*Journal of Fractional Calculus and Applications*
Vol. 15(2) July. 2024, No. 5
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

# FINITE INTEGRAL INVOLVING INCOMPLETE ALEPH-FUNCTIONS AND FRESNEL INTEGRAL

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Abstract. Special functions represent a class of mathematical functions that have achieved a distinct and recognized status within the realms of mathematical analysis, functional analysis, geometry, physics, and diverse practical applications. These functions have emerged as notable tools in these disciplines, owing to their unique properties and inherent significance. Over time, they have become firmly established due to their ability to address specific mathematical challenges and contribute valuable insights to various branches of science and engineering. The primary objective of this paper is to establish a thorough definition of comprehensive finite integrals through the incorporation of both the Fresnel integral and incomplete Aleph-functions. By adopting a unified and general approach, these integrals are shown to yield a diverse range of new outcomes, particularly in specific scenarios. To elucidate and underscore the significance of our contributions, we present a detailed exposition of our findings, accompanied by specific corollaries. These corollaries, in turn, are emphasized as special cases derived directly from the fundamental results outlined in our study.

## 1. Introduction

Formulas that incorporate special functions in integrals are crucial for addressing diverse challenges in science and engineering. Numerous researchers have derived noteworthy ordinary and fractional integrals associated with these special functions, for instance, Singh and Mishra [24] explored a range of integral formulas that incorporated the $\bar{H}$-function multiplied by algebraic functions. Sachan and Ayant [21] calculated general finite integrals that encompassed the elliptic integral of the first species, an extension of the Mittag-Leffler function, and incomplete Aleph-functions, employing well-known integral techniques. Suthar et al. [25] established integrals involving the product of the Aleph-function with exponential functions

2010 *Mathematics Subject Classification.* 33C05, 33C60.
*Key words and phrases.* Incomplete Gamma function, incomplete Aleph-function, Mellin-Barnes integrals contour, Fresnel integral.
Submitted December 25, 2023. Revised March 15, 2024.

and multi Gauss's hypergeometric function. Bhargava et al. [26] derived specific integrals involving Srivastava's polynomials and the Aleph-function. Furthermore, Sachan and Singh [22] established unified integrals involving the generalized Mittag-Leffler function, generalized $M$-series, $H$-function, and $I$-function of two variables. For more examples, see references such as [2, 4, 5, 6, 9, 17, 18, 19, 20, 23]. Srivastava et al. [14]) delved into the exploration of the incomplete Gamma-function and incomplete hypergeometric function. More recently, in another work by Srivastava et al. [15]), the focus shifted to the introduction and examination of the incomplete $H$-function and the incomplete $\overline{H}$-function. Various researchers, such as those cited in [7], [3], and [8], have investigated the incomplete Aleph-function, the incomplete $I$-function, and the incomplete $H$-function, providing calculations of integrals involving these functions. In this current study, we explore a generalized finite integral that involves the product of the incomplete Aleph-function and the Fresnel integral.

The incomplete Gamma functions $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ are defined in the following manner :

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u} \mathrm{d}u, \quad (\Re(\alpha) > 0; x \geqslant 0). \tag{1}$$

$$\Gamma(\alpha, x) = \int_x^\infty u^{\alpha-1} e^{-u} \mathrm{d}u, \quad (x \geqslant 0; \Re(\alpha) > 0 \text{ when } x = 0). \tag{2}$$

We have the following relation :

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha) \quad (\Re(\alpha) > 0). \tag{3}$$

Now, we give the expression of the incomplete Aleph-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ [7] :

$$ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) = {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left(z \;\middle|\; \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,n}, [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \\ (g_j, G_j)_{1,m}, [\tau_i(g_{ji}, G_{ji})]_{m+1,q_i} \end{matrix}\right)$$

$$= \frac{1}{2\pi\omega} \int_L \frac{\Gamma(1 - a_1 - A_1 s, x) \prod_{j=2}^n \Gamma(1 - a_j - A_j s) \prod_{j=1}^m \Gamma(g_j + G_j s)}{\sum_{i=1}^r \tau_i \left[\prod_{j=m+1}^{q_i} \Gamma(1 - g_{ji} - G_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + A_{ji} s)\right]} z^{-s} \mathrm{d}s, \tag{4}$$

and

$$ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) = {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left(z \;\middle|\; \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,n}, [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \\ (g_j, G_j)_{1,m}, [\tau_i(g_{ji}, G_{ji})]_{m+1,q_i} \end{matrix}\right)$$

$$= \frac{1}{2\pi\omega} \int_L \frac{\gamma(1 - a_1 - A_1 s, x) \prod_{j=2}^n \Gamma(1 - a_j - A_j s) \prod_{j=1}^m \Gamma(g_j + G_j s)}{\sum_{i=1}^r \tau_i \left[\prod_{j=m+1}^{q_i} \Gamma(1 - g_{ji} - G_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + A_{ji} s)\right]} z^{-s} \mathrm{d}s, \tag{5}$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ defined above exits for $x \geq 0$ and the following validities conditions.

The contour $L$ is in the $s$-plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma(1 - a_j - A_j s), j = 2, \cdots, n$ to the right of the contour $L$ and the poles of $\Gamma(g_j + G_j s), j = 1, \cdots, m$ to the left of the contour $L$. The parameters $\tau_i$, $m, n, p_i, q_i$ are positive numbers satisfying $0 \leq n \leq p_i$, $0 \leq m \leq q_i$ and $a_j, g_j, a_{ji}, g_{ji}$ are complex numbers. These poles of the integrand are assumed to be simple. We have the following conditions :

$$\Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i,\ i = 1, \cdots, r \tag{6}$$

$$\Omega_j \geq 0, |\arg(z)| < \frac{\pi}{2}\Omega_i \ \text{ and } \ \Re(\zeta_i) + 1 < 0 \tag{7}$$

where

$$\Omega_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} G_j - \tau_i \left( \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} G_{ji} \right) \tag{8}$$

and

$$\zeta_i = \sum_{j=1}^{m} g_j - \sum_{j=1}^{n} a_j + \tau_i \left( \sum_{j=m+1}^{q_i} g_{ji} - \sum_{j=n+1}^{p_i} a_{ji} \right) + \frac{p_i - q_i}{2},\ i = 1, \cdots, r \tag{9}$$

We have easily the following relation:

$$^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) +^{(\gamma)} \aleph^{m,n}_{p_i,q_i,\tau_i,r}(z) = \aleph^{m,n}_{p_i,q_i,\tau_i,r}(z). \tag{10}$$

Where $\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ is the Aleph-function defined by Sudland [16]. Let's see the special cases. Taking $\tau_i \to 1$, then the incomplete Aleph-functions $^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and $^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ reduce respectively to incomplete $I$-functions $^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and $^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ where:

$$^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z) = {}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}\left( z \left| \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,n}, (a_{ji}, A_{ji})_{n+1,p_i} \\ (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1,q_i} \end{array} \right. \right)$$

$$= \frac{1}{2\pi\omega} \int_L \frac{\Gamma(1 - a_1 - A_1 s, x) \prod_{j=2}^{n} \Gamma(1 - a_j - A_j s) \prod_{j=1}^{m} \Gamma(g_j + G_j s)}{\sum_{i=1}^{r} \left[ \prod_{j=m+1}^{q_i} \Gamma(1 - g_{ji} - G_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + A_{ji} s) \right]} z^{-s} \mathrm{d}s, \tag{11}$$

and

$$^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z) = {}^{(\gamma)}I^{m,n}_{p_i,q_i,r}\left( z \left| \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,n}, (a_{ji}, A_{ji})_{n+1,p_i} \\ (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1,q_i} \end{array} \right. \right)$$

$$= \frac{1}{2\pi\omega} \int_L \frac{\gamma(1 - a_1 - A_1 s, x) \prod_{j=2}^{n} \Gamma(1 - a_j - A_j s) \prod_{j=1}^{m} \Gamma(g_j + G_j s)}{\sum_{i=1}^{r} \left[ \prod_{j=m+1}^{q_i} \Gamma(1 - g_{ji} - G_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + A_{ji} s) \right]} z^{-s} \mathrm{d}s, \tag{12}$$

under the same conditions that above with $\tau_i \to 1$. Now, we suppose $r = 1$, the incomplete $I$-functions $^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and $^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ reduce respectively to incomplete $H$-functions $^{(\Gamma)}H^{m,n}_{p,q}(z)$ and $^{(\gamma)}H^{m,n}_{p,q}(z)$ where

$$^{(\Gamma)}H^{m,n}_{p,q}(z) = {}^{(\Gamma)}H^{m,n}_{p,q}\left( z \left| \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (g_j, G_j)_{1,q} \end{array} \right. \right)$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}\mathrm{d}s, \tag{13}$$

and

$$^{(\gamma)}H_{p,q}^{m,n}(z) = {}^{(\gamma)}H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\(g_j,G_j)_{1,q}\end{matrix}\right)$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}\mathrm{d}s, \tag{14}$$

under the same conditions verified by the incomplete $I$-functions with $r = 1$. By using the formula (11) and (12), we have the following relations:

$$^{(\Gamma)}I_{p_i,q_i,r}^{m,n}(z) +^{(\gamma)} I_{p_i,q_i,r}^{m,n}(z) = I_{p_i,q_i,r}^{m,n}(z) \tag{15}$$

the function $I_{p_i,q_i,r}^{m,n}(z)$ being the function defined by Saxena [12] and

$$^{(\Gamma)}H_{p,q}^{m,n}(z) +^{(\gamma)} H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z) \tag{16}$$

The Fresnel integral is defined as follows:

$$C(x) = \int_0^x \cos(t^2)\mathrm{d}t \tag{17}$$

See [1] and [11] for more precisions.

## 2. Required Integral

In this section, we give a general finite integral, see ([10], 4.5.3 Eq.2 page 191).

**Lemma 2.1.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\mathrm{d}x = 2^{-2s-\frac{3}{4}}\; a^{2s+\frac{7}{4}}\; b^{\frac{1}{2}} \frac{\Gamma\left(2s+\frac{9}{4}\right)}{\Gamma\left(2s+\frac{11}{4}\right)} \times {}_2F_3\left(\begin{matrix}\frac{1}{4},2s+\frac{9}{4}\\ \frac{1}{2},\frac{5}{4},2s+\frac{11}{4}\end{matrix}\,\middle|\, -\frac{ab^2}{8}\right) \tag{18}$$

where $\Re(s) > -\frac{9}{8}, a > 0$.

## 3. Main integral

In this section, we study a generalization of the finite integral involving the incomplete Aleph-funtion with $x' > 0$.

**Theorem 3.1.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\; {}^{(\Gamma)}\aleph_{p_i,q_i,\tau_i,r}^{m,n}(zx^A(a-x)^A)\mathrm{d}x$$

$$= 2^{-2s-\frac{3}{4}}\, a^{2s+\frac{7}{4}}\, b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}}\left(-\frac{ab^2}{8}\right)^{n'}$$

$$\times {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z\left(\frac{a}{2}\right)^{2A}\,\middle|\,\begin{array}{c}(a_1,A_1,x'),T_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},T_2\end{array}\right) \tag{19}$$

where $T_1=\left(-\frac{5}{4}-2s-n';2A\right)$ and $T_2=\left(-\frac{7}{4}-2s-n';2A\right)$
provided $\Re(s)>-\frac{9}{8}$, $\Re(s)+A\min\limits_{1\leqslant j\leqslant m}\Re\left(\dfrac{g_j}{G_j}\right)>-\dfrac{9}{8}$, $\Omega_i>0,|\arg(z)|<\frac{\pi}{2}\Omega_i, i=1,\cdots,r$ or $\Omega_i\geq 0,|\arg(z)|<\frac{\pi}{2}\Omega_i$ and $\Re(\zeta_i)+1<0$, $\Omega_i$ and $\zeta_i$ is defined by (8) and (9) respectively, $A>0,a>0$.

*Proof.* To prove the theorem, expressing the modified incomplete Gamma Aleph-function in Mellin-Barnes contour integral with the help of (4) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collecting the power of $x$, we obtain $I$.

$$\begin{aligned}I&=\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(zx^A(a-x)^A)\mathrm{d}x\\&=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^n\Gamma(1-a_j-A_jt)\prod_{j=1}^m\Gamma(g_j+G_jt)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}z^{-t}\\&\times\int_0^a x^{s-At+\frac{1}{2}}(a-x)^{s-At}C\left(b\sqrt[4]{x(a-x)}\right)\mathrm{d}x\mathrm{d}t\end{aligned}$$

We use the lemma, this gives :

$$\begin{aligned}I&=\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(zx^A(a-x)^A)\mathrm{d}x\\&=\ 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}}\\&\times\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^n\Gamma(1-a_j-A_jt)\prod_{j=1}^m\Gamma(g_j+G_jt)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}z^{-t}\\&\times\frac{\Gamma\left(2s-2At+\frac{9}{4}\right)}{\Gamma\left(2s-2At+\frac{11}{4}\right)}2^{2At}a^{-2At}{}_2F_3\left(\begin{array}{c}\frac{3}{4},2s-2At+\frac{9}{4}\\\frac{3}{2},\frac{7}{4},2s-2At+\frac{11}{4}\end{array}\,\middle|\,-\frac{ab^2}{8}\right)\end{aligned}$$

We replace the Gauss hypergeometric function by the series $\sum\limits_{n=0}^{\infty}$ [13], under the hypothesis, we can interchanged this series and the $t$- integrals, we have:

$$\begin{aligned}I&=\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(zx^A(a-x)^A)\mathrm{d}x\\&=2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}}\sum_{n'=0}^{\infty}\ \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}}\left(-\frac{ab^2}{8}\right)^{n'}\\&\times\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^n\Gamma(1-a_j-A_jt)\prod_{j=1}^m\Gamma(g_j+G_jt)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}\ z^{-t}\\&\times\frac{\Gamma\left(2s-2At+\frac{9}{4}\right)}{\Gamma\left(2s-2At+\frac{11}{4}\right)}2^{2At}a^{-2At}\frac{\left(2s-2At+\frac{9}{4}\right)_{n'}}{\left(2s-2At+\frac{11}{4}\right)_{n'}}\ \mathrm{d}t\end{aligned}$$

Now we apply the relation $\Gamma(a)(a)_n = \Gamma(a+n)$, $a \neq 0, -1, -2, \cdots$ ,this gives:

$$I = \int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(zx^A(a-x)^A)\mathrm{d}x$$

$$= 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}} \left(-\frac{ab^2}{8}\right)^{n'}$$

$$\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^{n}\Gamma(1-a_j-A_jt)\prod_{j=1}^{m}\Gamma(g_j+G_jt)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}\ z^{-t}$$

$$\times \frac{\Gamma\left(2s-2At+\frac{11}{4}+n'\right)}{\Gamma\left(2s-2At+\frac{13}{4}+n'\right)} 2^{2At}a^{-2At}\mathrm{d}t$$

We interpret this contour integral of the Incomplete Gamma Aleph-function, we obtain the desired result. □

We have the same result with the incomplete gamma Aleph-function.

**Theorem 3.2.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(zx^A(a-x)^A)\mathrm{d}x$$

$$= 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}} \left(-\frac{ab^2}{8}\right)^{n'}$$

$$\times {}^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left(z\left(\frac{a}{2}\right)^{2A}\ \middle|\ \begin{matrix}(a_1,A_1,x'),T_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},T_2\end{matrix}\right) \tag{20}$$

under the same conditions that the Theorem 3.1. $T_1$ and $T_2$ are same as given with Theorem 3.1.
In the following section, we cite several particular cases.

## 4. Special cases

We consider the incomplete $I$-function.

**Corollary 4.0.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(zx^A(a-x)^A)\mathrm{d}x$$

$$= 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}} \left(-\frac{ab^2}{8}\right)^{n'}$$

$$\times {}^{(\Gamma)}I^{m,n+1}_{p_i+1,q_i+1,r}\left(z\left(\frac{a}{2}\right)^{2A}\ \middle|\ \begin{matrix}(a_1,A_1,x'),T_1,(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i},T_2\end{matrix}\right) \tag{21}$$

where $T_1$ and $T_2$ are same as given with Theorem 3.1.
provided $\Re(s) > -\frac{9}{8}$, $\Re(s) - A\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) > -\frac{9}{8}$, $\Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i, i =$

$1, \cdots, r$ or $\Omega_i \geq 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$ and $\Re(\zeta_i) + 1 < 0,\ A > 0, a > 0$ and

$$\Omega_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} G_j - \left( \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} G_{ji} \right) \tag{22}$$

and

$$\zeta_i = \sum_{j=1}^{m} g_j - \sum_{j=1}^{n} a_j + \left( \sum_{j=m+1}^{q_i} g_{ji} - \sum_{j=n+1}^{p_i} a_{ji} \right) + \frac{p_i - q_i}{2} \tag{23}$$

**Corollary 4.0.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(zx^A(a-x)^A)\mathrm{d}x$$
$$= 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}} \left(-\frac{ab^2}{8}\right)^{n'}$$
$$\times {}^{(\gamma)}I^{m,n+1}_{p_i+1,q_i+1,r}\left( z\left(\frac{a}{2}\right)^{2A} \middle| \begin{matrix} (a_1, A_1, x'), T_1, (a_j, A_j)_{2,n}, (a_{ji}, A_{ji})_{n+1,p_i} \\ (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1,q_i}, T_2 \end{matrix} \right) \tag{24}$$

under the conditions cited in the Corollary 4.0. Where $T_1$ and $T_2$ are same as given with Theorem 3.1.
Our function reduces to incomplete $H$-function, this gives

**Corollary 4.0.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\Gamma)}H^{m,n}_{p,q}(zx^A(a-x)^A)\mathrm{d}x$$
$$= 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}} \left(-\frac{ab^2}{8}\right)^{n'}$$
$$\times {}^{(\Gamma)}H^{m,n+1}_{p+1,q+1}\left( z\left(\frac{a}{2}\right)^{2A} \middle| \begin{matrix} (a_1, A_1, x'), \left(-\frac{5}{4} - 2s - n'; 2A\right), (a_j, A_j)_{2,p} \\ (g_j, G_j)_{1,q}, \left(-\frac{7}{4} - 2s - n'; 2A\right) \end{matrix} \right) \tag{25}$$

provided $\Re(s) > -\frac{9}{8}$, $\Re(s) - A \min_{1\leqslant j\leqslant m} \Re\left(\frac{g_j}{G_j}\right) > -\frac{9}{8}$, $\Omega > 0, |\arg(z)| < \frac{\pi}{2}\Omega$ and $\Re(\zeta) + 1 < 0,\ A > 0, a > 0$ and

$$\Omega = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} G_j - \left( \sum_{j=n+1}^{p} A_j + \sum_{j=m+1}^{q} G_j \right) \tag{26}$$

and

$$\zeta = \sum_{j=1}^{m} g_j - \sum_{j=1}^{n} a_j + \left( \sum_{j=m+1}^{q} g_j - \sum_{j=n+1}^{p} a_j \right) + \frac{p - q}{2} \tag{27}$$

**Corollary 4.0.**

$$\int_0^a x^{s+\frac{1}{2}}(a-x)^s C\left(b\sqrt[4]{x(a-x)}\right)\ {}^{(\gamma)}H^{m,n}_{p,q}(zx^A(a-x)^A)\mathrm{d}x$$
$$= 2^{-2s-\frac{3}{4}}\ a^{2s+\frac{7}{4}}\ b^{\frac{1}{2}} \sum_{n'=0}^{\infty} \frac{\left(\frac{1}{4}\right)_{n'}}{\left(\frac{1}{2}\right)_{n'}\left(\frac{5}{4}\right)_{n'}} \left(-\frac{ab^2}{8}\right)^{n'}$$

$$\times {}^{(\gamma)}H^{m,n+1}_{p+1,q+1}\left(z\left(\frac{a}{2}\right)^{2A}\;\middle|\;\begin{matrix}(a_1,A_1,x'),\left(-\frac{5}{4}-2s-n';2A\right),(a_j,A_j)_{2,p}\\(g_j,G_j)_{1,q},\left(-\frac{7}{4}-2s-n';2A\right)\end{matrix}\right) \tag{28}$$

under the conditions cited in the Corollary 4.0.

**Remarks.**

We encounter a unified finite integral structure involving the Aleph-function [16], the $I$-function [12], and Fox's $H$-function. All three mathematical entities contribute to a common generalized finite integral framework.

## 5. Conclusion

The significance of all our findings lies in their extensive applicability. By customizing the parameters and variables within the incomplete aleph functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$, we derive numerous results encompassing a remarkably diverse range of useful special functions (or products thereof). These outcomes can be expressed in terms of the $I$-function [12], $H$-function, Meijer's $G$-function, $E$-function, and hypergeometric function of a single variable. Furthermore, by specializing the parameters of the Fresnel integrals $C(.)$, we can generate a plethora of integrals involving the incomplete Aleph-functions. In essence, our research unveils a wealth of mathematical relationships and applications, showcasing the versatility and utility of the explored functions and integrals.

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