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NUMERICAL SOLUTION OF SOME CLASS OF GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS USING HAAR WAVELET

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Abstract. This paper is concerned with a numerical technique which comprises of Haar wavelet operational matrix of fractional order to solve some class of generalized fractional differential equations. Obtained numerical solutions are compared with the other, which gives the accuracy of the proposed method. Some of the examples are presented to indicate that the method is very efficient and accurate.

1. Introduction

Recently, the fractional calculus can be defined as that of derivatives as well as integrals of any order possible in the mathematical theory. Mainly, fractional calculus is a generalized form of an integer order calculus. Fractional calculus has been exploited as a crucial tool for applications that concern science and engineering. Fractional derivatives have proved to be tools in the modeling of many physical phenomena. These applications of fractional calculus have been elaborated previously by several authors. Most of the problems in the field of fluid mechanics, science and engineering studied through model them as fractional differential equations (FDEs) ([1] - [4]). Henceforth researchers have been involved in solving FDEs. Fractional differential equations have been found to be more realistic in modeling a variety of physical phenomena, engineering processes, biological systems, and financial products such as signal identification and image processing, optical systems, thermal system materials, control systems, etc. ([5], [6]). The great mathematicians such as Abel, Caputo, Euler, Fourier, Heaviside, Laplace, Leibniz, Riemann, Riesz and many more made foremost contributions to the fractional calculus.

Many of the researchers had been working for the solutions of FDEs, some of them as follows. Podlubny [7] has investigated the solution and offered a numerical technique in his book. Diethelm and Ford [8] solved the problem with Adams predictor and corrector method. Ray and Bera [9] constructed an analytical method using

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Adomian decomposition method for solving Bagley-Torvik equation. Mekkaoui and Hammouch [10] used the variational iteration method (VIM) and the fractional iteration method (FIM). Cenesiz et al. [11] proposed a new generalization of the Taylor collocation method for the numerical solution of a class of fractional-order differential equations. The Bessel collocation method used in [12] for solving Bagley-Torvik equation. A fractional Taylor vector approximation based numerical method developed and implemented by Krishnasamy and Razzaghi [13]. Pakdaman et al. [14] used an optimization technique (OT) based on training artificial neural network for solution of fractional-order differential equations. Balaji and Hariharan [15] applied wavelet method to solve fractional-order differential equations including the Bagley-Torvik equation.

Recently, the Haar wavelet method is applied for solving some of ordinary differential equations (ODEs) and partial differential equations (PDEs) by some of the researchers ([16] - [20]). Shiralashetti et al. [21] proposed Haar wavelet collocation method (HWCM) for the solution of multi-term FDEs. Continuation to that, in this paper we applied HWCM for the solution of some class of generalized FDEs and obtained solutions are compared with those exist in the literature and are excellent agreement with exact ones with higher accuracy.

The present paper is organized as follows; we start with brief summary about Haar wavelets and its operational matrix of integration of fractional order in section 2. Method of solution of the proposed technique is discussed in section 3. Section 4 provides illustrative examples with error analysis. Lastly, the conclusion of the proposed method is given section 5.

2. Haar wavelets

We have the simplest wavelet function i. e Haar wavelet. The scaling function $H_1(x)$ for the family of the Haar wavelets is defined as

$$H_1(x) = \begin{cases} 1 & for \quad x \in [0,1)\,, \\ 0 & Otherwise. \end{cases} \tag{1}$$

The Haar wavelet family for $x \in [0,1)$is defined as

$$H_i(x) = \begin{cases} 1 & for \quad x \in \left[\frac{l}{n}, \frac{l+0.5}{n}\right), \\ -1 & for \;\; x \in \left[\frac{l+0.5}{n}, \frac{l+1}{n}\right) \\ 0 & Otherwise. \end{cases} , \tag{2}$$

where $n = 2^m$, $m = 0, 1, \ldots, J$, is the resolution level of the wavelet and the numeral $l = 0, 1, \ldots, n-1$ represents the translation parameter. The maximum resolution level is J. The index i in left hand side (LHS) of Eq. (2) is computed by $i = n + l + 1$. In case of the minimal values of $n = 1$, $l = 0$ then $i = 2$. The maximum value of i is $N = 2^{J+1}$. Now we define the collocation points $x_j = \frac{j-0.5}{N}$, $j = 1, 2, \ldots, N$, and discretize the Haar function $H_i(x)$, then we can get Haar matrix $\mathrm{H}(i,j) = H_i(x)$, with the dimension $N \times N$.

If $J = 2 \Rightarrow N = 8$, using Eq. (2) we have Haar matrix

$$\mathrm{H}\,(8,8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

2.1. **Fractional order operational matrix of integration.** Now, we establish an operational matrix F_α of integration of the fractional order α using Eq. (2) is given by

$$F_{\alpha,\,i}(x) = \begin{cases} f_1 & for \quad x \in \left[\frac{l}{n}, \frac{l+0.5}{n}\right), \\ f_2 & for \quad x \in \left[\frac{l+0.5}{n}, \frac{l+1}{n}\right), \\ f_3 & for \quad x \in \left[\frac{l+1}{n}, 1\right), \\ 0 & Otherwise, \end{cases} \tag{3}$$

where $f_1 = \frac{1}{\Gamma(\alpha+1)}\left(x - \frac{l}{n}\right)^\alpha$, $f_2 = \frac{1}{\Gamma(\alpha+1)}\left\{\left(x - \frac{l}{n}\right)^\alpha - 2\left(x - \frac{l+0.5}{n}\right)^\alpha\right\}$ and

$$f_3 = \frac{1}{\Gamma(\alpha+1)}\left\{\left(x - \frac{l}{n}\right)^\alpha - 2\left(x - \frac{l+0.5}{n}\right)^\alpha + \left(x - \frac{l+1}{n}\right)^\alpha\right\}.$$

The integral matrix F_α have the elements $F_{\alpha,\,i}\,(x) = F_\alpha(i,j)$, If $J = 2 \Rightarrow N = 8$, from Eq. (3)

For instance$\alpha = 1/2$, we have

$$F_{1/2,\,i}\,(x) = F_{1/2}\,(8,8) = \begin{pmatrix} 0.2821 & 0.4886 & 0.6308 & 0.7464 & 0.8463 & 0.9356 & 1.0171 & 1.0925 \\ 0.2821 & 0.4886 & 0.6308 & 0.7464 & 0.2821 & -0.0416 & -0.2445 & -0.4002 \\ 0.2821 & 0.4886 & 0.0666 & -0.2309 & -0.1332 & -0.0685 & -0.0447 & -0.0323 \\ 0 & 0 & 0 & 0 & 0.2821 & 0.4886 & 0.0666 & -0.2309 \\ 0.2821 & -0.0756 & -0.0643 & -0.0266 & -0.0156 & -0.0106 & -0.0078 & -0.0061 \\ 0 & 0 & 0.2821 & -0.0756 & -0.0643 & -0.0266 & -0.0156 & -0.0106 \\ 0 & 0 & 0 & 0 & 0.2821 & -0.0756 & -0.0643 & -0.0266 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2821 & -0.0756 \end{pmatrix}$$

and for $\alpha = 3/2$, we have

$$F_{3/2,\,i}\,(x) = F_{3/2}\,(8,\,8) = \begin{pmatrix} 0.0118 & 0.0611 & 0.1314 & 0.2177 & 0.3174 & 0.4288 & 0.5509 & 0.6828 \\ 0.0118 & 0.0611 & 0.1314 & 0.2177 & 0.2938 & 0.3067 & 0.2881 & 0.2475 \\ 0.0118 & 0.0611 & 0.1079 & 0.0955 & 0.0663 & 0.0545 & 0.0476 & 0.0429 \\ 0 & 0 & 0 & 0 & 0.0118 & 0.0611 & 0.1079 & 0.0955 \\ 0.0118 & 0.0376 & 0.0210 & 0.0159 & 0.0134 & 0.0118 & 0.0107 & 0.0098 \\ 0 & 0 & 0.0118 & 0.0376 & 0.0210 & 0.0159 & 0.0134 & 0.0118 \\ 0 & 0 & 0 & 0 & 0.0118 & 0.0376 & 0.0210 & 0.0159 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0118 & 0.0376 \end{pmatrix}.$$

3. Method of Solution

Here, we can see the method of solution for the proposed scheme to solve generalized Bagley-Torvik equation, which has a special role in the study of behaviour of real materials where the motion of a rigid plate plunged in a Newtonian fluid. Consider the generalized Bagley-Torvik equation of the form,

$$a\,D^{\alpha}y(x) + bD^{3/2}y(x) + cy(x) = f(x), \tag{4}$$

with the initial conditions $y(0) = A,\ y'(0) = B$, where$1 < \alpha \leq 2,\ a, b, c \in R$ and $a \neq 0$.

The solution $y\,(x)$of the Eq. (4) can be obtained through the following procedure,

Step 1: Assume that

$$D^{\alpha}y\,(x) = \sum_{i=1}^{N} a_{\mathrm{i}}H_i(x), \tag{5}$$

where a_i's , $i = 1, 2, \ldots, N$ are Haar coefficients to be determined.

Step 2: By integrating twice Eq. (5) and using initial conditions, we get

$$Dy\,(x) = B + \sum_{i=1}^{N} a_{\mathrm{i}}F_{\alpha_1,\ i}(x),\ 0 <\ \alpha_1 \leq 1, \tag{6}$$

$$y\,(x) = A + Bx + \sum_{i=1}^{N} a_{\mathrm{i}}F_{\alpha,\ i}(x), \tag{7}$$

where $F_{\alpha_1,\ i}(x)$ and $F_{\alpha,\ i}(x)$ are operational matrices of integration of fractional order.

Step 3: Also, assume that

$$D^{3/2}y\,(x) = \sum_{i=1}^{N} a_{\mathrm{i}}F_{\alpha - 3/2,\ i}(x) \tag{8}$$

Step 4: Substituting Eq. (5) to Eq. (8) in Eq. (4), we get the following system of equations,

$$a\sum_{i=1}^{N} a_{\mathrm{i}}H_i(x) + b\sum_{i=1}^{N} a_{\mathrm{i}}F_{\alpha - 3/2,\ i}(x) + c\left(A + Bx + \sum_{i=1}^{N} a_{\mathrm{i}}F_{\alpha,\ i}(x)\right) = f(x). \tag{9}$$

Step 5: Solving Eq. (9) using MatLab, we get Haar wavelet coefficients a_i's. By substituting the values of a_i's in Eq. (7), we obtain the desired numerical solution of the given problem Eq. (4) using HWCM. The error will be calculated by using $E = |y_e - y_a|$ and $E_{\max} = \max|y_e - y_a|$, where $y_e \& y_a$ are exact and approximate solutions respectively.

4. Numerical Examples

Here, we applied the HWCM as the method discussed in the section 3 for the following examples.

Example 1. Consider the generalized Bagley-Torvik equation (4) with $a = b = c = 1$ and $f(x) = 1 + x$, we get

$$D^{\alpha}y(x) + D^{3/2}y(x) + y(x) = 1 + x \tag{10}$$

with the initial conditions $y(0) = 1,\ y'(0) = 1$.

As per the method explained in section 3, we obtained the HWCM solution $y(x)$ of Eq. (10) and is excellent agreement with the exact solution $y\,(x) = 1+x$ and better than the existing methods ([14], [22]) also with higher accuracy than the accuracy in figure 7 [14], which is shown in Fig. 1 and in Table 1. Comparison of numerical solution with the exact solution is presented in Fig. 2.

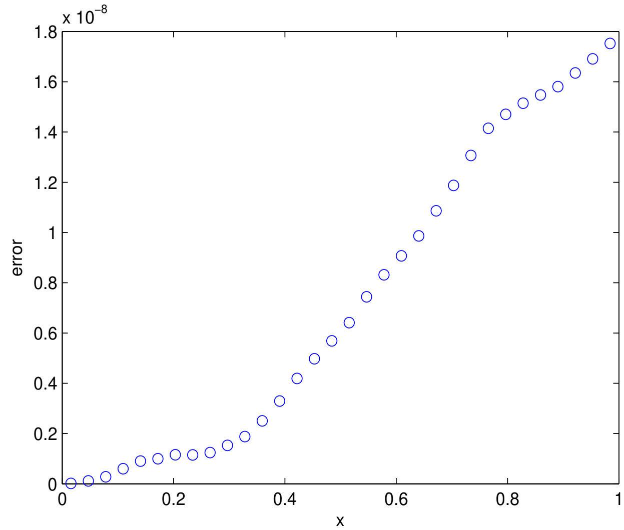

Fig. 1. Error analysis of HWCM of Example 1.

Table 1. Comparison of Numerical solution and maximum error of Example 1.

x	GA [22]	GA-PS [22]	OT [14]	HWCM	Exact	E_{GA} [22]	E_{GP} [22]	E_{OT} [14]	E_{HWCM}
0	1.024862	1.016007	1.000000	0.9999999944	1.0	2.30e−2	1.60e−2	0	5.6e−9
0.2	1.220821	1.199804	1.199997	1.1999999987	1.2	3.13e−2	1.95e−4	2.18e−6	1.3e−9
0.4	1.426952	1.401629	1.399996	1.3999999946	1.4	3.45e−2	1.62e−3	3.24e−6	5.4e−9
0.6	1.634569	1.607429	1.599997	1.5999999911	1.6	1.36e−2	7.42e−3	2.91e−6	8.9e−9
0.8	1.828738	1.799987	1.799997	1.7999999905	1.8	2.30e−2	1.27e−5	2.71e−6	9.5e−9
1	1.985057	1.953762	1.999995	1.9999999953	2.0	3.13e−2	4.62e−2	4.03e−6	4.7e−9

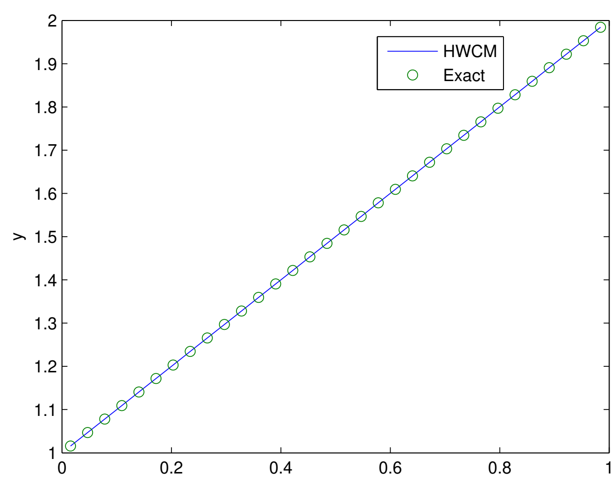

Fig. 2. Comparison of HWCM solution with exact solution for$J = 4$ of Example 1.

Example 2. Now consider the equation (4) with $a = b = c = 1$ and $f(x) = \Gamma(\alpha+1) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-3/2+1)} x^{\alpha-3/2} + x^{\alpha}$, we get,

$$D^{\alpha}y(x) + D^{3/2}y(x) + y(x) = \Gamma(\alpha+1) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-3/2+1)} x^{\alpha-3/2} + x^{\alpha} \qquad (11)$$

subjected to initial conditions $y(0) = 0$, $y'(0) = 0$.
As per the procedure explained in section 3, we get the HWCM solution $y(x)$ of Eq. (11) and is excellent agreement with the exact solution $y(x) = x^{\alpha}$ in higher accuracy, which is shown in Table 2. Comparison of numerical solutions with the exact solution is presented in Fig. 3. Also error analysis is shown in Fig. 4.

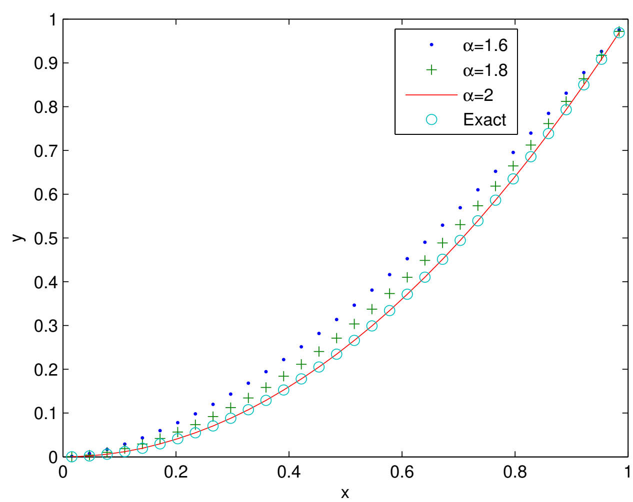

Fig. 3. Comparison of numerical solutions with exact solution of Example 2.

Table 2. The error analysis of HWCM for different values of α of Example 2.

$E_{\max} = \max\lvert y_e - y_a\rvert$			
J	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
3	1.0765e-07	3.0934e-08	8.8942e-09
4	5.2916e-08	2.6935e-08	7.2716e-09
5	3.1508e-08	1.4320e-08	7.1636e-09
6	1.3511e-08	1.2341e-08	3.4150e-09

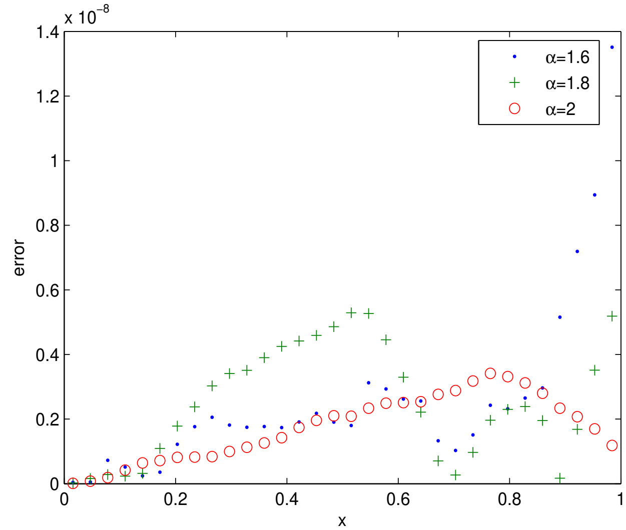

Fig. 4. Error analysis of HWCM of Example 2.

Example 3. Next, consider the α^{th} order Cauchy-type generalized FDE,

$$D^{\alpha}y(x) = y(x) + \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} - x^2, 0 < \alpha \leq 1 \tag{12}$$

with the initial condition $y(0) = 0$.
As per the previous examples, we have obtained the required HWCM solution $y(x)$ of Eq. (12) for different values of α and is excellent agreement with the exact solution $y(x) = x^2$, when $\alpha = 1$ and is presented in Fig. 5. The error analysis for different values of α is given in Table 3 and in Fig. 6.

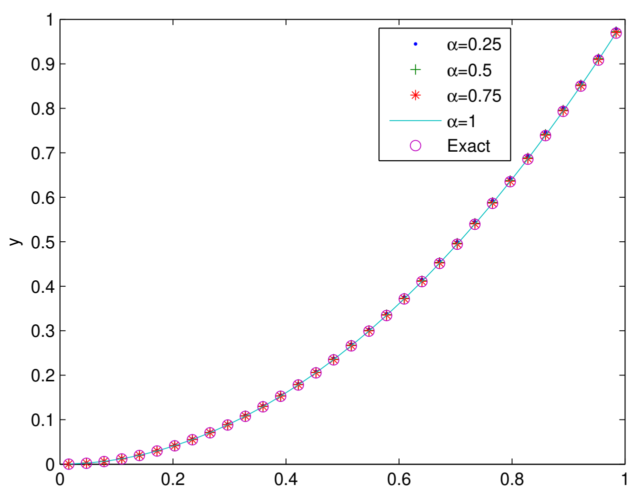

Fig. 5. Comparison of HWCM solution for different values of α with exact solution for $J = 4$ of Example 3.

Table 3. The error analysis of HWCM for different values of α of Example 3.

$E_{\max} = \max\|y_e - y_a\|$				
J	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
3	2.3381e-02	9.8169e-03	4.2784e-03	2.5750e-03
4	9.6934e-03	3.3744e-03	1.2124e-03	6.5349e-04
5	4.0308e-03	1.1640e-03	3.4414e-04	1.6463e-04
6	1.6789e-03	4.0427e-04	9.7818e-05	4.1317e-05

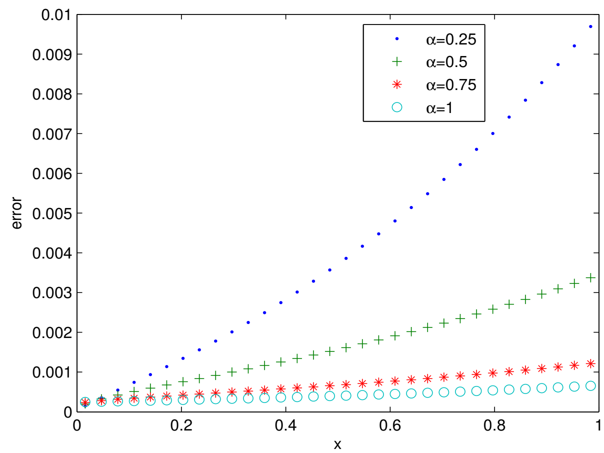

Fig. 6. Comparison of error analysis for different values of α of Example 3.

Example 4. Finally, consider the nonlinear Caputa-Katugampola FDE,

$$aD^2y(x) + bD^{\alpha_1}y(x) + cD^{\alpha_2}y(x) + Dy^3(x) = f(x), 0 < \alpha \leq 1 \tag{13}$$

with the initial condition $y(0) = 0$, $y'(0) = 0$, where $0 < \alpha_1 \leq 1$, $1 < \alpha_2 \leq 2$, $a, b, c, d \in R$.
As per previous examples, we obtained the desired HWCM solution $y(x)$ of Eq. (13) and is presented with the exact solution $y(x) = \frac{1}{3}x^3$ when $f(x) = \frac{2}{\Gamma(2)}x + \frac{2}{\Gamma(4-\alpha_1)}x^{3-\alpha_1} + \frac{2}{\Gamma(4-\alpha_2)}x^{3-\alpha_2} + \left(\frac{1}{3}x^3\right)^3$for different values of α_1 & α_2 and is shown in Fig. 7. The error analysis is given in Table 4 and in Fig. 8 for different values of α_1 & α_2.

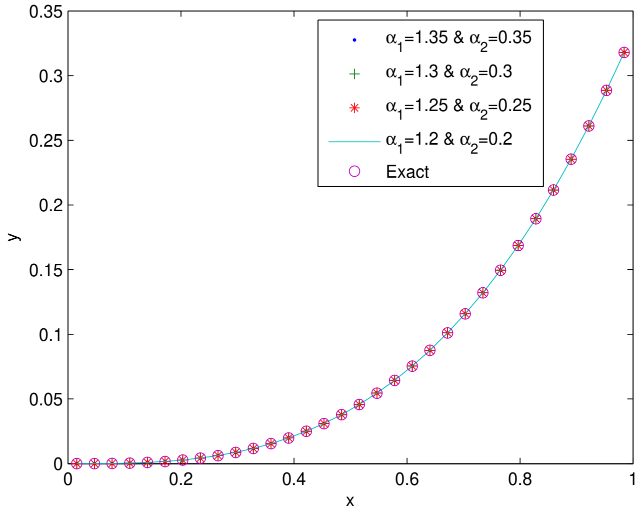

Fig. 7. Comparison of HWCM solution for different values of α_1 & α_2 with exact solution for $J = 4$ of Example 4.

Table 4. The error analysis of HWCM for different values of α_1 & α_2 of Example 4.

$E_{\max} = \max\|y_e - y_a\|$				
J	$\alpha_1 = 1.2$ & $\alpha_2 = 0.2$	$\alpha_1 = 1.25$ & $\alpha_2 = 0.25$	$\alpha_1 = 1.3$ & $\alpha_2 = 0.3$	$\alpha_1 = 1.35$ & $\alpha_2 = 0.35$
3	1.0373e-04	1.1859e-04	1.3547e-04	1.5442e-04
4	2.5398e-05	3.0038e-05	3.5360e-05	4.3427e-05
5	6.6382e-06	8.0853e-06	1.1583e-05	1.9663e-05
6	1.8429e-06	2.4181e-06	4.7515e-06	7.6689e-06

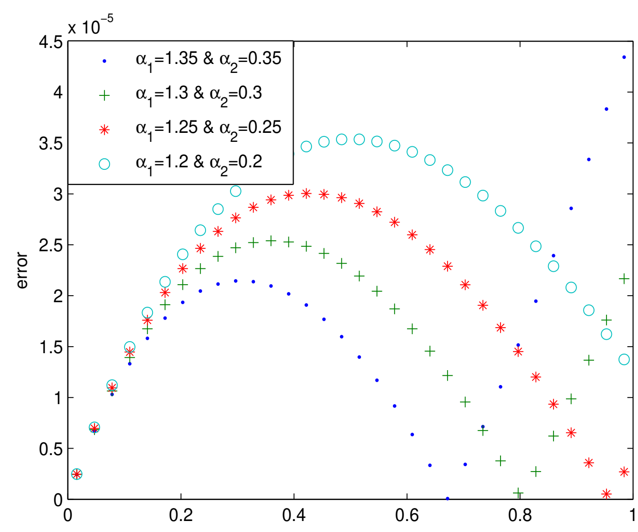

Fig. 8. Comparison of error analysis for different values of α_1 & α_2 of Example 4.

5. Conclusions

In this paper, we apply a numerical procedure for solving the mentioned some class of fractional differential equations using collocation method via Haar wavelets with the help of operational matrix of integration of fractional order. The obtained numerical solution using the proposed scheme is very efficient and effective which had been shown through tables and figures. Hence, the proposed technique is an effective approach to solve differential equations of fractional as well as integer order which have been modeled in the science and engineering.

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References

[1] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, vol. 198 Academic Press, San Diego, 1999.
[2] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[3] D. Baleanu, K. Diethelm, E. Scalas, J. Trujillo, Fractional Calculus Models and Numerical Methods, Series on Complexity, Nonlinearity and Chaos, World Scientific Publishing Company, Singapore, 2012.
[4] P. J. Torvik, R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, ASME J. Appl. Mech., 51 (2), 294-298, 1984.
[5] J. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, Commun. Nonlinear Sci. Numer. Simul., 16 (3), 1140-1153, 2011.
[6] C. Yang, T. Ji, Differential quadrature method for fractional Logistic differential equation, IAENG Int. J. Appl. Math., 48 (3), 342-348, 2018.
[7] I. Podlubny, Fractional Differential Equations, vol. 198. Academic press, New York, 1998.
[8] K. Diethelm, J. Ford, Numerical solution of the Bagley–Torvik equation, BIT Numer. Math., 42 (3), 490-507, 2002.
[9] S. S. Ray, R. K. Bera, Analytical solution of the Bagley–Torvik equation by Adomian decomposition method, Appl. Math. Comput., 168 (1), 398-410, 2005.
[10] T. Mekkaouii, Z. Hammouch, Approximate analytical solutions to the Bagley-Torvik equation by the fractional iteration method, Math. Comput. Sci., 39, 251-256, 2012.
[11] Y. Çenesiz, Y. Keskin, A. Kurnaz, The solution of the Bagley–Torvik equation with the generalized Taylor collocation method, J. Frankl. Inst., 347 (2), 452-466, 2010.
[12] S. Yuzbasi, Numerical solution of the Bagley–Torvik equation by the Bessel collocation method, Math. Methods Appl. Sci., 36 (3), 300-312, 2013.
[13] V. S. Krishnasamy, M. Razzaghi, The numerical solution of the Bagley–Torvik equation with fractional Taylor method, ASME J. Comput. Nonlinear Dyn., 11 (051010), 1-6, 2016.
[14] M. Pakdaman, A. Ahmadian, S. Effati, S. Salahshour, D. Baleanu, Solving differential equations of fractional order using an optimization technique based on training artificial neural network, Appl. Math. Comp., 293, 81-95, 2017.
[15] S. Balaji, G. Hariharan, An efficient operational matrix method for the numerical solutions of the fractional Bagley–Torvik equation using wavelets, J. Math. Chem., 57, 1885-1901, 2019.
[16] U. Lepik, Application of the Haar wavelet transform to solving integral and differential equations, Proc. Estonian Acad. Sci. Phys. Math., 56 (1), 28-46, 2007.
[17] S. Islam, I. Aziz, B. Sarler, The numerical solution of second-order boundary-value problems by collocation method with the Haar wavelets, Math. Comput. Model., 50, 1577-1590, 2010.
[18] S. C. Shiralashetti, A. B. Deshi, P. B. Mutalik Desai, Haar wavelet collocation method for the numerical solution of singular initial value problems, AIN Shams Engg. J., 7, 663-670, 2016.

[19] S. C. Shiralashetti, L. M. Angadi, A. B. Deshi, M. H. Kantli, Haar wavelet method for the numerical solution of Klein-Gordan equations, Asian-European J. Math. 9 (1), 1650012 (1-14), 2016.

[20] S. C. Shiralashetti, M. H. Kantli, A. B. Deshi, Haar wavelet based numerical solution of nonlinear differential equations arising in fluid dynamics, Int. J. Comput. Materials Sci. and Engg., 5 (2), 1650010 (1-13), 2016.

[21] S. C. Shiralashetti, A. B. Deshi, An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations, Nonlinear Dyn., 83, 293-303, 2016.

[22] M. A. Z. Raja, J. A. Khan, I. M. Qureshi, Solution of Fractional Order System of Bagley-Torvik Equation Using Evolutionary Computational Intelligence, Math. Probl. Engg., 2011, 675075 (1-18), 2011.

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