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ABSTRACT BIVARIATE RIGHT FRACTIONAL PSEUDO-POLYNOMIAL MONOTONE CONSTRAINED APPROXIMATION AND APPLICATIONS

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ABSTRACT. Here we extend our earlier right side bivariate high order simultaneous fractional monotone constrained approximation theory by pseudo-polynomials to right side abstract bivariate high order simultaneous fractional monotone constrained approximation by pseudo-polynomials, with applications to right side bivariate Prabhakar fractional calculus and non-singular kernel fractional calculi.

So we deal with the following general two-dimensional problem: Let f be a two variable continuously differentiable real valued function of a given order, let L^* be a linear right side abstract fractional mixed partial differential operator and suppose that $L^*(f) \geq 0$ on the critical region of $[-1,0]^2$. Then for specific and sufficiently large $n, m \in \mathbb{N}$, we can find a sequence of pseudo-polynomials $Q^*_{n,m}$ in two variables with the property $L^*\left(Q^*_{n,m}\right) \geq 0$ on this critical region of $[-1,0]^2$ such that f is approximated with rates fractionally and simultaneously by $Q^*_{n,m}$ in the uniform norm on the whole domain of f. This constrained approximation is given via inequalities involving the mixed modulus of smoothness $\omega_{s,q}$, $s, q \in \mathbb{N}$, of highest order integer partial derivative of f.

1. Introduction

The topic of monotone approximation theory started in [20] and it has become a major trend of approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [7] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1. *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume*

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$a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j} \right] \tag{1}$$

and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{4}$$

where C is independent of n or f.

Next let $n, m \in \mathbb{Z}_+$, P_θ denote the space of algebraic polynomials of degree $\leq \theta$. Consider the tensor product spaces $P_n \otimes C([-1, 1])$, $C([-1, 1]) \otimes P_m$ and their sum $P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m$, that is

$$P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m =$$
$$\left\{ \sum_{i=0}^{n} x^i A_i(y) + \sum_{j=0}^{m} B_j(x) y^j;\ A_i, B_j \in C([-1, 1]),\ x, y \in [-1, 1] \right\}. \tag{5}$$

This is the space of pseudo-polynomials of degree $\leq (n, m)$, first introduced by A. Marchaud in 1924-1927 (see [15], [16]). Here $f^{(k,l)}$ denotes $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}$, the (k, l)-partial derivative of f.

Here we consider the space $C^{r,p}\left([-1, 1]^2\right) = \{f : [-1, 1]^2 \to \mathbb{R};\ f^{(k,l)}$ is continuous for $0 \leq k \leq r,\ 0 \leq l \leq p\}$. Let $f \in C\left([-1, 1]^2\right)$; for $\delta_1, \delta_2 \geq 0$, define the mixed modulus of smoothness of order (s, q), $s, q \in \mathbb{N}$ (see [19], pp. 516-517) by

$$\omega_{s,q}(f; \delta_1, \delta_2) \equiv \sup \left\{ \left| {}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x, y) \right| : (x, y), \right. \tag{6}$$
$$\left. (x + sh_1, y + qh_2) \in [-1, 1]^2, |h_i| \leq \delta_i,\ i = 1, 2 \right\}.$$

Here

$${}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x, y) \equiv \sum_{\sigma=0}^{s} \sum_{\mu=0}^{q} (-1)^{s+q-\sigma-\mu} \binom{s}{\sigma} \binom{q}{\mu} f(x + \sigma h_1, y + \mu h_2) \tag{7}$$

is a mixed difference of order (s, q).

We mention

Theorem 2. *(H.H. Gonska [10]). Let $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, and $f \in C^{r,p}\left([-1, 1]^2\right)$. Let $n, m \in \mathbb{N}$ with $n \geq \max\{4(r + 1), r + s\}$ and $m \geq \max\{4(p + 1), p + q\}$. Then there exists a linear operator $Q_{n,m}$ from $C^{r,p}\left([-1, 1]^2\right)$ into the space of pseudopolynomials $(P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m)$ such that*

$$\left| (f - Q_{n,m}(f))^{(k,l)}(x, y) \right| \leq \tag{8}$$

$$M_{r,s} \cdot M_{p,q} \left(\Delta_n \left(x\right)\right)^{r-k} \cdot \left(\Delta_m \left(y\right)\right)^{p-l} \cdot \omega_{s,q} \left(f^{(r,p)}; \Delta_n \left(x\right), \Delta_m \left(y\right)\right),$$

for all $(0,0) \leq (k,l) \leq (r,p)$, $x, y \in [-1,1]$, *where*

$$\Delta_\theta \left(z\right) = \frac{\sqrt{1-z^2}}{\theta} + \frac{1}{\theta^2}, \quad \theta = n, m; \; z = x, y \in [-1,1]. \tag{9}$$

The constants $M_{r,s}$, $M_{p,q}$, *are independent of* f, (x,y) *and* (n,m); *they depend only on* (r,s), (p,q), *respectively.*

See also [11], saying that $Q_{n,m}^{(r,p)}(f)$ is continuous on $[-1,1]^2$.

We need the following result which is an easy consequence of the last theorem (see [19], p. 517).

Corollary 3. *([1]) Let* $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, *and* $f \in C^{r,p}\left([-1,1]^2\right)$. *Let* $n, m \in \mathbb{N}$ *with* $n \geq \max\{4(r+1), r+s\}$ *and* $m \geq \max\{4(p+1), p+q\}$. *Then there exists a pseudopolynomial* $Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$ *such that*

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{\widetilde{C}}{n^{r-k} m^{p-l}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{10}$$

for all $(0,0) \leq (k,l) \leq (r,p)$. *Here the constant* $\widetilde{C}$ *depends only on* r, p, s, q.

Corollary 3 was used in the proof of the motivational result that follows.

Theorem 4. *([1]) Let* h_1, h_2, v_1, v_2, r, p *be integers,* $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ *and let* $f \in C^{r,p}\left([-1,1]^2\right)$, *with* $f^{(r,p)}$ *having a mixed modulus of smoothness* $\omega_{s,q}\left(f^{(r,p)}; x, y\right)$ *there,* $s, q \in \mathbb{N}$. *Let* $\alpha_{i,j}(x,y)$, $i = h_1, h_1+1, ..., v_1$; $j = h_2, h_2+1, ..., v_2$ *be real-valued functions, defined and bounded in* $[-1,1]^2$ *and suppose* $\alpha_{h_1 h_2}$ *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$ *throughout* $[-1,1]^2$. *Take the operator*

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \tag{11}$$

and assume, throughout $[-1,1]^2$ *that*

$$L(f) \geq 0. \tag{12}$$

Then for any integers n, m *with* $n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1), p+q\}$, *there exists a pseudopolynomial*

$$Q_{n,m} \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that $L(Q_{m,n}) \geq 0$ *throughout* $[-1,1]^2$ *and*

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{C}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{13}$$

for all $(0,0) \leq (k,l) \leq (h_1,h_2)$. *Moreover we get*

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{C}{n^{r-k} m^{p-l}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{14}$$

for all $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$. *Also (14) is valid whenever* $0 \leq k \leq h_1$, $h_2+1 \leq l \leq p$ *or* $h_1+1 \leq k \leq r$, $0 \leq l \leq h_2$. *Here* C *is a constant independent of* f *and* n, m. *It depends only on* r, p, s, q, L.

We are also motivated by [2].

We mention

Definition 5. *(see* [13]*) Let* $[-1,1]^2$; $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C\left([-1,1]^2\right)$, $x = (x_1, x_2)$, $t = (t_1, t_2) \in [-1,1]^2$. *We define the right mixed Riemann-Liouville fractional two dimensional integral of order* α

$$\left(I_{1-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2, \tag{15}$$

with $x_1, x_2 < 1$.

Notice here that $I_{1-}^{\alpha}(|f|) < \infty$.

Definition 6. *(*[3]*, pp. 15-31) Let* $\alpha_1, \alpha_2 > 0$ *with* $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, *(*$\lceil \cdot \rceil$ *ceiling of the number). Let here* $f \in C^{m_1, m_2}\left([-1,1]^2\right)$. *We consider the right Caputo type fractional partial derivative:*

$$D_{1-}^{(\alpha_1, \alpha_2)} f(x) := \frac{(-1)^{m_1 + m_2}}{\Gamma(m_1 - \alpha_1)\Gamma(m_2 - \alpha_2)} \cdot$$

$$\int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{m_1 - \alpha_1 - 1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1 + m_2} f(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} dt_1 dt_2, \tag{16}$$

$\forall\ x = (x_1, x_2) \in [-1,1]^2$, *where* Γ *is the gamma function*

$$\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu - 1} dt, \quad \nu > 0. \tag{17}$$

We set

$$D_{1-}^{(0,0)} f(x) := f(x), \quad \forall\ x \in [-1,1]^2; \tag{18}$$

$$D_{1-}^{(m_1, m_2)} f(x) := (-1)^{m_1 + m_2} \frac{\partial^{m_1 + m_2} f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}}, \ \forall\ x \in [-1,1]^2. \tag{19}$$

Definition 7. *(*[3]*, pp. 15-31) We also set*

$$D_{1-}^{(0, \alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^{1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_2} f(x_1, t_2)}{\partial t_2^{m_2}} dt_2, \tag{20}$$

$$D_{1-}^{(\alpha_1, 0)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^{1} (t_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1} f(t_1, x_2)}{\partial t_1^{m_1}} dt_1, \tag{21}$$

and

$$D_{1-}^{(m_1, \alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^{1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1 + m_2} f(x_1, t_2)}{\partial x_1^{m_1} \partial t_2^{m_2}} dt_2, \tag{22}$$

$$D_{1-}^{(\alpha_1, m_2)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^{1} (t_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1 + m_2} f(t_1, x_2)}{\partial t_1^{m_1} \partial x_2^{m_2}} dt_1. \tag{23}$$

In [3], pp. 15-31, we extended Theorem 4 to the fractional level. Indeed there L is replaced by $\overline{L}$, a linear right Caputo fractional mixed partial differential operator. Now the monotonicity property holds true only on the critical square of $[-1,0]^2$. Simultaneously fractional convergence remains true on all of $[-1,1]^2$.

So we have proved there

Theorem 8. *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}\left([-1,1]^2\right)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}\left(f^{(r,p)}; x, y\right)$ there, $s, q \in \mathbb{N}$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, ..., v_1$; $j = h_2, h_2+1, ..., v_2$ be real valued functions, defined and bounded in $[-1,1]^2$ and suppose $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[-1,0]^2$. Assume that $h_1 + h_2 = 2\gamma, \gamma \in \mathbb{Z}_+$. Here $n, m \in \mathbb{N} : n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1), p+q\}$. Set*

$$l_{ij} := \sup_{(x,y)\in[-1,1]^2} \left|\alpha_{h_1 h_2}^{-1}(x,y)\, \alpha_{ij}(x,y)\right| < \infty, \tag{24}$$

for all $h_1 \leq i \leq v_1$, $h_2 \leq j \leq v_2$. Let $\alpha_{1i}, \alpha_{2j} \geq 0$ with $\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0, 1, ..., r$; $j = 0, 1, ..., p$, ($\lceil \cdot \rceil$ ceiling of the number), $\alpha_{10} = 0$, $\alpha_{20} = 0$.

Consider the right fractional bivariate differential operator

$$\overline{L} := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y)\, D_{1-}^{(\alpha_{1i}, \alpha_{2j})}. \tag{25}$$

Assume $\overline{L} f(x,y) \geq 0$, on $[-1,0]^2$.

Then there exists

$$Q_{n,m}^* \equiv Q_{n,m}^*(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that $\overline{L} Q_{n,m}^*(x,y) \geq 0$, on $[-1,0]^2$.

Furthermore it holds:

1)

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})}(f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty,[-1,1]^2} \leq$$

$$\frac{\widetilde{C} 2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\,\Gamma(j-\alpha_{2j}+1)\, n^{r-i} m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{26}$$

where $\widetilde{C}$ is a constant that depends only on r, p, s, q; $(h_1+1, h_2+1) \leq (i,j) \leq (r,p)$, or $0 \leq i \leq h_1$, $h_2+1 \leq j \leq p$, or $h_1+1 \leq i \leq r$, $0 \leq j \leq h_2$,

2)

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})}(f) - D_{1-}^{(\alpha_{11}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty,[-1,1]^2} \leq$$

$$\frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{27}$$

for $(1,1) \leq (i,j) \leq (h_1, h_2)$, where $c_{ij} = \widetilde{C} A_{ij}$, with

$$A_{ij} :=$$

$$\left\{ \left[\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} \frac{l_{\tau\mu} 2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau - a_{1\tau}+1)\,\Gamma(\mu-\alpha_{2\mu}+1)} \right] \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\,\Gamma(h_1-\alpha_{1i}-k+1)} \right) \cdot \right. \tag{28}$$

$$\left. \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\,\Gamma(h_2-\alpha_{2j}-\lambda+1)} \right) + \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\,\Gamma(j-\alpha_{2j}+1)} \right\},$$

3)

$$\left\| f - Q_{n,m}^* \right\|_{\infty,[-1,1]^2} \leq \frac{c_{00}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{29}$$

where $c_{00} := \widetilde{C} A_{00}$, *with*

$$A_{00} := \frac{1}{h_1! h_2!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau - a_{1\tau} + 1)\Gamma(\mu - \alpha_{2\mu} + 1)} \right) + 1,$$

4)

$$\left\| D_{1-}^{(0,\alpha_{2j})}(f) - D_{1-}^{(0,\alpha_{2j})} Q_{n,m}^{*} \right\|_{\infty,[-1,1]^2} \leq$$

$$\frac{c_{0j}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{30}$$

where $c_{0j} = \widetilde{C} A_{0j}$, $j = 1, ..., h_2$, *with*

$$A_{0j} := \left[\frac{1}{h_1!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau - a_{1\tau} + 1)\Gamma(\mu - \alpha_{2\mu} + 1)} \right) \right. \tag{31}$$

$$\left. \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2 - \alpha_{2j} - \lambda + 1)} \right) + \frac{2^{j-\alpha_{2j}}}{\Gamma(j - \alpha_{2j} + 1)} \right],$$

and

5)

$$\left\| D_{1-}^{(\alpha_{1i},0)}(f) - D_{1-}^{(\alpha_{1i},0)} Q_{n,m}^{*} \right\|_{\infty,[-1,1]^2} \leq$$

$$\frac{c_{i0}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{32}$$

where $c_{i0} = \widetilde{C} A_{i0}$, $i = 1, ..., h_1$, *with*

$$A_{i0} := \left[\frac{1}{h_2!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau - a_{1\tau} + 1)\Gamma(\mu - \alpha_{2\mu} + 1)} \right) \right. \cdot \tag{33}$$

$$\left. \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1 - \alpha_{1i} - k + 1)} \right) + \frac{2^{i-\alpha_{1i}}}{\Gamma(i - \alpha_{1i} + 1)} \right].$$

In this article we establish right side bivariate abstract fractional calculus high order monotone (constrained) approximation theory by pseudo-polynomials of Caputo type, and then we apply our results to bivariate Prabhakar fractional Calculus, bivariate generalized non-singular fractional calculus, and bivariate parametrized Caputo-Fabrizio non-singular fractional calculus.

Next, we build the related necessary fractional calculi background.

2. Bivariate right side Fractional Calculi

Here we need to be very specific in preparation for our main results.

2.1. **Bivariate right Abstract Fractional Calculus.** Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}\left([-1,1]^2\right)$. Here $h_1 \leq i \leq v_1$, $h_2 \leq j \leq v_2$. Let $\alpha_{1i}, \alpha_{2j} \geq 0$, $\alpha_{1i}, \alpha_{2j} \notin \mathbb{Z}_+ : \lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0, 1, ..., r$; $j = 0, 1, ..., p$, ($\lceil \cdot \rceil$ is the ceiling of number), $\alpha_{10} = 0$, $\alpha_{20} = 0$.

Consider also the integrable functions $k_{1i} := K_{\alpha_{1i}}, k_{2j} := K_{\alpha_{2j}} : [0,2] \to \mathbb{R}_+$, $i = 0, 1, ..., r$; $j = 0, 1, ..., p$.

We consider the abstract right side Caputo type bivariate fractional partial derivative of orders $(\alpha_{1i}, \alpha_{2j})$

$$ {}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{1-} f(x) := (-1)^{i+j} \int_{x_1}^{1} \int_{x_2}^{1} k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2)\, \frac{\partial^{i+j} f(t_1, t_2)}{\partial t_1^i \partial t_2^j} dt_1 dt_2, \tag{34} $$

$\forall\ x = (x_1, x_2) \in [-1,1]^2$.

We set

$$ {}^{k_{10}}_{k_{20}}D^{(0,0)}_{1-} f(x) := f(x), $$
$$ {}^{k_{1i}}_{k_{2j}}D^{(i,j)}_{1-} f(x) := (-1)^{i+j} \frac{\partial^{i+j} f(x_1,x_2)}{\partial x_1^i \partial x_2^j}, \quad \forall\ x = (x_1, x_2) \in [-1,1]^2. \tag{35} $$

We also set

$$ {}^{k_{1i}}_{k_{2j}}D^{(i,\alpha_{2j})}_{1-} f(x) := (-1)^{j} \int_{x_2}^{1} k_{2j}(t_2 - x_2) \frac{\partial^{i+j} f(x_1, t_2)}{\partial x_1^i \partial t_2^j} dt_2, \tag{36} $$

$$ {}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},j)}_{1-} f(x) := (-1)^{i} \int_{x_1}^{1} k_{1i}(t_1 - x_1) \frac{\partial^{i+j} f(t_1, x_2)}{\partial t_1^i \partial x_2^j} dt_1, \tag{37} $$

and in particular we define:

$$ {}^{k_{10}}_{k_{2j}}D^{(0,\alpha_{2j})}_{1-} f(x) := (-1)^{j} \int_{x_2}^{1} k_{2j}(t_2 - x_2) \frac{\partial^{j} f(x_1, t_2)}{\partial t_2^j} dt_2, \tag{38} $$

$$ {}^{k_{1i}}_{k_{20}}D^{(\alpha_{1i},0)}_{1-} f(x) := (-1)^{i} \int_{x_1}^{1} k_{1i}(t_1 - x_1) \frac{\partial^{i} f(t_1, x_2)}{\partial t_1^i} dt_1, \tag{39} $$

$\forall\ x = (x_1, x_2) \in [-1,1]^2$.

We will assume that

$$ \int_0^1 k_{ih_i}(z)\, dz \geq 1, \ \text{ when } h_i \neq 0, \tag{40} $$

where $i = 1, 2$.

In [3], we got that $0 < \Gamma(h_1 - \alpha_{1h_1} + 1), \Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1$, where Γ is the gamma function, and there it is

$$ k_{ih_i}(z) = \frac{z^{h_i - \alpha_{ih_i} - 1}}{\Gamma(h_i - \alpha_{ih_i})}, \quad i = 1, 2; \ \forall\ z \in [0,2], $$

and (40) is fulfilled.

2.2. **About right Bivariate Fractional Calculus.** We consider the Prabhakar function (also known as the three parameter Mittag-Leffler function), (see [12], p. 97; [9])

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha k+\beta)}z^k, \tag{41}$$

where $\alpha,\beta>0$, $\gamma\in\mathbb{R}$, $z\in\mathbb{R}$, and $(\gamma)_k=\gamma(\gamma+1)...(\gamma+k-1)$, it is $E_{\alpha,\beta}^{0}(z)=\frac{1}{\Gamma(\beta)}$.

Let $a,b,c,d\in\mathbb{R}$, $a<b$, $c<d$; $f\in C^{N_1,N_2}([a,b]\times[c,d])$, $\rho_i,\mu_i>0$; $\gamma_i,\omega_i\in\mathbb{R}$; $N_i=\lceil\mu_i\rceil$, $\mu_i\notin\mathbb{N}$; $i=1,2$.

We define the bivariate Prabhakar-Caputo right partial fractional derivative of orders (μ_1,μ_2) as follows ($x=(x_1,x_2)\in[a,b]\times[c,d]$):

$$\left({}^{C}D_{(\rho_1,\rho_2),(\mu_1,\mu_2),(\omega_1,\omega_2),(b-,d-)}^{(\gamma_1,\gamma_2)}f\right)(x)= \tag{42}$$

$$(-1)^{N_1+N_2}\int_{x_1}^{b}\int_{x_2}^{d}(t_1-x_1)^{N_1-\mu_1-1}(t_2-x_2)^{N_2-\mu_2-1}$$

$$E_{\rho_1,N_1-\mu_1}^{-\gamma_1}\left[\omega_1(t_1-x_1)^{\rho_1}\right]E_{\rho_2,N_2-\mu_2}^{-\gamma_2}\left[\omega_2(t_2-x_2)^{\rho_2}\right]\frac{\partial^{N_1+N_2}f(t_1,t_2)}{\partial t_1^{N_1}\partial t_2^{N_2}}dt_1dt_2,$$

with

$$\left({}^{C}D_{(\rho_1,\rho_2),(0,0),(\omega_1,\omega_2),(b-,d-)}^{(\gamma_1,\gamma_2)}f\right)(x):=f(x);$$

$$\left({}^{C}D_{(\rho_1,\rho_2),(N_1,N_2),(\omega_1,\omega_2),(b-,d-)}^{(\gamma_1,\gamma_2)}f\right)(x):=(-1)^{N_1+N_2}\frac{\partial^{N_1+N_2}f(x_1,x_2)}{\partial x_1^{N_1}\partial x_2^{N_2}}, \tag{43}$$

where $N_1,N_2\in\mathbb{N}$, etc.

For the related univariate theory see [5], [17].

2.3. **About right generalized non-singular Fractional Calculus.** Here we use the multivariate analogue of generalized Mittag-Leffler function, see [18], defined for $\lambda,\gamma_j,\rho_j,z_j\in\mathbb{C}$, $Re(\rho_j)>0$ $(j=1,...,m)$ in terms of a multiple series of the form:

$$E_{(\rho_j),\lambda}^{(\gamma_j)}(z_1,...,z_m)=E_{(\rho_1,...,\rho_m),\lambda}^{(\gamma_1,...,\gamma_m)}(z_1,...,z_m)=$$

$$\sum_{k_1,...,k_m=0}^{\infty}\frac{(\gamma_1)_{k_1}...(\gamma_m)_{k_m}}{\Gamma\left(\lambda+\sum_{j=1}^{m}k_j\rho_j\right)}\frac{z_1^{k_1}...z_m^{k_m}}{k_1!...k_m!}, \tag{44}$$

where $(\gamma_j)_{k_j}$ is the Pochhammer symbol. By [21], p. 157, (44) converges for $Re(\rho_j)>0$, $j=1,...,m$.

In particular we will use $\theta=1,2$; $E_{(\rho_\theta,...,\rho_\theta),\lambda_\theta}^{(\gamma_{\theta1},...,\gamma_{\theta m})}\left[\omega_{\theta1}t_\theta^{\rho_\theta},...,\omega_{\theta m}t_\theta^{\rho_\theta}\right]$, denoted by $E_{(\rho_\theta),\lambda_\theta}^{(\gamma_{\theta j})}\left[\omega_{\theta1}t_\theta^{\rho_\theta},...,\omega_{\theta m}t_\theta^{\rho_\theta}\right]$, where $0<\rho_\theta<1$, $t_\theta\geq0$, $\lambda_\theta>0$, $\gamma_{\theta j}\in\mathbb{R}$ with $(\gamma_{\theta j})_{k_{\theta j}}:=\gamma_{\theta j}(\gamma_{\theta j}+1)...(\gamma_{\theta j}+k_{\theta j}-1)$, $\omega_{\theta j}\in\mathbb{R}-\{0\}$, for $j=1,...,m$.

Let $f\in C^{N_1,N_2}\left([-1,1]^2\right)$, $0<\mu_\theta\notin\mathbb{N}$ and $\lceil\mu_\theta\rceil=N_\theta\in\mathbb{N}$; $\theta=1,2$.

We define the bivariate Caputo type generalized right partial fractional derivative with non-singular kernel of order (μ_1,μ_2), as follows:

$$D_{1-}^{(\mu_1,\mu_2)}f(x):={}_{(\gamma_{2j})(\omega_{2j})}^{(\gamma_{1j})(\omega_{1j})CA}D_{1-}^{(\mu_1,\mu_2),(\lambda_1,\lambda_2)}f(x):=\frac{A(1-n_1+\mu_1,1-N_2+\mu_2)}{(N_1-\mu_1)(N_2-\mu_2)}$$

$$(-1)^{N_1+N_2}\int_{x_1}^{1}\int_{x_2}^{1}\prod_{\theta=1}^{2}E_{(1-N_\theta+\mu_\theta),\lambda_\theta}^{(\gamma_{\theta j})}\left[\frac{-\omega_{\theta 1}(1-N_\theta+\mu_\theta)}{N_\theta-\mu_\theta}(t_\theta-x_\theta)^{1-N_\theta+\mu_\theta}\right.$$

$$\left.,...,\frac{-\omega_{\theta m}(1-N_\theta+\mu_\theta)}{N_\theta-\mu_\theta}(t_\theta-x_\theta)^{1-N_\theta+\mu_\theta}\right]\frac{\partial^{N_1+N_2}f(t_1,t_2)}{\partial t_1^{N_1}\partial t_2^{N_2}}dt_1dt_2, \quad (45)$$

$\forall\, x=(x_1,x_2)\in[-1,1]^2$, where $A:=A(1-n_1+\mu_1,1-N_2+\mu_2)$ is a normalizing constant. Without loss of generality we assume that $A>0$.

We set $D_{1-}^{(0,0)}f=f$, $D_{1-}^{(N_1,N_2)}f=(-1)^{N_1+N_2}\frac{\partial^{N_1+N_2}f}{\partial x_1^{N_1}\partial x_2^{N_2}}$, when $N_1,N_2\in\mathbb{N}$, etc.

For the univariate theory see the related [4], [6], [8].

2.4. **Bivariate right parametrized Caputo-Fabrizio type non-singular kernel left partial fractional derivative of orders** (μ_1,μ_2) **:.** Let $f\in C^{N_1,N_2}\left([-1,1]^2\right)$, $\mathbb{N}\not\ni\mu_1,\mu_2>0$, $\lceil\mu_\theta\rceil=N_\theta\in\mathbb{N}$; $\omega_\theta<0$; $\theta=1,2$.

It is given by

$${}_{(\omega_1,\omega_2)}^{CF}D_{1-}^{(\mu_1,\mu_2)}f(x):=\frac{(-1)^{N_1+N_2}}{(N_1-\mu_1)(N_2-\mu_2)}$$

$$\int_{x_1}^{1}\int_{x_2}^{1}\prod_{\theta=1}^{2}\left[\exp\left(-\frac{(1-N_\theta+\mu_\theta)\,\omega_\theta}{N_\theta-\mu_\theta}(t_\theta-x_\theta)\right)\right]\frac{\partial^{N_1+N_2}f(t_1,t_2)}{\partial t_1^{N_1}\partial t_2^{N_2}}dt_1dt_2, \quad (46)$$

$\forall\, x=(x_1,x_2)\in[-1,1]^2$, and ${}_{(\omega_1,\omega_2)}^{CF}D_{1-}^{(0,0)}f=f$,
${}_{(\omega_1,\omega_2)}^{CF}D_{1-}^{(N_1,N_2)}f=(-1)^{N_1+N_2}\frac{\partial^{N_1+N_2}f}{\partial x_1^{N_1}\partial x_2^{N_2}}$, etc.

For the univariate case see [6], [14].

We make

Remark 9. *Right Fractional Calculi 2.2-2.4 are special cases of the right abstract fractional calculus 2.1. The abstract important condition (40) is fulfilled by: in section 2.2 for large enough $\omega_\theta>0$, $\theta=1,2$; in section 2.3 for small enough $\omega_{\theta j}<0$, $\theta=1,2$; $j=1,...,m$; and in section 2.4 for small enough $\omega_\theta<0$, $\theta=1,2$. For details see* [6].

3. Main Result

We present the following right bivariate abstract fractional monotone (constrained) approximation result.

Theorem 10. *Let h_1,h_2,v_1,v_2,r,p be integers, $0\leq h_1\leq v_1\leq r$, $0\leq h_2\leq v_2\leq p$ and let $f\in C^{r,p}\left([-1,1]^2\right)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}\left(f^{(r,p)};\delta_1,\delta_2\right),\delta_1,\delta_2>0$ there, $s,q\in\mathbb{N}$. Let $\alpha_{ij}(x,y)$, $i=h_1,h_1+1,...,v_1$; $j=h_2,h_2+1,...,v_2$ be real-valued functions, defined and bounded in $[-1,1]^2$ and suppose $\alpha_{h_1h_2}$ is either $\geq\alpha>0$ or $\leq\beta<0$ throughout $[-1,0]^2$. Assume that h_1, h_2 are even. Here $n,m\in\mathbb{N}:n\geq\max\{4(r+1),r+s\}$, $m\geq\max\{4(p+1),p+q\}$. Set*

$$l_{ij}:=\sup_{(x,y)\in[-1,1]^2}\left|\alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)\right|<\infty \quad (47)$$

for all $h_1\leq i\leq v_1$, $h_2\leq j\leq v_2$. Let $0<\alpha_{1i},\alpha_{2j}\notin\mathbb{N}$ with $\lceil\alpha_{1i}\rceil=i$, $\lceil\alpha_{2j}\rceil=j$, $j=1,...,r$; $j=1,...,p$ ($\lceil\cdot\rceil$ is the ceiling of the number) and $\alpha_{10}=\alpha_{20}=0$.

Consider the abstract right fractional bivariate differential operator

$$L^{*}:=\sum_{i=h1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{ij}\left(x,y\right)\ {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}. \tag{48}$$

Assume $L^{*}f\left(x,y\right)\geq 0,$ *on* $\left[-1,0\right]^{2}$ *.There exists a pseudo-polynomial of degree* $\leq (n,m)$

$$Q_{n,m}^{*}:=Q_{n,m}^{*}\left(f\right)\in\left(P_{n}\otimes C\left(\left[-1,1\right]\right)+C\left(\left[-1,1\right]\right)\otimes P_{m}\right)$$

such that $L^{*}Q_{n,m}^{*}\left(x,y\right)\geq 0,$ *on* $\left[-1,0\right]^{2}.$

We set

$$\lambda_{1i}:=\int_{0}^{2}k_{1i}\left(z\right)dz,\quad \lambda_{2j}:=\int_{0}^{2}k_{2j}\left(z\right)dz, \tag{49}$$

for $i=1,...,r;\ j=1,...,p.$

Set also $\lambda_{10}=\lambda_{20}=1.$

Furthermore it holds:

1) if $(h_1+1,h_2+1)\leq(i,j)\leq(r,p),$ *or* $0\leq i\leq h_1,\ h_2+1\leq j\leq p,$ *or* $h_1+1\leq i\leq r,\ 0\leq j\leq h_2,$ *then*

$$\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f-\ {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq$$

$$\lambda_{1i}\lambda_{2j}\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{50}$$

2) if $(1,1)\leq(i,j)\leq(h_1,h_2),$ *then*

$$\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f-\ {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}$$

$$\left\{\left(\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}\right)\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right\|_{\infty,[-1,1]^2}+\lambda_{1i}\lambda_{2j}\right\}, \tag{51}$$

3) it holds

$$\left\|f-Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}\left[\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}+1\right], \tag{52}$$

4) when $i=0,\ j=1,...,h_2,$ *we get*

$$\left\|{}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}f-\ {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}$$

$$\left[\left(\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}\right)\left\|{}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}y^{h_2}\right\|_{\infty,[-1,1]}+\lambda_{2j}\right], \tag{53}$$

and

5) the case of $j=0,\ i=1,...,h_1$ *follows, it holds:*

$$\left\|{}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}f-\ {}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}$$

$$\left[\left(\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2} l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}\right)\left\| {}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}x^{h_1}\right\|_{\infty,[-1,1]} + \lambda_{1i}\right]. \tag{54}$$

Proof. By Corollary 3 there exists

$$Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that

$$\left\| f^{(i,j)} - Q_{n,m}^{(i,j)}\right\|_\infty \leq \frac{\widetilde{C}}{n^{r-i}m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{55}$$

for all $(0,0) \leq (i,j) \leq (r,p)$, while $Q_{n,m} \in C^{r,p}\left([-1,1]^2\right)$. Here $\widetilde{C}$ depends only on r,p,s,q, where $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$, with $r,p \in \mathbb{Z}_+,\ s,q \in \mathbb{N},\ f \in C^{r,p}\left([-1,1]^2\right)$.

Indeed by [11] we have that $Q_{n,m}^{(r,p)}$ is continuous on $[-1,1]^2$.

We observe the following $(i = 1,...,r;\ j = 1,...,p)$

$$\left| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f(x_1,x_2) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}(x_1,x_2)\right| =$$

$$\Bigg|(-1)^{i+j}\int_{x_1}^1\int_{x_2}^1 k_{1i}(t_1-x_1)\,k_{2j}(t_2-x_2)\,\frac{\partial^{i+j}f(t_1,t_2)}{\partial t_1^i\partial t_2^j}dt_1dt_2 -$$

$$(-1)^{i+j}\int_{x_1}^1\int_{x_2}^1 k_{1i}(t_1-x_1)\,k_{2j}(t_2-x_2)\,\frac{\partial^{i+j}Q_{n,m}(t_1,t_2)}{\partial t_1^i\partial t_2^j}dt_1dt_2\Bigg| = \tag{56}$$

$$\left|\int_{x_1}^1\int_{x_2}^1 k_{1i}(t_1-x_1)\,k_{2j}(t_2-x_2)\left(\frac{\partial^{i+j}f(t_1,t_2)}{\partial t_1^i\partial t_2^j} - \frac{\partial^{i+j}Q_{n,m}(t_1,t_2)}{\partial t_1^i\partial t_2^j}\right)dt_1dt_2\right| \leq$$

$$\int_{x_1}^1\int_{x_2}^1 k_{1i}(t_1-x_1)\,k_{2j}(t_2-x_2)\left|\frac{\partial^{i+j}f(t_1,t_2)}{\partial t_1^i\partial t_2^j} - \frac{\partial^{i+j}Q_{n,m}(t_1,t_2)}{\partial t_1^i\partial t_2^j}\right|dt_1dt_2 \leq$$

$$\left(\int_{x_1}^1\int_{x_2}^1 k_{1i}(t_1-x_1)\,k_{2j}(t_2-x_2)\,dt_1dt_2\right)\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) =$$

$$\left(\int_{x_1}^1 k_{1i}(t_1-x_1)\,dt_1\right)\left(\int_{x_2}^1 k_{2j}(t_2-x_2)\,dt_2\right)\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) = \tag{57}$$

$$\left(\int_0^{1-x_1} k_{1i}(z)\,dz\right)\left(\int_0^{1-x_2} k_{2j}(z)\,dz\right)\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \leq$$

$$\left(\int_0^2 k_{1i}(z)\,dz\right)\left(\int_0^2 k_{2j}(z)\,dz\right)\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right).$$

We have proved that

$$\left| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f(x_1,x_2) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}(x_1,x_2)\right| \leq \tag{58}$$

$$\left(\int_0^2 k_{1i}(z)\,dz\right)\left(\int_0^2 k_{2j}(z)\,dz\right)\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right),$$

$\forall\ (x_1,x_2) \in [-1,1]^2;\ i = 1,...,r;\ j = 1,...,p.$

So we have proved that there exists $Q_{n,m}$ such that

$$\left\| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}(f) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}(Q_{n,m}) \right\|_{\infty,[-1,1]^2} \leq \tag{59}$$

$$\left(\int_0^2 k_{1i}(z)\,dz\right)\left(\int_0^2 k_{2j}(z)\,dz\right)\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right),$$

$i=1,...,r;\ j=1,...,p$.

We call

$$\lambda_{1i} := \int_0^2 k_{1i}(z)\,dz, \quad \lambda_{2j} := \int_0^2 k_{2j}(z)\,dz, \tag{60}$$

for $i=1,...,r;\ j=1,...,p$, as in (49).

We also set $\lambda_{10}=\lambda_{20}=1$.

Thus the following inequality is valid in general:

$$\left\| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}(f) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}(Q_{n,m}) \right\|_{\infty,[-1,1]^2} \leq \tag{61}$$

$$\lambda_{1i}\lambda_{2j}\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right),$$

for $i=0,1,...,r;\ j=0,1,...,p$.

Define

$$\rho_{n,m} := \widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\left[\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\left(l_{ij}\lambda_{1i}\lambda_{2j}n^{i-r}m^{j-p}\right)\right]. \tag{62}$$

I) Suppose, throughout $[-1,0]^2$, $\alpha_{h_1h_2}(x,y) \geq \alpha > 0$.
Let $Q^*_{n,m}(x,y) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$, $(x,y) \in [-1,1]^2$, as in (61), so that

$$\left\| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(f(x,y) + \rho_{n,m}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q^*_{n,m}(x,y) \right\|_{\infty,[-1,1]^2} \leq$$

$$\lambda_{1i}\lambda_{2j}\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right) =: T_{ij}, \tag{63}$$

for $i=0,1,...,r;\ j=0,1,...,p$.

If $(h_1+1,h_2+1) \leq (i,j) \leq (r,p)$, or $0 \leq i \leq h_1$, $h_2{+}1 \leq j \leq p$, or $h_1{+}1 \leq i \leq r$, $0 \leq j \leq h_2$ we get from the last

$$\left\| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}(f) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q^*_{n,m} \right\|_{\infty,[-1,1]^2} \leq$$

$$\lambda_{1i}\lambda_{2j}\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{64}$$

proving (50).

If $(0,0) \leq (i,j) \leq (h_1,h_2)$, we get that

$$\left\| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f + \frac{\rho_{n,m}}{h_1!h_2!}\,{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right) - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q^*_{n,m}(x,y) \right\|_{\infty,[-1,1]^2}$$

$$\leq T_{ij}. \tag{65}$$

That is for $(1,1) \leq (i,j) \leq (h_1,h_2)$, we have

$$\left\| {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f - {}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q^*_{n,m} \right\|_{\infty,[-1,1]^2} \leq$$

$$\frac{\rho_{n,m}}{h_1!h_2!}\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right\|_{\infty,[-1,1]^2}+T_{ij}=$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{h_1!h_2!}\left[\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}\left(l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}n^{i_*-r}m^{j_*-p}\right)\right]$$

$$\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right\|_{\infty,[-1,1]^2}+\lambda_{1i}\lambda_{2j}\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)= \quad (66)$$

$$\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\left\{\frac{1}{h_1!h_2!}\left[\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}n^{i_*-r}m^{j_*-p}\right]\right.$$

$$\left.\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right\|_{\infty,[-1,1]^2}+\frac{\lambda_{1i}\lambda_{2j}}{n^{r-i}m^{p-j}}\right\}\leq$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}\left\{\left(\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}\right)\right. \quad (67)$$

$$\left.\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right\|_{\infty,[-1,1]^2}+\lambda_{1i}\lambda_{2j}\right\},$$

proving (51).

If $i=j=0$, from (63) we obtain

$$\left\|f+\frac{\rho_{n,m}}{h_1!h_2!}x^{h_1}y^{h_2}-Q_{n,m}^*\right\|_{\infty,[-1,1]^2}\leq\frac{\widetilde{C}}{n^rm^p}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \quad (68)$$

and

$$\left\|f-Q_{n,m}^*\right\|_{\infty,[-1,1]^2}\leq\frac{\rho_{n,m}}{h_1!h_2!}+\frac{\widetilde{C}}{n^rm^p}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)=$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{h_1!h_2!}\left[\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}n^{i_*-r}m^{j_*-p}\right]$$

$$+\frac{\widetilde{C}}{n^rm^p}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)=$$

$$\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\left[\frac{1}{h_1!h_2!}\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}n^{i_*-r}m^{j_*-p}+\frac{1}{n^rm^p}\right]\leq \quad (69)$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}\left[\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}+1\right],$$

proving (52).

Next case is for $i=0$, $j=1,...,h_2$, from (63) we get

$$\left\|{}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}f+\frac{\rho_{n,m}x^{h_1}}{h_1!h_2!}\ {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}y^{h_2}-\ {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}Q_{n,m}^*\right\|_{\infty,[-1,1]^2}\leq \quad (70)$$

$$\lambda_{2j}\frac{\widetilde{C}}{n^rm^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right).$$

Therefore we have that

$$\left\|{}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}f-\ {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}Q_{n,m}^*\right\|_{\infty,[-1,1]^2}\leq$$

$$\frac{\rho_{n,m}}{h_1!h_2!}\left\| {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}y^{h_2}\right\|_{\infty,[-1,1]}+\lambda_{2j}\frac{\widetilde{C}}{n^r m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)=$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{h_1!h_2!}\left[\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}\left(l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}n^{i_*-r}m^{j_*-p}\right)\right]$$

$$\left\| {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}y^{h_2}\right\|_{\infty,[-1,1]}+\lambda_{2j}\frac{\widetilde{C}}{n^r m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)=$$

$$\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\left[\frac{\left[\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}\left(l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}n^{i_*-r}m^{j_*-p}\right)\right]}{h_1!h_2!}\right.$$

$$\left.\left\| {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}y^{h_2}\right\|_{\infty,[-1,1]}+\frac{\lambda_{2j}}{n^r m^{p-j}}\right]\leq \tag{71}$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}\left[\frac{\left(\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}\right)}{h_1!h_2!}\right.$$

$$\left.\left\| {}_{k_{2j}}^{k_{10}}D_{1-}^{(0,\alpha_{2j})}y^{h_2}\right\|_{\infty,[-1,1]}+\lambda_{2j}\right],$$

proving (53).

The case of $j=0$, $i=1,...,h_1$, is wet similarly as in (71). Namely we get

$$\left\| {}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}f-{}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}Q_{n,m}^*\right\|_{\infty,[-1,1]^2}\leq$$

$$\frac{\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}\left[\left(\frac{\sum_{i_*=h_1}^{v_1}\sum_{j_*=h_2}^{v_2}l_{i_*j_*}\lambda_{1i_*}\lambda_{2j_*}}{h_1!h_2!}\right)\right. \tag{72}$$

$$\left.\left\| {}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}x^{h_1}\right\|_{\infty,[-1,1]}+\lambda_{1i}\right],$$

proving (54).

So if $(x,y)\in[-1,0]^2$, we can write

$$\alpha_{h_1h_2}^{-1}(x,y)\,L^*\left(Q_{n,m}^*(x,y)\right)=\alpha_{h_1h_2}^{-1}(x,y)\,L^*\left(f(x,y)\right)+$$

$$\frac{\rho_{n,m}}{h_1!h_2!}\,{}_{k_{2h_2}}^{k_{1h_1}}D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)+\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)\cdot$$

$$\left[{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^*(x,y)-{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f(x,y)-\frac{\rho_{n,m}}{h_1!h_2!}\,{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right] \tag{73}$$

(by $L^*f\geq0$)

$$\overset{(63)}{\geq}\frac{\rho_{n,m}}{h_1!h_2!}\,{}_{k_{2h_2}}^{k_{1h_1}}D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)-$$

$$\left(\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}l_{ij}\lambda_{1i}\lambda_{2j}n^{i-r}m^{j-p}\right)\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)=$$

$$\rho_{n,m}\left(\frac{{}_{k_{2h_2}}^{k_{1h_1}}D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)}{h_1!h_2!}-1\right)= \tag{74}$$

$$\rho_{n,m}\left[\frac{\left({}_{k_{20}}^{k_{1h_1}}D_{1-}^{(\alpha_{1h_1},0)}x^{h_1}\right)\left({}_{k_{2h_2}}^{k_{10}}D_{1-}^{(0,\alpha_{2h_2})}y^{h_2}\right)}{h_1!h_2!}-1\right]=:\varphi.$$

If $h_1=h_2=0$, then $\alpha_{1h_1}=\alpha_{2h_2}=0$, and $\varphi=0$.

If $h_1=0$, and $h_2\neq 0$, then

$$\varphi=\rho_{n,m}\left[\frac{{}_{k_{2h_2}}^{k_{10}}D_{1-}^{(0,\alpha_{2h_2})}y^{h_2}}{h_2!}-1\right]=$$

$$\rho_{n,m}\left[(-1)^{h_2}\int_y^1 k_{2h_2}(t-y)\,dt-1\right]\overset{(h_2\text{ is even})}{=}\rho_{n,m}\left(\int_0^{1-y}k_{2h_2}(z)\,dz-1\right)\geq \tag{75}$$

$$\rho_{n,m}\left(\int_0^1 k_{2h_2}(z)\,dz-1\right)\geq 0,$$

by the assumption (40).

Similarly, we treat the case $h_1\neq 0$, $h_2=0$.

When $h_1,h_2\neq 0$, then we have

$$\varphi\overset{(h_1,h_2\text{ are even})}{=}\rho_{n,m}\left[\left(\int_x^1 k_{1h_1}(t-x)\,dt\right)\left(\int_y^1 k_{2h_2}(t-y)\,dt\right)-1\right]= \tag{76}$$

$$\rho_{n,m}\left[\left(\int_0^{1-x}k_{1h_1}(z)\,dz\right)\left(\int_0^{1-y}k_{2h_2}(z)\,dz\right)-1\right]\geq$$

$$\rho_{n,m}\left[\left(\int_0^1 k_{1h_1}(z)\,dz\right)\left(\int_0^1 k_{2h_2}(z)\,dz\right)-1\right]\geq 0,$$

by the assumption (40).

So in all four cases we get that

$$L^*\left(Q_{n,m}^*(x,y)\right)\geq 0,\ \forall\ (x,y)\in[-1,0]^2. \tag{77}$$

II) Suppose, throughout $[-1,0]^2$, $\alpha_{h_1h_2}(x,y)\leq\beta<0$. Let $Q_{n,m}^*(x,y)\in(P_n\otimes C([-1,1])+C([-1,1])\otimes P_m)$, $(x,y)\in[-1,1]^2$, as in (61), so that

$$\left\|{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(f(x,y)-\rho_{n,m}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)-{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^{**}(x,y)\right\|_{\infty,[-1,1]^2}\leq$$

$$\lambda_{1i}\lambda_{2j}\frac{\widetilde{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{78}$$

for $i=0,1,...,r$, $j=0,1,...,p$.

Similarly, we get as in the first case $\geq\alpha>0$, the inequalities of simultaneous fractional convergence, see {(51), (67)}, {(52), (69)}, {(53),(71)}, {(54), (72)}.

So if $(x,y)\in[-1,0]^2$ we can write

$$\alpha_{h_1h_2}^{-1}(x,y)\,L^*\left(Q_{n,m}^*(x,y)\right)=\alpha_{h_1h_2}^{-1}(x,y)\,L^*\left(f(x,y)\right)-$$

$$\frac{\rho_{n,m}}{h_1!h_2!}\,{}_{k_{2h_2}}^{k_{1h_1}}D_{1-}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1}y^{h_2}\right)+\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)\cdot \tag{79}$$

$$\left[{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^*(x,y)-{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}f(x,y)+\frac{\rho_{n,m}}{h_1!h_2!}\,{}_{k_{2j}}^{k_{1i}}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(x^{h_1}y^{h_2}\right)\right]$$

(by $L^{*}f \geq 0$)

$$\overset{(78)}{\leq} -\frac{\rho_{n,m}}{h_1!h_2!} \, {}_{k_{2h_2}}^{k_{1h_1}}D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right) +$$

$$\left(\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}\lambda_{1i}\lambda_{2j}n^{r-i}m^{p-j}\right)\widetilde{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right) = \tag{80}$$

$$\rho_{n,m}\left[1-\frac{{}_{k_{2h_2}}^{k_{1h_1}}D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)}{h_1!h_2!}\right] =$$

$$\rho_{n,m}\left[1-\frac{\left({}_{k_{20}}^{k_{1h_1}}D_{1-}^{\left(\alpha_{1h_1},0\right)}x^{h_1}\right)\left({}_{k_{2h_2}}^{k_{10}}D_{1-}^{\left(0,\alpha_{2h_2}\right)}y^{h_2}\right)}{h_1!h_2!}\right] =: \psi.$$

If $h_1 = h_2 = 0$, then $\alpha_{1h_1} = \alpha_{2h_2} = 0$, and $\psi = 0$.

If $h_1 = 0$, and $h_2 \neq 0$, then

$$\psi = \rho_{n,m}\left[1-\frac{{}_{k_{2h_2}}^{k_{10}}D_{1-}^{\left(0,\alpha_{2h_2}\right)}y^{h_2}}{h_2!}\right] = \rho_{n,m}\left[1-\int_y^1 k_{2h_2}\left(t-y\right)dt\right] = \tag{81}$$

$$\rho_{n,m}\left[1-\int_0^{1-y} k_{2h_2}\left(z\right)dz\right] \leq \rho_{n,m}\left[1-\int_0^1 k_{2h_2}\left(z\right)dz\right] \leq 0,$$

see (40).

Similarly, we treat the case $h_1 \neq 0$, $h_2 = 0$.

When $h_1, h_2 \neq 0$, then we have

$$\psi = \rho_{n,m}\left[1-\left(\int_x^1 k_{1h_1}\left(t-x\right)dt\right)\left(\int_y^1 k_{2h_2}\left(t-y\right)dt\right)\right] =$$

$$\rho_{n,m}\left[1-\left(\int_0^{1-x} k_{1h_1}\left(z\right)dz\right)\left(\int_0^{1-y} k_{2h_2}\left(z\right)dz\right)\right] \leq \tag{82}$$

$$\rho_{n,m}\left[1-\left(\int_0^1 k_{1h_1}\left(z\right)dz\right)\left(\int_0^1 k_{2h_2}\left(z\right)dz\right)\right] \leq 0,$$

by the assumption (40).

So in all four cases we proved again that

$$L^{*}\left(Q_{n,m}^{*}\left(x,y\right)\right) \geq 0, \ \forall \ \left(x,y\right) \in \left[-1,0\right]^2. \tag{83}$$

The proof is complete. □

Conclusion 11. *Clearly Theorem 10 generalizes Theorem 8 to many right side fractional calculi, opening new avenues of fractional research activity. The approximating pseudo-polynomial* $Q_{n,m}^{*}$ *depends on* $f, \rho_{n,m}, h_1, h_2$; *which* $\rho_{n,m}$ *depends on* $\widetilde{C}$ *(which depends on* r, p, s, q*),* $f, n, m, l_{ij}, \lambda_{1i}, \lambda_{2j}$; *and which:* λ_{1i} *depends on* k_{1i} *and* λ_{2j} *depends on* k_{2j}. *That is* $Q_{n,m}^{*}$ *depends on the type of bivariate right side fractional calculus we use.*

Consequently, Theorem 10 is valid at least for the following important bivariate right fractional linear differential operators:

1)

$$L_1^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \left[{}^{C}D^{(\gamma_1,\gamma_2)}_{(\rho_1,\rho_2),(\alpha_{1i},\alpha_{2j}),(\omega_1,\omega_2),(1-,1-)} \right], \qquad (84)$$

where $\rho_1, \rho_2 > 0$*,* $\gamma_1, \gamma_2 < 0$*, and* $\omega_1, \omega_2 > 0$ *large enough (from right bivariate Prabhakar fractional calculus, see (42));*

2)

$$L_2^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \left[D^{(\alpha_{1i},\alpha_{2j})}_{1-} \right], \qquad (85)$$

(see (45)) where $\theta = 1,2$*;* $\gamma_{\theta j} > 0$*,* $j = 1,...,m$*;* $\lambda_\theta = 1$*,* $0 < \rho_\theta < 1$*; and small enough* $\omega_{\theta j} < 0$*,* $j = 1,...,m$ *(from right bivariate generalized non-singular fractional calculus);*

and

3)

$$L_3^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \left[{}^{CF}_{(\omega_1,\omega_2)}D^{(\alpha_{1i},\alpha_{2j})}_{1-} \right], \qquad (86)$$

(see (46)) for small enough $\omega_1, \omega_2 < 0$ *(from right bivariate parametrized Caputo-Fabrizio non-singular kernel fractional calculus).*

Our developed right bivariate abstract fractional monotone approximation theory by pseudopolynomials with its applications, involves weaker conditions than the one with ordinary partial derivatives, see Theorem 4, and can manage many diverse general cases in a multitude of complex settings and environments.

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