda([a,b],\mathbb{R})$, i.e.

$$\lim_{n\to\infty}\int_{[a,b)}|\psi_n(t)-\widetilde{\varphi}(t)|^p dt=0.\tag{12}$$

We define $\left(u_n^i\right)_{n\in\mathbb{N}}$ by:

$$u_n^i := t_i - \frac{\mu\left(t_i\right)}{\left(b-a\right)2^n}\left(t_i-\alpha_i\right), \text{ for all } i\in I_1.$$

Then, for all $i \in I_1$, we have

$$u_n^i \in (\alpha_i, t_i), \text{ for all } n\in\mathbb{N}.$$

Set

$$A_n := \bigcup_{i\in I_1}[u_n^i, t_i],\ n\in\mathbb{N},$$

then

$$\lambda\left(A_n\right) = \sum_{i\in I_1}\lambda\left(\left[u_n^i, t_i\right]\right) \leq \frac{b-a}{2^n}. \tag{13}$$

Firstly, we show that $(\chi_{A_n}\psi_n)_{n\in\mathbb{N}}$ converges to 0 in $L_\lambda^p\left(\left[a,b\right],\mathbb{R}\right)$. It is enough to show that $(\chi_{A_n}\psi_n)_{n\in\mathbb{N}}$ converges in measure to 0 and $\{\chi_{A_n}\psi_n, n\in\mathbb{N}\}$ is equi-integrable in $L_\lambda^p\left(\left[a,b\right],\mathbb{R}\right)$. For all $\varepsilon>0$ we have

$$\begin{aligned}\lambda\left(\left\{t\in\left[a,b\right],\left|\left(\chi_{A_n}\psi_n\right)(t)\right|>\varepsilon\right\}\right) &= \lambda\left(\left\{t\in A_n, \left|\psi_n\left(t\right)\right|>\varepsilon\right\}\right)\\ &\leq \lambda\left(A_n\right) \leq \frac{b-a}{2^n}.\end{aligned}$$

Hence $(\psi_n)_{n\in\mathbb{N}}$ converges in $L_\lambda^p([a,b],\mathbb{R})$, this implies that $\{\psi_n, n\in\mathbb{N}\}$ is equi-integrable in $L_\lambda^p\left(\left[a,b\right],\mathbb{R}\right)$.
Then there exists $\delta$, such that for all $A$ in the Borel $\sigma$-field $\mathcal{B}\left(\left[a,b\right]\right)$, with $\lambda\left(A\right)\leq\delta$, we obtain

$$\sup_{n\in\mathbb{N}}\int_A\left|\psi_n\left(t\right)\right|^p dt \leq \varepsilon \Rightarrow \sup_{n\in\mathbb{N}}\int_A\left|\left(\chi_{A_n}\psi_n\right)(t)\right|^p dt\leq\varepsilon.$$

This implies

$$\lim_{n\to+\infty}\int_{A_n}\left|\psi_n\left(t\right)\right|^p dt = 0.$$

Since the intervals $\left\{\left[u_n^i, t_i\right]\right\}_{i\in I_1}$ are disjointed, then

$$\lim_{n\to+\infty}\sum_{i\in I_1}\int_{[u_n^i,t_i]}\left|\psi_n\left(t\right)\right|^p dt = 0.$$

From Rolle's Theorem, there exists $\theta_n^i\in\left(u_n^i, t_i\right)$, for all $i\in I_1$, $n\in\mathbb{N}$, such that

$$\begin{aligned}\int_{[u_n^i,t_i]}\left|\psi_n\left(t\right)\right|^p dt &= \left(t_i-u_n^i\right)\left|\psi_n\left(\theta_n^i\right)\right|^p\\ &\geq \left(t_i-\theta_n^i\right)\left|\psi_n\left(\theta_n^i\right)\right|^p.\end{aligned}$$

Therefore

$$\lim_{n\to+\infty}\sum_{i\in I_1}\left(t_i-\theta_n^i\right)\left|\psi_n\left(\theta_n^i\right)\right|^p = 0. \tag{14}$$

Using the fact that $\left\{\left(t_j,\sigma\left(t_j\right)\right)\right\}_{j\in J}$ are disjointed intervals, then

$$\begin{aligned}\sum_{j\in J}\int_{(t_j,\sigma(t_j))}\left|\psi_n\left(t\right)-\varphi\left(t_j\right)\right|^p dt &= \int_{\bigcup_{j\in J}(t_j,\sigma(t_j))}\left|\psi_n\left(t\right)-\widetilde{\varphi}\left(t\right)\right|^p dt\\ &\leq \int_{[a,b[}\left|\psi_n\left(t\right)-\widetilde{\varphi}\left(t\right)\right|^p dt.\end{aligned}$$

By Rolle's Theorem, there exists $\left(t_j^n\right)_{n\in\mathbb{N}} \in (t_j, \sigma(t_j))$, for all $j \in J$, such that

$$\int_{(t_j,\sigma(t_j))} |\psi_n(t) - \varphi(t_j)|^p \, dt = \mu(t_j) \left|\psi_n\left(t_j^n\right) - \varphi(t_j)\right|^p .$$

Hence

$$\lim_{n\to\infty} \sum_{j\in J} \mu(t_j) \left|\psi_n\left(t_j^n\right) - \varphi(t_j)\right|^p = 0. \tag{15}$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\varphi_n(t) := \begin{cases} \psi_n\left(t_j^n\right), & \text{if } t = t_j, \text{ for all } j \in J, \\ \dfrac{\psi_n(t_i^n) - \psi_n\left(\theta_n^i\right)}{t_i - \theta_n^i}(t - t_i) + \psi_n(t_i^n), & \text{if } \theta_n^i \le t < t_i, \text{ for all } i \in I_1, \\ \psi_n(t), & \text{if } t \in \mathcal{F} \bigcup_{i\in I_1} \left[\alpha_i, \theta_n^i\right). \end{cases} \tag{16}$$

First we show that $(\varphi_n)_{n\in\mathbb{N}}$ is rd-continuous. It is clear that $(\varphi_n)_{n\in\mathbb{N}}$ is continuous on $\bigcup_{i\in I_1} (\alpha_i, t_i]$. Set

$$k_n := \sup_{t\in[a,b]} \left|\psi_n'(t)\right|, \quad \text{for all} \quad n \in \mathbb{N}.$$

If $t \in \mathcal{F} \cup \{\alpha_i, i \in I\}$, by (11), we have for all $\varepsilon > 0$, there exists $\alpha > 0$, such that

$$|\sigma(s) - t| < \frac{\varepsilon}{3k_n}, \text{ for all } s \in \mathcal{U}_\alpha(t).$$

Set

$$\delta_n := \frac{1}{3} \inf\left(\alpha, \frac{\varepsilon}{k_n}\right), \; n \in \mathbb{N}.$$

For all $s \in \mathcal{U}_{\delta_n}(t)$, we have

- For $s \in \mathcal{F} \bigcup_{i\in I_1} \left[\alpha_i, \theta_n^i\right]$, that

$$|\psi_n(t) - \varphi_n(s)| = |\psi_n(t) - \psi_n(s)| \le k_n |t - s| < \varepsilon.$$

- For $s \in \left[\theta_n^i, t_i\right]$, with $i \in I_1$, hence

$$|t - t_i^n| \le \tfrac{\varepsilon}{3k_n}, |t_i - t_i^n| \le \tfrac{\varepsilon}{3k_n}, \text{ and } |s - t_i| \le \tfrac{\varepsilon}{3k_n}.$$

  Therefore

$$\begin{aligned} |\psi_n(t) - \varphi_n(s)| &\le |\psi_n(t) - \psi_n(t_i^n)| + \frac{\left|\psi_n(t_i^n) - \psi_n\left(\theta_n^i\right)\right|}{t_i - \theta_n^i} |s - t_i| \\ &\le k_n |t - t_i^n| + k_n \frac{\left|t_i^n - \theta_n^i\right|}{t_i - \theta_n^i} |s - t_i| \\ &\le k_n |t - t_i^n| + k_n |s - t_i| + k_n |t_i^n - t_i| \le \varepsilon. \end{aligned}$$

- For $s = t_j$, with $j \in J$, then $\left|t - t_j^n\right| \le \frac{\varepsilon}{3k_n}$, hence

$$|\psi_n(t) - \varphi_n(s)| = \left|\psi_n(t) - \psi_n\left(t_j^n\right)\right| \le k_n \left|t - t_j^n\right| < \varepsilon.$$

This implies that for all $t \in \mathcal{F} \cup \{\alpha_i, i \in I\}$.

$$\lim_{s\longrightarrow t} \varphi_n(s) = \psi_n(t).$$

Now we show that $(\varphi_n)_{n\in\mathbb{N}}$ converges to $\varphi$ in $L^p_\Delta(\mathbb{T},\mathbb{R})$. It is enough to show that $(\varphi_n)_{n\in\mathbb{N}}$ converges to $\varphi$ in $L^p_\lambda(\mathbb{T},\mathbb{R})$. We have

$$\begin{aligned}\int_{\mathbb{T}}|\varphi_n(t)-\varphi(t)|^p\,dt &= \int_{\bigcup\limits_{i\in I_1}[\theta_n^i,t_i]}|\varphi_n(t)-\varphi(t)|^p\,dt+\int_{\mathcal{F}\bigcup\limits_{i\in I_1}[\alpha_i,\theta_n^i[}|\varphi_n(t)-\varphi(t)|^p\,dt\\ &\le \int_{\bigcup\limits_{i\in I_1}[\theta_n^i,t_i]}|\varphi_n(t)-\varphi(t)|^p\,dt+\int_{[a,b[}|\psi_n(t)-\widetilde{\varphi}(t)|^p\,dt\\ &\le 2^{p-1}\int_{\bigcup\limits_{i\in I_1}[\theta_n^i,t_i]}|\varphi_n(t)|^p\,dt+2^{p-1}\int_{\bigcup\limits_{i\in I_1}[\theta_n^i,t_i]}|\varphi(t)|^p\,dt\\ &\quad+\int_{[a,b[}|\psi_n(t)-\widetilde{\varphi}(t)|^p\,dt,\end{aligned}$$

where the following inequality has been used

$$(x+y)^p\le 2^{p-1}(x^p+y^p),\ \text{true}\ \forall x;y\in[0;+\infty[;\ \forall p\in[1;+\infty[.$$

For all $n\in\mathbb{N}$, we defined $B_n:=\bigcup\limits_{i\in I_1}\left[\theta_n^i,t_i\right]$, we show that $(\chi_{B_n}\widetilde{\varphi})_{n\in\mathbb{N}}$ converges to 0 in $L^p_\lambda([a,b],\mathbb{R})$. Since $B_n\subseteq A_n$, by (13) we obtain that $\lambda(B_n)\le\dfrac{b-a}{2^n}$. Hence for all $\varepsilon>0$, we have

$$\lambda(\{t\in[a,b],|\chi_{B_n}\widetilde{\varphi}(t)|>\varepsilon\})\le\lambda(B_n)\le\frac{b-a}{2^n}.$$

Since

$$|\chi_{B_n}\widetilde{\varphi}|\le|\widetilde{\varphi}|,\ \text{for all}\ n\in\mathbb{N}.$$

Then $\{\chi_{B_n}\widetilde{\varphi},n\in\mathbb{N}\}$ is equi-integrable in $L^p_\lambda([a,b],\mathbb{R})$, thus

$$\lim_{n\longrightarrow+\infty}\int_{B_n}|\varphi(t)|^p\,dt=0. \tag{17}$$

Then

$$\int_{\bigcup\limits_{i\in I_1}[\theta_n^i,t_i]}|\varphi_n(t)|^p\,dt=\sum_{i\in I_1}\int_{[\theta_n^i,t_i]}|\varphi_n(t)|^p\,dt.$$

Also we obtain

$$\begin{aligned}\int_{\theta_n^i}^{t_i}|\varphi_n(t)|^p\,dt &\le 2^{p-1}\int_{\theta_n^i}^{t_i}\left(\frac{\left|\psi_n(t_i^n)-\psi_n\left(\theta_n^i\right)\right|}{t_i-\theta_n^i}\right)^p(t_i-t)^p+|\psi_n(t_i^n)|^p\,dt\\ &\le 2^{p-1}\left|\psi_n(t_i^n)-\psi_n\left(\theta_n^i\right)\right|^p\left(t_i-\theta_n^i\right)+2^{p-1}|\psi_n(t_i^n)|^p\left(t_i-\theta_n^i\right)\\ &\le 2^{2p}\left|\psi_n\left(\theta_n^i\right)\right|^p\left(t_i-\theta_n^i\right)+2^{2p}|\psi_n(t_i^n)|^p\left(t_i-\theta_n^i\right).\end{aligned}$$

It is enough to prove that the second term converges to 0.

$$\begin{aligned}\sum_{i\in I_1}|\psi_n(t_i^n)|^p\left(t_i-\theta_n^i\right) &\le \sum_{i\in I_1}|\psi_n(t_i^n)|^p\left(t_i-u_n^i\right)\\ &\le \frac{1}{2^n}\sum_{i\in I_1}\mu(t_i)\,|\psi_n(t_i^n)|^p,\end{aligned}$$

thus

$$\lim_{n\longrightarrow+\infty}\int_{\bigcup\limits_{i\in\mathbb{I}_1}[\theta_n^i,t_i]}|\varphi_n(t)|^p\,dt=0. \tag{18}$$

Then, by (12), (17), and (18) we have

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{T}}|\varphi_n(t)-\varphi(t)|^p\,dt=0. \tag{19}$$

Thus, by (3), (15), and (19) we get that

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{T}\cap[a,b[}|\varphi_n(t)-\varphi(t)|^p\,\Delta t=0.$$

□

The following results are consequence of Theorem 3.2.

**Corollary 3.3.** *Let $p\in(1,\infty)$, then $L^p_\Delta(\mathbb{T},\mathbb{R})$ is dense in $L^1_\Delta(\mathbb{T},\ \mathbb{R})$.*

*Proof.* Let $\varphi\in L^p_\Delta(\mathbb{T},\mathbb{R})$, then by (3) we have

$$\|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})}=\|\widetilde{\varphi}\|_{L^p_\lambda([a,b],\mathbb{R})}. \tag{20}$$

Let $q\in(1,+\infty)$, such that $\frac{1}{p}+\frac{1}{q}=1$, according to Hölder's inequality, we obtain

$$\|\widetilde{\varphi}\|_{L^1_\lambda([a,b],\mathbb{R})}\le(b-a)^{\frac{1}{q}}\|\widetilde{\varphi}\|_{L^p_\lambda([a,b],\mathbb{R})}. \tag{21}$$

By (20) and (21), we have

$$\|\varphi\|_{L^1_\Delta(\mathbb{T},\mathbb{R})}\le(b-a)^{\frac{1}{q}}\|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})}.$$

Then

$$L^p_\Delta(\mathbb{T},\mathbb{R})\hookrightarrow L^1_\Delta(\mathbb{T},\mathbb{R}).$$

Since

$$C_{rd}(\mathbb{T},\mathbb{R})\subset L^p_\Delta(\mathbb{T},\mathbb{R})\subset L^1_\Delta(\mathbb{T},\mathbb{R}),$$

then $L^p_\Delta(\mathbb{T},\mathbb{R})$ is dense in $L^1_\Delta(\mathbb{T},\mathbb{R})$. □

**Proposition 3.4.** *$C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $C(\mathbb{T},\mathbb{R})$.*

*Proof.* Let $\varphi\in C(\mathbb{T},\mathbb{R})$, then $\overline{\varphi}\in C([a,b],\mathbb{R})$. Using the density of $C^1([a,b],\mathbb{R})$ in $C([a,b],\mathbb{R})$, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}\in C^1([a,b],\mathbb{R})$ converging to $\overline{\varphi}$ in $C([a,b],\mathbb{R})$.
Let $\varphi_n:\mathbb{T}\longrightarrow\mathbb{R}$, $t\longrightarrow\varphi_n(t)=\psi_n(t)$, for all $n\in\mathbb{N}$, thus $\varphi_n$ is $\Delta-$differentiable on $\mathbb{T}^k$, and $\varphi_n^\Delta$ is given by:

$$\varphi_n^\Delta(t)=\begin{cases}\psi_n'(t) & \text{if } t\in\mathbb{T}^k\setminus\mathcal{R}\\ \dfrac{\psi_n(\sigma(t_j))-\psi_n(t_j)}{\mu(t_j)} & \text{if } t=t_j\in\mathbb{T}^k,\text{ for all } j\in J.\end{cases}$$

Now, we show that $\varphi_n^\Delta$ is rd-continuous, for all $n\in\mathbb{N}$. Let $t\in\mathbb{T}^k$ a left-dense or a right-dense point and prove that

$$\lim_{s\to t}\varphi_n^\Delta(s)=\psi_n'(t),\text{ for all } n\in\mathbb{N}.$$

Since $\psi_n\in C^1([a,b],\mathbb{R})$, then for all $\varepsilon>0$, there exists $\delta_1>0$, such that

$$|\psi_n'(t)-\psi_n'(s)|<\varepsilon,\ \text{ for all } s\in(t-\delta_1,t+\delta_1). \tag{22}$$

We define $\Psi_n$ on $(t-\delta_1, t+\delta_1)$ by:

$$\Psi_n(s) := \psi_n(s) - \psi_n^{'}(t)(s-t).$$

By (22) we have

$$\left|\Psi_n^{'}(s)\right| < \varepsilon, \text{ for all } s \in (t-\delta_1, t+\delta_1).$$

Then $\Psi_n$ is an $\varepsilon$-lipschitz function on $(t-\delta_1, t+\delta_1)$, so we get

$$\left|\psi_n^{'}(t) - \frac{\psi_n(\tau) - \psi_n(s)}{\tau - s}\right| < \varepsilon, \text{ for all } \tau, s \in (t-\delta_1, t+\delta_1), \text{ and } \tau \neq s.$$

By (11), there exists $\delta_2 > 0$, such that

$$|\sigma(s) - t| < \delta_1, \text{ for all } s \in \mathcal{U}_{\delta_2}(t).$$

Put $\delta := \inf(\delta_1, \delta_2)$, for all $s \in \mathcal{U}_\delta(t)$. We consider the following two cases

- If $s$ is right-dense, then

$$\left|\psi_n^{'}(t) - \varphi_n^{\Delta}(s)\right| = \left|\psi_n^{'}(t) - \psi_n^{'}(s)\right| < \varepsilon.$$

- If $s$ is right-scattered, one has $\sigma(s), s \in \mathcal{U}_{\delta_1}(t)$, then

$$\left|\psi_n'(t) - \varphi_n^{\Delta}(s)\right| = \left|\psi_n^{'}(t) - \frac{\psi_n(\sigma(s)) - \psi_n(s)}{\sigma(s) - s}\right| < \varepsilon.$$

Finally we obtain that $\varphi_n^{\Delta}$ is a continuous function at right-dense points in $\mathbb{T}$, and its left-sided limits exist at left dense points in $\mathbb{T}$.
Using the inequality

$$\|\varphi_n - \varphi\|_{C(\mathbb{T},\mathbb{R})} \leq \|\psi_n - \overline{\varphi}\|_\infty,$$

we prove that $(\varphi_n)_{n\in\mathbb{N}}$ converges to $\varphi$ in $C(\mathbb{T}, \mathbb{R})$.

□

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