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ON A NONLOCAL PROBLEM FOR PARTIAL STOCHASTIC FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. This paper is concerned with the existence of mild solutions for a class of stochastic functional integro-differential equations with nonlocal conditions in the α-norm. The linear part of the equations is assumed to generate an analytic resolvent operator, and the nonlinear part satisfies some Lipschitz conditions with respect to the α-norm. By using Schaefer's fixed point theorem, we establish a existence result, which generalizes the recent conclusions on this issue. In the end, an example is given to illustrate the theory.

1. Introduction

In this paper, we shall consider the existence of mild solutions for the following stochastic functional integro-differential equations with nonlocal conditions

$$dx(t) = A\left[x(t) + \int_0^t f(t-s)x(s)ds\right]dt$$
$$+F\left(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))ds\right)w(t), \quad t \in J, \tag{1}$$
$$x(0) + g(x) = x_0, \tag{2}$$

where $J = [0, b]$, the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, A is the infinitesimal generator of a compact, analytic resolvent operator $R(t), t > 0$ on H, and $f(t), t \in J$ is a bounded linear operator. Let K be another separable Hilbert space with inner product $(\cdot, \cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. We are also employing the same notation $\|\cdot\|$ for the norm $L(K; H)$, where $L(K; H)$ denotes the space of all bounded linear operators from K into H. The functions $F, h, g, \sigma_i (i = 1, \ldots, n+1)$, are continuous functions and will be specified later.

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Stochastic differential equations are playing an increasingly important role in applications to finance, numerical analysis, physics, and biology. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. Recently, much attention has been paid to existence, uniqueness and stability for stochastic differential and integro-differential equations in the infinite dimensions case, see the monographs [10],[16],[23], the papers [4],[22],[26],[27],[31] and the references therein. Some classes of stochastic evolution equations have been considered by Taniguchi et al. [28], El-Borai et al. [13], Bao and Zhou [5], Govindan [15], Ren and Chen [25], Chang et al. [9] and the references therein.

The study of abstract nonlocal semilinear initial value problems was initiated by Byszewski [6],[8]. Subsequently, many authors are devoted to the study of nonlocal Cauchy problems because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems, we refer the reader to [1],[7],[11],[14],[20],[21],[24],[30] and the references contained therein. Very recently, several papers have appeared on the nonlocal problem of existence of solutions for semilinear stochastic differential equations and integro-differential equations in Hilbert spaces. For example, Balasubramaniam and Ntouyas [2] investigated global existence of solutions for a semilinear stochastic delay evolution equation with nonlocal conditions. Keck and McKibben [19] showed the global existence and convergence properties of mild solutions to a class of abstract semilinear functional stochastic integro-differential equations. In paper [3], the authors discussed the existence of mild and strong solutions of semilinear neutral functional differential evolution equations with nonlocal conditions by using fractional power of operators and Sadovskii fixed point theorem. The purpose of this paper is that we continue the study of these authors. We get the existence results for mild solutions of problem (1)-(2) with α-norm as in [14] when the nonlocal item g is only depends upon the continuous properties. Our results are based on the Banach contraction principle and Schaefer's fixed point theorem combined with theories of analytic resolvent operators. The nonlocal Cauchy problems for nonlinear integro-differential equations with resolvent operators considered here serve as an abstract formulation of partial integro-differential equations which arise in various applications such as viscoelasticity, heat equations and many other physical phenomena [17],[18],[21].

This paper will be organized as follows. In Section 2, we will briefly recall some basic definitions and preliminary facts to be used in the following sections. Section 3 is devoted to the existence of mild solutions of problem (1)-(2). Finally, a concrete example is presented in Section 4 to show the application of our main results.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P; \mathbb{F})(\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to H$ and the collection of random variables $S = \{x(t, w) : \Omega \to H | t \in J\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t, w)$ and $x(t) : J \to H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \le s \le t\}$ is the σ-algebra generated by w and $\mathcal{F}_b = \mathcal{F}$.

Let $L(K;H)$ denote the space of all bounded linear operators from K into H. For $h_1, h_2 \in L(K;H)$, we define $(h_1, h_2) = \mathrm{Tr}(h_1 Q h_2^*)$ where h_2^* is the adjoint of the operator h_2 and Q is the nuclear operator associated with the Wiener process, where $Q \in L_n^n(K)$, the space of positive nuclear operator in K. For $\psi \in L(K;H)$ we define

$$\| \psi \|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \psi e_n \|^2 .$$

If $\| \psi \|_Q < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K;H)$ denote the space of all Q-Hilbert-Schmidt operators ψ. The completion $L_Q(K;H)$ of $L(K;H)$ with respect to the topology induced by the norm $\| \cdot \|_Q$ where $\| \psi \|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology. For more details, we refer the reader to Da Prato and Zabczyk [10].

Now, we give knowledge on the resolvent operator which appeared in Grimmer and Pritchard [18].

Definition 2.1. A family of bounded linear operator $R(t) \in L(H)$ for $t \in J$ is called a resolvent operator for

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t f(t-s)x(s)ds\right] \tag{3}$$

if

(a) $R(0) = I$, the identity operator on H.

(b) For each $x \in H, R(t)x$ is continuous for $t \in J$.

(c) $R(t) \in L(Y), t \in J$, where Y is the Banach space formed from $D(S)$ endowed with the graph norm. For $y \in Y, R(\cdot)y \in C^1(J,H) \cap C(J,Y)$ and

$$\frac{d}{dt}R(t)y = A\left[R(t)y + \int_0^t f(t-s)R(s)yds\right] = R(t)Ay + \int_0^t R(t-s)Af(s)yds,\ t \in J.$$

Let $0 \in \rho(A)$, then it is possible to define the fractional power A^α, for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in H and the expression $\| x \|_\alpha = \| A^\alpha x \|, \quad x \in D(A^\alpha)$, defines a norm on $D(A^\alpha)$ which will be denoted by H_α.

Lemma 2.1([18]). Under the above conditions, we have:

(1) $A^\alpha : H_\alpha \to H$, then H_α is a Banach space for $0 \le \alpha \le 1$.

(2) If the resolvent operator of A is compact then $H_\alpha \to H_\beta$ is continuous and compact for $0 < \beta \le \alpha$.

(3) For every $0 < \alpha \le 1$, there exists a constant $M_\alpha > 0$ such that

$$\| A^\alpha R(t) \| \le \frac{M_\alpha}{t^\alpha}, \quad 0 < t \le b.$$

Let $L_2(\Omega, \mathcal{F}_{t,H})$ denote the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in H. Let $L_2^{\mathcal{F}}(J,H)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in H. Let $\mathcal{C}(J)$ be the Banach space $C(J, H_\alpha)$ and $\mathcal{C}$ denote the Banach space $\mathcal{C}(J, L_2(\Omega, \mathcal{F}, H))$ the family of all $\mathcal{F}_t$-measurable, $\mathcal{C}(J)$-valued random variables x with the norm

$$\| x \|_{\mathcal{C}} = \sup_{0 \le t \le b} (E \| x(t) \|_\alpha^2)^{\frac{1}{2}}.$$

Let $L_2^0(\Omega, \mathcal{C})$denote the family of all $\mathcal{F}_0$-measurable, $\mathcal{C}$-valued random variables $x(0)$.
Definition 2.2. A stochastic process $x \in \mathcal{C}$ is called a mild solution of (1)-(2) if
(i) $x_0, g(x) \in L_2^0(\Omega, \mathcal{C})$;
(ii) $x_0 + g(x) = x_0$;
(iii) $x(t) \in H$ has càdlàg paths on $t \in J$ a.s., and it satisfies the following integral equation:

$$x(t) = R(t)[x_0 - g(x)] + \int_0^t R(t-s) \times F\left(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)), \int_0^s h(s, \tau, x(\sigma_{n+1}(\tau))) d\tau\right) dw(s), \ t \in J. \quad (4)$$

Lemma 2.2 (Schaefer's fixed point theorem [12]). Let X be a normed linear space. Let $Q : X \to X$ be a completely continuous operator, so that , it is continuous and the image of any bounded set is contained in a compact set and let

$$\zeta(Q) = \{x \in X : x = \lambda Q x \text{ for some } 0 < \lambda < 1\}$$

that $\zeta(Q)$ is unbounded or Q has a fixed point.
For some $\alpha \in (0, 1)$, we assume the following hypotheses:
(H1) A is the infinitesimal generator of a compact, analytic resolvent operator $R(t), t \ge 0$ in the Hilbert space H and there exists constant M such that

$$\| R(t) \|^2 \le M, \ t \in J.$$

(H2) The function $F : J \times H_\alpha^{n+1} \to L(K; H)$ is continuous and there exist constants $l_F^{(1)} > 0, l_1 > 0$, such that for all $x_i, y_i \in H_\alpha, i = 1, \ldots, n+1$, we have

$$\| F(t, x_1, x_2, \ldots, x_{n+1}) - F(t, y_1, y_2, \ldots, y_{n+1}) \|^2 \le l_F^{(1)} \left[\sum_{i=1}^{n+1} \| x_i - y_i \|_\alpha^2\right],$$

and

$$l_1 = \max_{t \in J} \| F(t, 0, \ldots, 0) \|^2 .$$

(H3) The function $F : J \times H_\alpha^{n+1} \to L(K; H_\alpha)$ is continuous and there exist constants $l_F^{(2)} > 0, l_2 > 0$, such that for all $x_i, y_i \in H_\alpha, i = 1, \ldots, n+1$, we have

$$\| F(t, x_1, x_2, \ldots, x_{n+1}) - F(t, y_1, y_2, \ldots, y_{n+1}) \|_\alpha^2 \le l_F^{(2)} \left[\sum_{i=1}^{n+1} \| x_i - y_i \|_\alpha^2\right],$$

and

$$l_2 = \max_{t \in J} \| F(t, 0, \ldots, 0) \|_\alpha^2 .$$

(H4) There exists a positive number β with $\alpha \le \beta \le 1$ such that $F : J \times H_\alpha^{n+1} \to$

$L(K;H_\beta)$ is continuous and there exist constants $l_F^{(3)} > 0, l_3 > 0$, such that for all $x_i, y_i \in H_\alpha, i = 1, \ldots, n+1$, we have

$$\| F(t, x_1, x_2, \ldots, x_{n+1}) - F(t, y_1, y_2, \ldots, y_{n+1}) \|_\beta^2 \leq l_F^{(3)} \left[\sum_{i=1}^{n+1} \| x_i - y_i \|_\alpha^2 \right],$$

and

$$l_3 = \max_{t \in J} \| F(t, 0, \ldots, 0) \|_\beta^2 .$$

(H5) The function $h : J \times J \times H_\alpha \to H_\alpha$ is continuous and there exist constants $l_h > 0, l_h^{(1)} > 0$, such that for all $x, y \in H_\alpha$,

$$\| h(t,s,x) - h(t,s,y) \|_\alpha^2 \leq l_h \| x - y \|_\alpha^2,$$

and

$$l_h^{(1)} = \max_{0 \leq s \leq t \leq b} \| h(t, s, 0) \|_\alpha^2 .$$

(H6) $\sigma_i : J \to J, i = 1, \ldots, n+1$, are continuous functions such that $\sigma_i(t) \leq t, i = 1, \ldots, n+1$.
(H7) The function $g(\cdot) : \mathcal{C} \to H_\alpha$ is continuous and there exists a $\delta \in (0, b)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in \mathcal{C}$ with $\phi = \psi$ on $[\delta, b]$.
(H8) There is a constant $c > 0$ such that

$$0 \leq \limsup_{\|\phi\|_{\mathcal{C}} \to \infty} \frac{E \| g(\phi) \|_\alpha^2}{\| \phi \|_{\mathcal{C}}^2} \leq c, \qquad \phi \in \mathcal{C}.$$

3. Main results

Theorem 3.1. Let $x(0) \in L_2^0(\Omega, \mathcal{C})$. If the assumptions (H1), (H2) and (H5)-(H8) hold and

$$16Mce^{\frac{16 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n+4l_h b^2) b^{1-2\alpha}}{1-2\alpha}} < 1, \tag{5}$$

then the nonlocal Cauchy problem (1)-(2) has a mild solution on J.
Proof. Let $l_0 > 0$ be a constant chosen such that

$$\gamma := \sup_{t \in J} \left\{ M_\alpha^2 l_F^{(1)} \mathrm{Tr}(Q)(n + l_h b^2) \int_0^t e^{-l_0(t-s)} (t-s)^{-2\alpha} ds \right\} < 1,$$

and we introduce in the space $\mathcal{C}$ the equivalent norm defined as

$$\| \phi \|_V := \sup_{t \in J} (e^{-l_0 t} E \| \phi(t) \|_\alpha^2)^{\frac{1}{2}}.$$

Then, it is easy to see that $V := (\mathcal{C}, \| \cdot \|_V)$ is a Banach space. Fix $v \in \mathcal{C}$ and for $t \in J, \phi \in V$, we now define an operator

$$(P_v \phi)(t) = R(t)(t)[x_0 - g(v)] + \int_0^t R(t-s)$$
$$\times F\left(s, \phi(\sigma_1(s)), ..., \phi(\sigma_n(s)), \int_0^s h(s, \tau, \phi(\sigma_{n+1}(\tau))) d\tau \right) dw(s). \tag{6}$$

Since $R(\cdot)(x_0 - g(v)) \in \mathcal{C}$, it follows, from (H2),(H5) and (H6), that $(P_v\phi)(t) \in V$ for all $\phi \in V$. Let $\phi, \psi \in V$, we have

$$\begin{aligned}
&e^{-l_0 t}E \parallel (P_v\phi)(t) - (P_v\psi)(t) \parallel_\alpha^2 \\
&\leq e^{-l_0 t}E\bigg\| \int_0^t R(t-s)\bigg[F\bigg(s, \phi(\sigma_1(s)), ..., \phi(\sigma_n(s)), \int_0^s h(s,\tau,\phi(\sigma_{n+1}(\tau)))d\tau\bigg) \\
&\quad - F\bigg(s, \psi(\sigma_1(s)), ..., \psi(\sigma_n(s)), \int_0^s h(s,\tau,\psi(\sigma_{n+1}(\tau)))d\tau\bigg)\bigg]dw(s)\bigg\|_\alpha^2 \\
&\leq M_\alpha^2 l_F^{(1)} \mathrm{Tr}(Q) \int_0^t e^{-l_0 t}(t-s)^{-2\alpha} E\bigg[\parallel \phi(\sigma_1(s)) - \psi(\sigma_1(s)) \parallel_\alpha^2 \\
&\quad + \cdots + \parallel \phi(\sigma_n(s)) - \psi(\sigma_n(s)) \parallel_\alpha^2 \\
&\quad + b\int_0^s \parallel h(s,\tau,\phi(\sigma_{n+1}(\tau))) - h(s,\tau,\psi(\sigma_{n+1}(\tau))) \parallel_\alpha^2 d\tau\bigg]ds \\
&\leq M_\alpha^2 l_F^{(1)} \mathrm{Tr}(Q) \int_0^t e^{-l_0 t}(t-s)^{-2\alpha}\bigg[e^{l_0\sigma_1(s)} \sup_{s\in J} e^{-l_0 s}E \parallel \phi(s) - \psi(s) \parallel_\alpha^2 \\
&\quad + \cdots + e^{l_0\sigma_n(s)} \sup_{s\in J} e^{-l_0 s}E \parallel \phi(s) - \psi(s) \parallel_\alpha^2 \\
&\quad + l_h bE\int_0^s \parallel \phi(\sigma_{n+1}(\tau)) - \psi(\sigma_{n+1}(\tau)) \parallel_\alpha^2 d\tau\bigg]ds \\
&\leq M_\alpha^2 l_F^{(1)} \mathrm{Tr}(Q) \int_0^t e^{-l_0 t}(t-s)^{-2\alpha}\bigg[ne^{l_0 s} \sup_{s\in J} e^{-l_0 s}E \parallel \phi(s) - \psi(s) \parallel_\alpha^2 \\
&\quad + l_h b^2 e^{l_0\sigma_{n+1}(s)} \sup_{s\in J} e^{-l_0 s}E \parallel \phi(s) - \psi(s) \parallel_\alpha^2\bigg]ds \\
&\leq M_\alpha^2 l_F^{(1)} \mathrm{Tr}(Q) \int_0^t e^{-l_0(t-s)}(t-s)^{-2\alpha}\bigg[n \sup_{s\in J} e^{-l_0 s}E \parallel \phi(s) - \psi(s) \parallel_\alpha^2 \\
&\quad + l_h b^2 \sup_{s\in J} e^{-l_0 s}E \parallel \phi(s) - \psi(s) \parallel_\alpha^2\bigg]ds \\
&\leq M_\alpha^2 l_F^{(1)} \mathrm{Tr}(Q)(n + l_h b^2) \int_0^t e^{-l_0(t-s)}(t-s)^{-2\alpha}ds \parallel \phi - \psi \parallel_V^2 \\
&\leq \gamma \parallel \phi - \psi \parallel_V^2, \quad t \in J,
\end{aligned}$$

which implies that

$$e^{-l_0 t}E \parallel (P_v\phi)(t) - (P_v\psi)(t) \parallel_\alpha^2 \leq \gamma \parallel \phi - \psi \parallel_V^2, \quad t \in J.$$

Thus

$$\parallel P_v\phi - P_v\psi \parallel_V^2 \leq \gamma \parallel \phi - \psi \parallel_V^2, \quad \phi, \psi \in V.$$

Therefore, P_v is a strict contraction. By the Banach contraction principle we conclude that P_v has a unique fixed point $\phi_v \in V$ and Eq. (6) has a unique mild solution on $[0, b]$. Set

$$\tilde{v}(t) := \begin{cases} v(t) & \text{if } t \in (\delta, b], \\ v(\delta) & \text{if } t \in [0, \delta]. \end{cases}$$

From (6), we have

$$\begin{aligned}\phi_{\tilde{v}}(t) = R(t)[x_0 - g(\tilde{v})] + \int_0^t R(t-s) \\ \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg)ds.\end{aligned} \tag{7}$$

Consider the map $\Psi : \mathcal{C}_\delta = \mathcal{C}([\delta, b], L_2(\Omega, \mathcal{F}, H)) \to \mathcal{C}_\delta$ defined by

$$(\Psi v)(t) = \phi_{\tilde{v}}(t), \quad t \in [\delta, b]. \tag{8}$$

We shall show that Ψ satisfies in all conditions of Lemma 2.2. The proof will be given in several steps.

Step 1. The set $G = \{v \in \mathcal{C}_\delta : \lambda \in (0,1), v = \lambda\Psi(v)\}$ is bounded.

Indeed, let $\lambda \in (0,1)$ and let $v \in \mathcal{C}_\delta$ be a possible solution of $v = \lambda\Psi(v)$ for some $0 < \lambda < 1$. This implies, by (7) and (8), that for each $t \in (0, b]$ we have

$$\begin{aligned}v(t) = \lambda\phi_{\tilde{v}}(t) = \lambda R(t)[x_0 - g(\tilde{v})] + \lambda \int_0^t R(t-s) \\ \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg)dw(s).\end{aligned} \tag{9}$$

From conditions (H8) and (5), we conclude that there exist positive constants ϵ and γ^* such that, for all $\| \phi \|_{\mathcal{C}} > \gamma^*$,

$$E \| g(\phi) \|_\alpha^2 \leq (c+\epsilon) \| \phi \|_{\mathcal{C}}^2, \quad 16M(c+\epsilon)e^{\frac{16 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n+4l_h b)b^{1-2\alpha}}{1-2\alpha}} < 1. \tag{10}$$

Let

$$G_1 = \{\phi : \| \phi \|_{\mathcal{C}} \leq \gamma^*\}, \quad G_2 = \{\phi : \| \phi \|_{\mathcal{C}} > \gamma^*\},$$
$$C_1 = \max\{\| Eg(\phi) \|_\alpha^2, \phi \in G_1\}.$$

Thus,

$$E \| g(\phi) \|_\alpha^2 \leq C_1 + (c+\epsilon) \| \phi \|_{\mathcal{C}}^2 . \tag{11}$$

By (H2), (H5), (H6) and (11), from (9) we have for each $t \in (0, b]$, $\| v(t) \|_\alpha \leq \| \phi_{\tilde{v}}(t) \|_\alpha$ and

$$\begin{aligned}E \| \phi_{\tilde{v}}(t) \|_\alpha^2 &\leq 4E \| R(t)[x_0 - g(\tilde{v})] \|_\alpha^2 + 4E \bigg\| \int_0^t R(t-s) \\ &\quad \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg)dw(s) \bigg\|_\alpha^2 \\ &\leq 4ME[\| x_0 + g(\tilde{v}) \|_\alpha^2] + 4M_\alpha^2 b \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha} \\ &\quad \times E \bigg\| F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) \bigg\|^2 ds \\ &\leq 16ME[\| x_0 \|_\alpha^2 + \| g(\tilde{v}) \|_\alpha^2] + 16M_\alpha^2 \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha} \\ &\quad \times E\bigg[\bigg\| F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) \\ &\quad - F(s, 0, ..., 0) \bigg\|^2 + \| F(s, 0, \ldots, 0) \|^2 \bigg] ds\end{aligned}$$

$$\leq 16M[E \parallel x_0 \parallel_\alpha^2 + E \parallel g(\tilde{v}) \parallel_\alpha^2] + 16M_\alpha^2 \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha}$$
$$\times \Bigg\{ l_F^{(1)} \Bigg[\sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + \cdots + \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2$$
$$+ 4bE \int_0^s [\parallel h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) - h(s,\tau,0) \parallel_\alpha^2 + \parallel h(s,\tau,0) \parallel_\alpha^2] d\tau \Bigg] + l_1 \Bigg\} ds$$
$$\leq 16M[E \parallel x_0 \parallel_\alpha^2 + C_1 + (c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}^2] + 16M_\alpha^2 \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha}$$
$$\times \Bigg\{ l_F^{(1)} \Bigg[n \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + 4b^2 (l_h \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + l_h^{(1)}) \Bigg] + l_1 \Bigg\} ds$$
$$\leq M^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}^2$$
$$+ 16 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n + 4l_h b^2) \int_0^t (t-s)^{-2\alpha} \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 ds,$$

where $M^* = 16M[E \parallel x_0 \parallel_\alpha + C_1] + \frac{16M_\alpha^2 \mathrm{Tr}(Q) b^{1-2\alpha}(4b^2 l_F^{(1)} l_h^{(1)} + l_1)}{1-2\alpha}$. Using the Gronwall's inequality, we get

$$\sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(t) \parallel_\alpha^2 \leq [M^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}^2] e^\eta,$$

where $\eta = \frac{16 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n+4l_h b^2) b^{1-2\alpha}}{1-2\alpha}$, and the previous inequality holds. Consequently,

$$\parallel v \parallel_{\mathcal{C}}^2 \leq [M^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}^2] e^\eta,$$

therefore we have

$$\parallel v \parallel_{\mathcal{C}}^2 \leq \frac{M^* e^\eta}{1 - 16M(c+\epsilon) e^\eta} < \infty.$$

Thus the proof of boundedness of the set G is complete.

Step 2. Ψ maps bounded sets into equicontinuous sets of $\mathcal{C}_\delta$.

For each constant $r > 0$, let

$$v \in C_r(\delta) := \Bigg\{ \phi \in \mathcal{C}_\delta : \sup_{\delta \leq t \leq b} E \parallel \phi(t) \parallel_\alpha^2 \leq r \Bigg\}.$$

Then $C_r(\delta)$ is a bounded closed convex set in $\mathcal{C}_\delta$. Let $v \in C_r(\delta)$, $\delta \leq t_1 < t_2 \leq b$, and $\varepsilon > 0$ be small. Note that

$$E \Bigg\| F\Bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) d\tau \Bigg) \Bigg\|^2$$
$$\leq 4E \Bigg[\Bigg\| F\Bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) d\tau \Bigg)$$
$$- F(s,0,\ldots,0) \Bigg\|^2 + \parallel F(s,0,\ldots,0) \parallel^2 \Bigg]$$
$$\leq 4 l_F^{(1)} E \Bigg[\parallel \phi_{\tilde{v}}(\sigma_1(s)) \parallel_\alpha^2 + \cdots + \parallel \phi_{\tilde{v}}(\sigma_n(s)) \parallel_\alpha^2$$
$$+ \Bigg\| \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) d\tau \Bigg\|_\alpha^2 \Bigg] + 4 l_1$$

$$
\begin{aligned}
&\leq 4l_F^{(1)}\bigg[\sup_{s\in[\delta,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + \cdots + \sup_{s\in[\delta,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 \\
&\quad + 4bE\int_0^s [\parallel h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) - h(s,\tau,0) \parallel_\alpha^2 + \parallel h(s,\tau,0) \parallel_\alpha^2] d\tau\bigg] + 4l_1 \\
&\leq 4l_F^{(1)}\bigg[n \sup_{s\in[\delta,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + 4b^2[l_h \sup_{s\in[\delta,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + l_h^{(1)}]\bigg] + 4l_1 \\
&\leq 4l_F^{(1)}\bigg[(n+4l_hb^2) \sup_{s\in[\delta,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + 4b^2 l_h^{(1)}\bigg] + 4l_1 \\
&\leq 4l_F^{(1)}[(n+4l_hb^2)r + 4b^2 l_h^{(1)}] + 4l_1 := M^{**}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&E \parallel \Psi v(t_2) - \Psi v(t_1) \parallel_\alpha^2 \\
&\leq 16E \parallel [R(t_2)-R(t_1)][x_0 - g(\tilde{v})] \parallel_\alpha^2 + 16E\bigg\| \int_0^{t_1-\varepsilon} [R(t_2-s) - R(t_1-s)] \\
&\quad \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) ds\bigg\|_\alpha^2 \\
&\quad + 16E\bigg\| \int_{t_1-\varepsilon}^{t_1} [R(t_2-s) - R(t_1-s)] \\
&\quad \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) ds\bigg\|_\alpha^2 \\
&\quad + 16E\bigg\| \int_{t_1}^{t_2} R(t_2-s) \\
&\quad \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) ds\bigg\|_\alpha^2 \\
&\leq 16E \parallel [R(t_2)-R(t_1)][x_0 - g(\tilde{v})] \parallel_\alpha^2 \\
&\quad + 16M^{**}\mathrm{Tr}(Q)\bigg(\int_0^{t_1-\varepsilon} \parallel A^\alpha [R(t_2-s) - R(t_1-s)] \parallel^2 ds \\
&\quad + \int_{t_1-\varepsilon}^{t_1} \parallel A^\alpha [R(t_2-s) - R(t_1-s)] \parallel^2 ds + \int_{t_1}^{t_2} \parallel A^\alpha R(t_2-s) \parallel^2 ds\bigg) \\
&\leq 16E \parallel [R(t_2)-R(t_1)][x_0 - g(\tilde{v})] \parallel_\alpha^2 \\
&\quad + 16M^{**}\mathrm{Tr}(Q)\bigg(\int_0^{t_1-\varepsilon} \parallel A^\alpha [R(t_2-s) - R(t_1-s)] \parallel^2 ds \\
&\quad + \frac{M_\alpha^2}{1-2\alpha}[(t_2-t_1)^{1-2\alpha} - (t_2-t_1-\varepsilon)^{1-2\alpha} + \varepsilon^{1-2\alpha}] \\
&\quad + \frac{M_\alpha^2}{1-2\alpha}(t_2-t_1)^{1-2\alpha}\bigg).
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$, with ε is sufficiently small, since $R(t)$ is strongly continuous and the compactness of $R(t)$ for $t > 0$ implies $R(t)$,$A^\alpha R(t)$ the continuity in the uniform operator topology. Thus Ψ maps $C_r(\delta)$ into an equicontinuous family of functions.

Step 3. The set $W(t) = \{\Psi(v)(t) : v \in C_r(\delta)\}$ is relatively compact in H.

Let $\delta < t \le s \le b$ be fixed and ε a real number satisfying $0 < \varepsilon < t$. For $v \in C_r(\delta)$, we define

$$(\Psi_\varepsilon v)(t) = R(t)[x_0 - g(\tilde{v})] + \int_0^{t-\varepsilon} R(t-s) \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg)dw(s).$$

Since the compactness of $R(t)$ for $t>0$, we deduce that the set $U_\varepsilon(t) = \{(\Psi_\varepsilon v)(t) : v \in C_r(\delta)\}$ is relatively compact in H for every ε, $0 < \varepsilon < t$. Also, for every $v \in C_r(\delta)$, we have

$$\begin{aligned} &E \parallel (\Psi v)(t) - (\Psi_\varepsilon v)(t) \parallel_\alpha^2 \\ &\le E\bigg\| \int_{t-\varepsilon}^t R(t-s) \\ &\quad \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg)ds\bigg\|_\alpha^2 \\ &\le M_\alpha^2 \int_{t-\varepsilon}^t (t-s)^{-2\alpha} M^{**} ds \le \frac{M_\alpha^2 M^{**}}{1-2\alpha}\varepsilon^{1-2\alpha}. \end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we can see that there are relative compact sets arbitrarily close to the set $W(t) = \{(\Psi v) : v \in C_r(\delta)\}$, and $W(t)$ is a relatively compact in H. It is easy to see that $\Psi(C_r(\delta))$ is uniformly bounded. Since we have shown $\Psi(C_r(\delta))$ is equicontinuous collection, by the Arzelá-Ascoli theorem it suffices to show that Ψ maps $C_r(\delta)$ into a relatively compact set in H.

Step 4. $\Psi : \mathcal{C}_\delta \to \mathcal{C}_\delta$ is continuous.

From (6) and (H2),(H5), we deduce that for $v_1, v_2 \in C_r(\delta),\ t \in [0,b]$,

$$\begin{aligned} &E \parallel \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \parallel_\alpha^2 \\ &\le 4E \parallel R(t)[g(\tilde{v}_1) - g(\tilde{v}_2)] \parallel_\alpha^2 + 4E\bigg\| \int_0^t R(t-s) \\ &\quad \times \bigg[F\bigg(s, \phi_{\tilde{v}_1}(\sigma_1(s)), ..., \phi_{\tilde{v}_1}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)))d\tau\bigg) \\ &\quad - F\bigg(s, \phi_{\tilde{v}_2}(\sigma_1(s)), ..., \phi_{\tilde{v}_2}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))d\tau\bigg)\bigg]dw(s)\bigg\|_\alpha^2 \\ &\le 4ME \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel_\alpha^2 + 4l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha} \\ &\quad \times E\bigg[\parallel \phi_{\tilde{v}_1}(\sigma_1(s)) - \phi_{\tilde{v}_2}(\sigma_1(s)) \parallel_\alpha^2 + \cdots + \parallel \phi_{\tilde{v}_1}(\sigma_n(s)) - \phi_{\tilde{v}_2}(\sigma_n(s)) \parallel_\alpha^2 \\ &\quad + b\int_0^s \parallel h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau))) - h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau))) \parallel_\alpha^2 d\tau\bigg]ds \\ &\le 4ME \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel_\alpha^2 + 4l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha} \\ &\quad \times \bigg[\sup_{s\in[0,b]} E \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel_\alpha^2 + \cdots + \sup_{s\in[0,b]} E \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel_\alpha^2 \end{aligned}$$

$$
\begin{aligned}
&+ l_h b E \int_0^s \| \phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)) - \phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)) \|_\alpha^2 \, d\tau \Big] ds \\
&\le 4ME \| g(\tilde{v}_1) - g(\tilde{v}_2) \|_\alpha^2 + 4 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q) \int_0^t (t-s)^{-2\alpha} \\
&\times \Big[n \sup_{s\in[0,b]} E \| \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \|_\alpha^2 + l_h b^2 \sup_{s\in[0,b]} E \| \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \|_\alpha^2 \Big] ds \\
&\le 4ME \| g(\tilde{v}_1) - g(\tilde{v}_2) \|_\alpha^2 \\
&+ 4 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n + l_h b^2) \int_0^t (t-s)^{-2\alpha} \sup_{t\in[0,b]} E \| \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \|_\alpha^2 \, ds.
\end{aligned}
$$

Using again the Gronwall's inequality, we have that, for t, v_1, v_2 as above,

$$
\sup_{t\in[0,b]} E \| \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \|_\alpha^2 \le 4M e^{\frac{4 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n + l_h b^2) b^{1-2\alpha}}{1-2\alpha}} E \| g(\tilde{v}_1) - g(\tilde{v}_2) \|_\alpha^2,
$$

for all $t \in [0, b]$, which implies that

$$
\| \Psi v_1 - \Psi v_2 \|_{\mathcal{C}} \le 4M e^{\frac{4 l_F^{(1)} M_\alpha^2 \mathrm{Tr}(Q)(n + l_h b^2) b^{1-2\alpha}}{1-2\alpha}} E \| g(\tilde{v}_1) - g(\tilde{v}_2) \|_\alpha^2,
$$

for all $t \in [\delta, b]$, $v_1, v_2 \in C_r(\delta)$. Therefore, Ψ is continuous.

These arguments enable us to conclude that Ψ is completely continuous. We can now apply Lemma 2.2 to conclude that Ψ has at least fixed point $\tilde{v}_* \in \mathcal{C}_\delta$. Let $x = \phi_{\tilde{v}_*}$. Then, we have

$$
\begin{aligned}
x(t) &= R(t)[x_0 - g(\tilde{v}_*)] + \int_0^t R(t-s) \\
&\quad \times F\Big(s, x(\sigma_1(s)), ..., x(\sigma_n(s)), \int_0^s h(s, \tau, x(\sigma_{n+1}(\tau))) d\tau \Big) ds. \qquad (12)
\end{aligned}
$$

Note that $x = \phi_{\tilde{v}_*} = (P\tilde{v}_*)(t) = \tilde{v}_*$, $t \in [\delta, b]$. By (H7), we obtain

$$
g(x) = g(\tilde{v}_*).
$$

This implies, combined with (12), that $x(t)$ is a mild solution of the problem (1)-(2) and the proof of Theorem 3.1 is complete.

Theorem 3.2. Let $x(0) \in L_2^0(\Omega, \mathcal{C})$. If the assumptions (H1), (H3) and (H5)-(H8) hold and

$$
16Mc e^{16 l_F^{(2)} M \mathrm{Tr}(Q)(n + 4 l_h b^2)} < 1, \qquad (13)
$$

then the nonlocal Cauchy problem (1)-(2) has a mild solution on J.

Proof. Let $l_0 > 0$ be a constant chosen such that

$$
\gamma_1 := \sup_{t\in J} \left\{ M l_F^{(2)} \mathrm{Tr}(Q)(n + l_h b^2) \int_0^t e^{-l_0(t-s)} ds \right\} < 1,
$$

and $V, P_v, \tilde{v}$ as in Theorem 3.1. Therefore, P_v has a unique fixed point and Eq. (6) has a unique solution $\phi_v \in V$. Define the operator $\Psi : \mathcal{C}_\delta \to \mathcal{C}_\delta$ as (8). In order to apply Lemma 2.2, we first give the set G is priori bound. In fact, by (H3), (H5)

(H6) and (11), from (9) we have for each $t \in (0, b]$,

$$\begin{aligned}
&E \parallel \phi_{\tilde{v}}(t) \parallel_\alpha^2 \\
&\leq 4E \parallel R(t)[x_0 - g(\tilde{v})] \parallel_\alpha^2 + 4E\Bigg\| \int_0^t R(t-s) \\
&\quad \times F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) dw(s)\Bigg\|_\alpha^2 \\
&\leq 4ME[\parallel x_0 + g(\tilde{v}) \parallel_\alpha^2] + 4M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t E\bigg\| F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \dots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg)\bigg\|_\alpha^2 ds \\
&\leq 16ME[\parallel x_0 \parallel_\alpha^2 + \parallel g(\tilde{v}) \parallel_\alpha^2] + 16M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t E\bigg[\bigg\| F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), \dots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\bigg) \\
&\quad - F(s, 0, \dots, 0)\bigg\|_\alpha^2 + \parallel F(s,0,\dots,0) \parallel_\alpha^2\bigg] ds \\
&\leq 16M[E \parallel x_0 \parallel_\alpha^2 + E \parallel g(\tilde{v}) \parallel_\alpha^2] \\
&\quad + 16M\mathrm{Tr}(Q) \int_0^t \bigg\{ l_F^{(2)} \bigg[\sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + \dots + \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 \\
&\quad + 4bE \int_0^s [\parallel h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) - h(s,\tau,0) \parallel_\alpha + \parallel h(s,\tau,0) \parallel_\alpha^2] d\tau \bigg] + l_2 \bigg\} ds \\
&\leq 16M[E \parallel x_0 \parallel_\alpha^2 + C_1 + (c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}^2] + 16M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t \bigg\{ l_F^{(2)} \bigg[n \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + 4b^2(l_h \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + l_h^{(1)}) \bigg] + l_2 \bigg\} ds \\
&\leq M_1^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}} + 16 l_F^{(2)} M\mathrm{Tr}(Q)(n + 4l_h b^2) \\
&\quad \times \int_0^t \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 ds,
\end{aligned}$$

where $M_1^* = 16M[E \parallel x_0 \parallel_\alpha + C_1] + 16M\mathrm{Tr}(Q)[4b^2 l_F^{(2)} l_h^{(1)} + l_2]$. Using the Gronwall's inequality, we get

$$\sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(t) \parallel_\alpha^2 \leq [M_1^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}] e^{\eta_1},$$

where $\eta_1 = 16 l_F^{(2)} M\mathrm{Tr}(Q)(n + 4l_h b^2)$. Consequently,

$$\parallel v \parallel_{\mathcal{C}}^2 \leq [M_1^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}] e^{\eta_1}.$$

Since $16M(c+\epsilon)e^{\eta_1} < 1$, we have

$$\parallel v \parallel_{\mathcal{C}}^2 \leq \frac{M_1^* e^{\eta_1}}{1 - 16M(c+\epsilon)e^{\eta_1}} < \infty.$$

Thus the proof of boundedness of the set G is complete. The proofs of the other steps are similar to those in Theorem 3.1. Therefore we omit the details.

Theorem 3.3. Let $x(0) \in L_2^0(\Omega, \mathcal{C})$. If the assumptions (H1), (H4) and (H5)-(H8) hold and

$$16Mce^{16 l_F^{(3)} \| A^{\alpha-\beta} \| M\mathrm{Tr}(Q)(n+4l_h b^2)} < 1, \tag{14}$$

then the nonlocal Cauchy problem (1)-(2) has a mild solution on J.
Proof. Let $l_0 > 0$ be a constant chosen such that

$$\gamma_2 := \sup_{t\in J}\left\{Ml_F^{(3)} \parallel A^{\alpha-\beta} \parallel \mathrm{Tr}(Q)(n+l_h b^2)\int_0^t e^{-l_0(t-s)}ds\right\} < 1,$$

and $V, P_v, \tilde{v}$ as in Theorem 3.1. Therefore, P_v has a unique fixed point and Eq. (6) has a unique solution $\phi_v \in V$. Just as in the proof of Theorem 3.2, we only show that the set G is priori bound. In fact, by (H4), (H5), (H6) and (11), from (9) we have for each $t \in (0, b]$,

$$\begin{aligned}
&E \parallel \phi_{\tilde{v}}(t) \parallel_\alpha^2 \\
&\le 4E \parallel R(t)[x_0 - g(\tilde{v})] \parallel_\alpha^2 + 4E\left\| \int_0^t A^{\alpha-\beta}R(t-s)\right. \\
&\quad \left. \times A^\beta F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)dw(s)\right\|^2 \\
&\le 4ME[\parallel x_0 + g(\tilde{v}) \parallel_\alpha] + 4 \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t E\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)\right\|_\beta ds \\
&\le 16ME[\parallel x_0 \parallel_\alpha^2 + \parallel g(\tilde{v}) \parallel_\alpha^2] + 16 \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t E\left[\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)\right.\right. \\
&\quad \left.\left. - F(s,0,...,0)\right\|_\beta^2 + \parallel F(s,0,...,0) \parallel_\beta^2\right]ds \\
&\le 16M[E \parallel x_0 \parallel_\alpha^2 + E \parallel g(\tilde{v}) \parallel_\alpha^2] + 16 \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t \left\{ l_F^{(3)}\left[\sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + \cdots + \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 \right.\right. \\
&\quad \left.\left. + 4bE\int_0^s [\parallel h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) - h(s,\tau,0) \parallel_\alpha^2 + \parallel h(s,\tau,0) \parallel_\alpha^2]d\tau\right] + l_3\right\}ds \\
&\le 16M[E \parallel x_0 \parallel_\alpha^2 + C_1 + (c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}] + 4 \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q) \\
&\quad \times \int_0^t \left\{ l_F^{(3)}\left[n \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + 4b^2(l_h \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 + l_h^{(1)})\right] + l_3\right\}ds \\
&\le M_2^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}} + 16 l_F^{(3)} \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q)(n + 4l_h b^2) \\
&\quad \times \int_0^t \sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(s) \parallel_\alpha^2 ds,
\end{aligned}$$

where $M_2^* = 16M[E \parallel x_0 \parallel_\alpha + C_1] + 16 \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q)[4b^2 l_F^{(3)} l_h^{(1)} + l_3]$. Using the Gronwall's inequality, we get

$$\sup_{s\in(0,b]} E \parallel \phi_{\tilde{v}}(t) \parallel_\alpha^2 \le [M_2^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}]e^{\eta_2},$$

where $\eta_2 = 16 l_F^{(3)} \parallel A^{\alpha-\beta} \parallel M\mathrm{Tr}(Q)(n + 4l_h b^2)$. Consequently,

$$\parallel v \parallel_{\mathcal{C}} \le [M_2^* + 16M(c+\epsilon) \parallel \tilde{v} \parallel_{\mathcal{C}}]e^{\eta_2}.$$

Since $16M(c+\epsilon)e^{\eta_2} < 1$, we have

$$\| v \|_{\mathcal{C}} \leq \frac{M_2^* e^{\eta_2}}{1 - 16M(c+\epsilon)e^{\eta_2}} < \infty.$$

Thus the proof of boundedness of the set G is complete.

Remark 3.1. (H7)-(H8) are satisfied if there exist constants b_1, b_2, such that

$$\| g(\phi) \|_\alpha \leq b_1 + b_2 \| \phi \|_{\mathcal{C}}, \qquad \phi \in \mathcal{C} \tag{15}$$

or there exist constants $c_1, c_2, \mu \in [0,1)$, such that

$$\| g(\phi) \|_\alpha \leq c_1 + c_2 \| \phi \|_{\mathcal{C}}^\mu, \qquad \phi \in \mathcal{C}. \tag{16}$$

4. Application

As an application, we consider the stochastic partial integrodifferential equation of the following form

$$\begin{aligned} dz(t,x) = \frac{\partial^2}{\partial x^2}\left[z(t,x) + \int_0^t b(t-s)z(t,x)ds\right]dt \\ +\left[f_1\left(t, x, \int_0^\pi z(\sin t, x)dx, \int_0^t f_2(t,s,z(\sin s, x))ds\right)\right]w(t), \quad (17) \\ 0 \leq t \leq 1, 0 \leq x \leq \pi, \end{aligned}$$

$$z(t,0) = z(t,\pi) = 0, \ 0 \leq t \leq 1, \tag{18}$$

$$z(0,x) + \sum_{i=0}^{p} \int_0^\pi k(x,y) z^{\frac{1}{3}}(t_i, y)dy = z_0(x), \quad 0 \leq x \leq \pi, \tag{19}$$

where $w(t)$ denotes a one-dimensional standard Wiener process, and there exists a constant K_1 such that $|b(t-s)| \leq K_1$ and p is a positive integer, $0 < t_0 < t_1 < \cdots < t_p < 1$, $z_0(x) \in L^2([0,\pi])$ is $\mathcal{F}_0$ -measurable and satisfies $E \| z_0 \|^2 < \infty$.

Let $H = L^2([0,\pi])$ with the norm $\| \cdot \|$. Define the operator $A : D(A) \subset H \to H$ given by $Au = u''$ with

$$D(A) := H_0^2([0,\pi]) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}.$$

Then A generates a strongly continuous semigroup that is analytic, and resolvent operator $R(t)$ can be extracted from this analytic semigroup(see [18]). Furthermore, A has a discrete spectrum; the eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Then the following properties hold:

(i) If $u \in D(A)$, then

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n\rangle z_n,$$

(ii) For each $u \in H$,

$$A^{-\frac{1}{2}}u = \sum_{n=1}^{\infty} \frac{1}{n}\langle u, z_n\rangle z_n.$$

(iii)The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}u = \sum_{n=1}^{\infty} n\langle u, z_n\rangle z_n$$

on the space $D(A^{\frac{1}{2}}) = \{u(\cdot) \in H, \sum_{n=1}^{\infty} n\langle u, z_n\rangle z_n \in H\}$ and $\| A^{-\frac{1}{2}} \| = 1$.

Lemma 4.1([29]). If $m \in D(A^{\frac{1}{2}})$, then m is absolutely continuous, $m' \in H$ and $\| m' \| = \| A^{\frac{1}{2}} m \|$.

Let $H_{\frac{1}{2}} := (D(A^{\frac{1}{2}}), \| \cdot \|_{\frac{1}{2}})$, where $\| \cdot \|_{\frac{1}{2}} := \| A^{\frac{1}{2}} x \|$ for each $x \in D(A^{\frac{1}{2}})$.

We assume that the following conditions hold:

(a) The function $k(x, y)$ is measurable and

$$\eta^* = \left(\int_0^{\pi} \int_0^{\pi} k^2(x, y) dx dy \right)^{1/2} < \infty.$$

Moreover, for each $y \in [0, \pi]$ the function $x \mapsto \frac{\partial}{\partial t} k(x, y)$ is measurable, $k(0, y) = k(\pi, y) = 0$,

$$\eta_0 = \left(\int_0^{\pi} \int_0^{\pi} \left(\frac{\partial}{\partial x} k^2(x, y) \right)^2 dx dy \right)^{1/2} < \infty,$$

and there is a nonnegative function $\Theta \in L^1(0, 1)$ such that $|\frac{\partial}{\partial x} k(x, y)| \leq \Theta(t)$ for all $(x, y) \in [0, 1] \times [0, 1]$.

(b) The function $f_1 : [0, 1] \times [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $l_{f_1}^{(1)} > 0$ such that

$$|f_1(t, x, x_1, y_1,) - f_1(t, x, x_2, y_2)| \leq l_{f_1}^{(1)} [|x_1 - x_2| + |y_1 - y_2|],$$

for all $t \in [0, 1], x_i, y_i \in \mathbb{R}, i = 1, 2$.

(c) The function $f_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $l_{f_2}^{(1)} > 0$ such that

$$|f_2(t, s, y_1) - f_2(t, s, y_2)| \leq l_{f_2}^{(1)} |y_1 - y_2|,$$

for all $(t, s) \in [0, 1] \times [0, 1], y_i \in \mathbb{R}, i = 1, 2$.

Let $\mathcal{C}$ denote the Banach space $\mathcal{C}([0, 1], L_2(\Omega, \mathcal{F}, H))$ the family of all $\mathcal{F}_t$-measurable, $\mathcal{C}([0, 1])$-valued random variables x with the norm

$$\| x \|_{\mathcal{C}} = \sup_{0 \leq t \leq 1} (E \| x(t) \|_{\frac{1}{2}}^2)^{\frac{1}{2}}.$$

Here we choose $\alpha = \frac{1}{2}$. We can define respectively $F : [0, 1] \times H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to L(K; H), h : [0, 1] \times [0, 1] \times H_{\frac{1}{2}} \to H_{\frac{1}{2}}$ and $g : \mathcal{C} \to H_{\frac{1}{2}}$ by

$$\begin{aligned} F&\left(t, z(\sigma(t)), \int_0^t h(t, s, z(\sigma(s))) ds \right)(x) \\ &= f_1\left(t, x, \int_0^{\pi} z(\sin t, x) dx, \int_0^t f_2(t, s, z(\sin s, x)) ds \right), \\ &\qquad h(t, s, z(\sigma(s)))(x) = f_2(t, s, z(\sin s, x)), \end{aligned}$$

and

$$g(z)(\cdot) = \sum_{i=0}^{p} K_0 z^{\frac{1}{3}}(t_i)(\cdot), \quad z \in \mathcal{C},$$

where $K_0 : H_{\frac{1}{2}} \to H$ is defined by

$$(K_0 \xi_0)(x) = \int_0^{\pi} k(x, y) \xi_0(y) dy, \quad \xi_0 \in H_{\frac{1}{2}}.$$

Let $\sigma(t) = \sin t$. Then Eq. (17)-(19) takes the abstract form (1)-(2). Moreover, for $z_i, \tilde{z}_i \in H_{\frac{1}{2}}, i = 1, 2$ and $x \in [0, \pi]$, we have

$$\begin{aligned}
&\| F(t, z_1, \tilde{z}_1) - F(t, z_2, \tilde{z}_2) \|^2 \\
&= \Bigg[\Bigg(\int_0^\pi \Bigg| f_1\Bigg(t, x, \int_0^\pi z_1(\sin t, x)dx, \int_0^t f_2\Bigg(t, s, z_1(\sin s, x))ds\Bigg) \\
&\quad - f_1\Bigg(t, x, \int_0^\pi z_2(\sin t, x)dx, \int_0^t f_2(t, s, z_2(\sin s, x))ds\Bigg)\Bigg|^2 dx\Bigg)^{\frac{1}{2}}\Bigg]^2 \\
&\leq \Bigg[l_{f_1}^{(1)}\Bigg(\Bigg(\int_0^\pi \Bigg(\int_0^\pi z_1(\sin t, x)dx - \int_0^\pi z_2(\sin t, x)dx\Bigg)^2 dx\Bigg)^{\frac{1}{2}} \\
&\quad + l_{f_2}^{(1)}\Bigg(\int_0^\pi \Bigg(\int_0^t (z_1(\sin s, x) - z_2(\sin s, x))ds\Bigg)^2 dx\Bigg)^{\frac{1}{2}}\Bigg)\Bigg]^2 \\
&\leq [l_{f_1}^{(1)}(\pi \| A^{-\frac{1}{2}} \| \| A^{\frac{1}{2}}(z_1 - z_2) \| + l_{f_2}^{(1)} \| A^{-\frac{1}{2}} \| \| A^{\frac{1}{2}}(z_1 - z_2) \|)]^2 \\
&= [l_{f_1}^{(1)}(\pi + l_{f_2}^{(1)})]^2 \| z_1 - z_2 \|_{\frac{1}{2}}^2 .
\end{aligned}$$

This implies that F satisfies assumption (H2) and h satisfies assumption (H5). Now, if $z \in \mathcal{C}$, then

$$\langle g(z), z_n \rangle = \frac{1}{n}\sqrt{\frac{2}{n}}\Bigg\langle \sum_{i=0}^p \int_0^\pi k(x, y)z^{\frac{1}{3}}(t_i, y)dy, \cos(nx)\Bigg\rangle.$$

This shows that g take values in $\mathcal{C}$ in terms of properties (i) and (iii). Moreover, by Lemma 4.1, we have

$$\begin{aligned}
E \| g(z) \|_{\frac{1}{2}}^2 &= E \| A^{\frac{1}{2}} g(z)(\cdot) \|^2 = E \| g(z)'(\cdot) \|^2 \\
&\leq E\Bigg\| \sum_{i=0}^p (K_0 z^{\frac{1}{3}}(t_i))'(\cdot)\Bigg\|^2 \leq p^2 \sum_{i=0}^p E \| (K_0 z^{\frac{1}{3}}(t_i))'(\cdot) \|^2 \\
&\leq p^2 \sum_{i=0}^p \eta_0^2 E \| z^{\frac{1}{3}}(t_i)) \|^2 \leq p^2 \eta_0^2 \sum_{i=0}^p E[\| A^{-\frac{1}{2}} \| \| A^{\frac{1}{2}} z^{\frac{1}{3}}(t_i) \|]^2 \\
&\leq p^2 \eta_0^2 \sum_{i=0}^p E \| z^{\frac{1}{3}}(t_i) \|_{\frac{1}{2}}^2 \leq p^3 \eta_0^2 \| z^{\frac{1}{3}} \|_{\mathcal{C}}^2,
\end{aligned}$$

which verifies that g satisfies (H7)-(H8) with $c = 0$. Moreover, all the other conditions stated in Theorem 3.1 are satisfied. Hence, the nonlocal Cauchy problem (17)-(19) admits a mild solution on $[0, 1]$.

Next we add the following assumptions:

(d) The function $f_1 : [0, 1] \times [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f_1(\cdot, 0, \cdot, \cdot) = f_1(\cdot, \pi, \cdot, \cdot) = 0$, and it satisfies the differentiable with respect to the second argument and there exists $l_{f_1}^{(2)} > 0$ such that

$$\left|\frac{\partial}{\partial x} f_1(t, x, x_1, y_1) - \frac{\partial}{\partial x} f_1(t, x, x_2, y_2)\right| \leq l_{f_1}^{(2)}[|x_1 - x_2| + |y_1 - y_2|],$$

for all $t \in [0, 1], x_i, y_i \in \mathbb{R}, i = 1, 2$.

Here we choose $\alpha = \beta = \frac{1}{2}$. We can define $F : [0, 1] \times H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to L(K; H_{\frac{1}{2}})$, and similar computation of g shows that F is a function from $[0, 1] \times H_{\frac{1}{2}} \times H_{\frac{1}{2}}$ into

$L(K; H_{\frac{1}{2}})$. Moreover, for $z_i, \tilde{z}_i \in H_{\frac{1}{2}}, i = 1, 2$ and $x \in [0, \pi]$, we have

$$
\begin{aligned}
&\| F(t, z_1, \tilde{z}_1) - F(t, z_2, \tilde{z}_2) \|_{\frac{1}{2}}^2 \\
&= \Bigg[\Bigg(\int_0^\pi \Bigg| \frac{\partial}{\partial x} f_1 \Big(t, x, \int_0^\pi z_1(\sin t, x) dx, \int_0^t f_2(t, s, z_1(\sin s, x)) ds \Big) \\
&\quad - \frac{\partial}{\partial x} f_1 \Big(t, x, \int_0^\pi z_2(\sin t, x) dx, \int_0^t f_2(t, s, z_2(\sin s, x)) ds \Big) \Bigg|^2 dx \Bigg)^{\frac{1}{2}} \Bigg]^2 \\
&\le \Bigg[l_{f_1}^{(2)} \Bigg(\Bigg(\int_0^\pi \Big(\int_0^\pi z_1(\sin t, x) dx - \int_0^\pi z_2(\sin t, x) dx \Big)^2 dx \Bigg)^{\frac{1}{2}} \\
&\quad + l_{f_2}^{(1)} \Bigg(\int_0^\pi \Big(\int_0^t (z_1(\sin s, x) - z_2(\sin s, x)) ds \Big)^2 dx \Bigg)^{\frac{1}{2}} \Bigg) \Bigg]^2 \\
&\le [l_{f_1}^{(2)} (\pi \| A^{-\frac{1}{2}} \| \| A^{\frac{1}{2}} (z_1 - z_2) \| + l_{f_2}^{(1)} \| A^{-\frac{1}{2}} \| \| A^{\frac{1}{2}} (z_1 - z_2) \|)]^2 \\
&= [l_{f_1}^{(2)} (\pi + l_{f_2}^{(1)})]^2 \| z_1 - z_2 \|_{\frac{1}{2}}^2 .
\end{aligned}
$$

Hence F satisfies assumption (H4). By Theorem 3.3, the nonlocal Cauchy problem (17)-(19) admits a mild solution on $[0, 1]$ under the above assumptions.

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