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CONFORMABLE FRACTIONAL DOUBLE LAPLACE TRANSFORM AND ITS APPLICATIONS TO FRACTIONAL PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. A new integral transform for obtaining exact solutions of the fractional partial integro differential equations called "conformable double Laplace transform" is expressed. All the fractional derivatives and integrals are in conformable sense. Some basic properties and convolution theorem for conformable double Laplace transform are proved firstly in the literature. By using this newly defined integral transform and its properties, fractional partial integro-differential equations can be reduced into algebraic equations. Hence one can obtain the exact solutions quickly and effectively. Illustrative examples are included to demonstrate the validity and applicability of the presented transform.

1. Introduction

The idea of arbitrary order differentiation and integration started with the letter of L'Hospital to Leibniz in 1695. Although the integer order derivatives and integrals have clear physical and geometric descriptions, the fractional order differentiation and integration do not have admissible physical and geometrical meaning. Despite the absence of any physical or geometrical meaning, the number of studies on fractional calculus [1, 2, 3, 4] especially on the applications of fractional differentiation and integration have been growing day by day. Much of the studies using fractional differentiation and integration focussed on several areas such as engineering, physics, chemistry, biology, medicine and etc. which have huge amount of problems must be solved. The number of open problems which are expressed by using fractional differentiation and integration inspired many scientists. First of all scientists needed definitions of fractional derivatives and integrals. This requirement aroused the different definitions such as Grünwald-Letnikov, Riesz, Riemann-Liouville, Caputo [1, 2, 3, 4]. Despite the fact that each definition differs from the other one, Riemann-Liouville and Caputo are the most popular ones among them. Riemann-Liouville and Caputo fractional derivative involve integral forms in their definitions. These integral forms make calculations harder and any methods can

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not be applied to the problems. In addition to this scientists determined many deficiencies of these definitions. For instance [5]

(1) The Riemann-Liouville derivative does not satisfy $D_a^\alpha 1 = 0$(Caputo derivative satisfies), if α is not a natural number.
(2) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions.
$$D_a^\alpha(fg) = gD_a^\alpha(f) + fD_a^\alpha(g).$$
(3) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions.
$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{fD_a^\alpha(f) - gD_a^\alpha(g)}{g^2}.$$
(4) All fractional derivatives do not satisfy the chain rule.
$$D_a^\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t).$$
(5) All fractional derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general.
(6) In the Caputo definition it is assumed that the function f is differentiable.

As a result of these situations scientists began to look for a way out for expressing efficient, applicable, limpid and simple definition of arbitrary order derivation and integration. This search of the scientists gave a result. In 2014, a new, simple, well behaved fractional derivative and fractional integral definition which obey basic classical properties of known derivative and integral are expressed by Khalil *et. al.* [5].

Definition 1.1. Let $f : [0, \infty) \to \mathbb{R}$ be a function. The α^{th} order *"conformable fractional derivative"* of f is defined by,
$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$
for all $t > 0, \alpha \in (0, 1)$.

Definition 1.2. If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function f starting from $a \geq 0$ is defined as:
$$I_\alpha^a(f)(t) = \int_a^t f(x)d_\alpha x = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$$
where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

As we mentioned above, neither Riemann-Liouville fractional derivative nor Caputo fractional derivative has physical and geometrical descriptions. But this new fractional derivative has the physical and geometrical interpretations [6] and satisfies the following basic properties and theorems referred in [5, 7]

(1) $D_\alpha(cf + dg) = cD_\alpha(f) + cD_\alpha(g)$ for all $a, b \in \mathbb{R}$.
(2) $D_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
(3) $D_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
(4) $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$.
(5) $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$.

(6) If in addition to f is differentiable, then $D_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

After the expression of conformable derivative, many researchers paid attention to this new definition. Many applications involving this new definition are implemented by many scientists [8, 10, 9, 11] because of its easy applicability and lucidness. For instance Korkmaz *et. al.* [12] used $\exp(-\varphi(\epsilon))$ and modified Kudryashov methods to obtain the analytical solutions of time fractional parabolic equation with exponential nonlinearity. Hosseini *et. al.* [13] the conformable extracted the exact solutions of time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities by making use of the ansatz method. Khodadad *et. al.* [14] employed Riccati sub equation method to solve fractional Zakharov-Kuznetsov equation with dual-power law nonlinearity in the sense of the conformable derivative. Cao *et. al.* [15] solved time-fractional Burgers equation numerically. Yavuz [16] solved some initial boundary value problems with the help of conformable fractional Adomian decomposition method and conformable fractional modified homotopy perturbation method. Kaplan [17] presented analytical solutions of a nonlinear conformable time fractional equation with the aid of the modified simple equation method and the exponential rational function method. Souahi *et. al.* [18] discussed the Barbalat-type lemmas for conformable fractional order integrals which can be used to conclude the convergence of a function to zero. Kumar *et. al.* [19] studied to the sine-Gordon expansion method to have the analytical solutions of the Tzitzica equation, the Dodd-Bullough-Mikhailov (DBM) equation, the Tzitzica-Dodd-Bullough (TDB) equation, and the Liouville equation.

In this study the expresses transform is applied to fractional integro-differential equations which arise in the mathematical modeling of various physical events such as heat conduction in materials with memory, diffusion process and etc. The rest of the article is organized as follows. In section 2 the definition of newly defined conformable double Laplace transform and some theorems over basic properties of this definition are given. In section 3 the convolution theorem for conformable double Laplace transform is proven. In section 4 the exact solutions for partial integro-differential equations and integral equations are obtained as implementation of conformable double Laplace transform.

2. Conformable Double Laplace Transform

Definition 2.1. Let $u(x,t)$ be an exponential order, continuous function on the interval $[0,\infty)$ and for some $a,b \in \mathbb{R}$ $sup_{x>0,t>0}\frac{|u(x,t)|}{e^{\frac{ax^\beta}{\beta}+\frac{bt^\alpha}{\alpha}}} < \infty$ satisfied. Under these conditions conformable double Laplace transform is expressed by [20]

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta[u(x,t)] = U(p,s) = \int_0^\infty \int_0^\infty e^{-p\frac{x^\beta}{\beta}-s\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t d_\beta x \tag{1}$$

where $p,s \in \mathbb{C}$, $0 < \alpha,\beta \leq 1$ and the integrals are in the sense of conformable fractional integral.

Definition 2.2. (**Single Conformable Laplace Transform of a Function with Two Variables**)([20])
The single conformable Laplace transform of function $u(x,t)$ with respect to x of is denoted by

$$\mathscr{L}_x^\beta[u(x,t)] = U(p,t) = \int_0^\infty e^{-p\frac{x^\beta}{\beta}} u(x,t) d_\beta x \tag{2}$$

where the integral with respect to x is in conformable sense.
The symbol $\mathscr{L}_x^\beta[u(x,t)]$ indicates the conformable integral of (2), we consider the variable which the single conformable Laplace transform applied by the help of the subscript x on L. Alike the conformable Laplace transform of the function $u(x,t)$ with respect to variable t can be stated as

$$\mathscr{L}_t^\alpha[u(x,t)] = U(x,s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t \tag{3}$$

In the light of these definitions, double Laplace transformation denoted by $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta[u(x,t)]$ denoted in the equality (1). When the function $u(x,t)$ satisfies the adequate conditions [21], we can change the order of transformation, so

$$\int_0^\infty \int_0^\infty e^{-p\frac{x^\beta}{\beta} - s\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t d_\beta x = \int_0^\infty \int_0^\infty e^{-s\frac{t^\alpha}{\alpha} - p\frac{x^\beta}{\beta}} u(x,t) d_\beta x d_\alpha t$$

and symbolically can be shown as

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta[u(x,t)] = \mathscr{L}_x^\beta \mathscr{L}_t^\alpha[u(x,t)] = U(p,s).$$

2.1. **Some Properties of Conformable Double Laplace Transform.** Lets prove some of the properties of the conformable double Laplace Transform which can allows to find out further transform pairs $u(x,t), U(p,s)$.

Theorem 2.1. [20] *Let $u(x,t)$, $w(x,t)$ be two functions which have the conformable double Laplace transform. Thus,*

i. $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta[c_1 u(x,t) + c_2 w(x,t)] = c_1 \mathscr{L}_t^\alpha \mathscr{L}_x^\beta[u(x,t)] + c_2 \mathscr{L}_t^\alpha \mathscr{L}_x^\beta[w(x,t)]$ *where c_1 and c_2 are real constants.*
ii. $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta[e^{-d\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}} u(x,t)] = U(p+d, s+c)$.
iii. $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta[f(\gamma x, \sigma t)] = \frac{1}{r} U\left(\frac{p}{\gamma^\beta}, \frac{s}{\sigma^\alpha}\right)$ *where $r = \gamma^\beta \sigma^\alpha$.*
iv. $(-1)^{m+n} \mathscr{L}_t^\alpha \mathscr{L}_x^\beta\left[\frac{x^{m\beta}}{\beta^m} \frac{t^{n\alpha}}{\alpha^n} u(x,t)\right] = \frac{\partial^{m+n} U(p,s)}{\partial p^m \partial s^n}$.

Proof. i. By using the definition of conformable double Laplace transform the proof of (i) can be shown easily.
ii.

$$\begin{aligned}\mathscr{L}_t^\alpha \mathscr{L}_x^\beta\left[e^{-d\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}} u(x,t)\right] &= \int_0^\infty \int_0^\infty e^{-p\frac{x^\beta}{\beta} - s\frac{t^\alpha}{\alpha}} e^{-d\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t d_\beta x \\ &= \int_0^\infty e^{-p\frac{x^\beta}{\beta} - d\frac{x^\beta}{\beta}} \left(\int_0^\infty e^{-s\frac{t^\alpha}{\alpha} - c\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t\right) d_\beta x\end{aligned} \tag{4}$$

With the aid of conformable laplace transform definition

$$\int_0^\infty e^{-(s+c)\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t = U(x, s+c). \tag{5}$$

Now subrogating the Eqn. 5 into Eqn. 4 yields

$$\int_0^\infty e^{-(p+d)\frac{x^\beta}{\beta}} U(x, s+c) d_\beta x = U(p+d, s+c).$$

iii. Let $\tau = \gamma x$ and $\chi = \sigma t$, so the proof can be expressed as follows

$$\begin{aligned}
\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[u(\gamma x, \sigma t)] &= \int_0^{\infty}\int_0^{\infty} e^{-p\frac{x^\beta}{\beta}-s\frac{t^\alpha}{\alpha}} u(\gamma x, \sigma t) d_\alpha t d_\beta x \\
&= \int_0^{\infty} e^{-p\frac{x^\beta}{\beta}} \left(\int_0^{\infty} e^{-s\frac{t^\alpha}{\alpha}} u(\gamma x, \sigma t) d_\alpha t\right) d_\beta x \\
&= \frac{1}{\sigma^\alpha}\int_0^{\infty} e^{-p\frac{x^\beta}{\beta}} \left(\int_0^{\infty} e^{-s\frac{\chi^\alpha}{\sigma^\alpha \alpha}} u(\gamma x, \chi) d_\alpha \chi\right) d_\beta x \\
&= \frac{1}{\sigma^\alpha}\int_0^{\infty} e^{-p\frac{x^\beta}{\beta}} U\left(\gamma x, \frac{s}{\sigma^\alpha}\right) d_\beta x \\
&= \frac{1}{\sigma^\alpha \gamma^\beta}\int_0^{\infty} e^{-p\frac{\tau^\beta}{\gamma^\beta \beta}} U\left(\tau, \frac{s}{\sigma^\alpha}\right) d_\beta \tau \\
&= \frac{1}{\gamma^\beta \sigma^\alpha} U\left(\frac{p}{\gamma^\beta}, \frac{s}{\sigma^\alpha}\right).
\end{aligned}$$

iv. The order of differentiation and integration can be changed, due to convergence properties of the improper integral included. So we can differentiate with respect to p, s under the integral sign. Hence,

$$\frac{\partial^{m+n}U(p,s)}{\partial p^m \partial s^n} = \int_0^{\infty} \frac{\partial^m}{\partial p^m} e^{-p\frac{x^\beta}{\beta}} \left[\int_0^{\infty} \frac{\partial^n}{\partial s^n}\left(e^{-s\frac{t^\alpha}{\alpha}} u(x,t)\right) d_\alpha t\right] d_\beta x.$$

Repeating differentiation with respect to p and s, arises the following equation

$$\frac{\partial^{m+n}U(p,s)}{\partial p^m \partial s^n} = (-1)^{m+n}\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}\left[\frac{x^{m\beta}}{\beta^m}\frac{t^{n\alpha}}{\alpha^n}u(x,t)\right].$$

□

Theorem 2.2. *(The Convolution Theorem for Conformable Double Laplace Transform)If $U(p,s) = \mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[u(x,t)]$ and $V(p,s) = \mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[V(x,t)]$ both exist for $s > 0$ and $p > 0$, then*

$$\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[u(x,t) * v(x,t)] = U(p,s)V(p,s). \tag{6}$$

where $u(x,t) * v(x,t)$ denotes the convolution of the functions $u(x,t)$ and

Proof. Using the Lemma 5.2 in Ref. [7]

$$\begin{aligned}
\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[u*v](x,t) &= \mathscr{L}_t\mathscr{L}_x[u*v]((\beta x)^{\frac{1}{\beta}}, (\alpha t)^{\frac{1}{\alpha}}) \\
&= \int_0^{\infty}\int_0^{\infty} e^{-st-px}(u*v)((\beta x)^{\frac{1}{\beta}}, (\alpha t)^{\frac{1}{\alpha}}) dt dx \\
&= \int_0^{\infty}\int_0^{\infty} e^{-st-px}\left[\int_0^{x}\int_0^{t} u\left((\beta(x-\theta))^{\frac{1}{\beta}}, (\alpha(t-\eta))^{\frac{1}{\alpha}}\right) v\left((\beta\theta)^{\frac{1}{\beta}}, (\alpha\eta)^{\frac{1}{\alpha}}\right) d\eta d\theta\right] dt dx.
\end{aligned}$$

Making transformations $q = t - \eta$, $z = x - \theta$ yields

$$\begin{aligned}
&= \int_0^{\infty}\int_0^{\infty} e^{-s(q+\eta)-p(z+\theta)}\left[\int_0^{\infty}\int_0^{\infty} u\left((\beta z)^{\frac{1}{\beta}}, (\alpha q)^{\frac{1}{\alpha}}\right) v\left((\beta\theta)^{\frac{1}{\beta}}, (\alpha\eta)^{\frac{1}{\alpha}}\right) d\eta d\theta\right] dq dz \\
&= \int_0^{\infty}\int_0^{\infty} e^{-sq-pz} u\left((\alpha q)^{\frac{1}{\alpha}}, (\beta z)^{\frac{1}{\beta}}\right) dq dz \int_0^{\infty}\int_0^{\infty} e^{-s\eta-p\theta} v\left((\beta\theta)^{\frac{1}{\beta}}, (\alpha\eta)^{\frac{1}{\alpha}}\right) d\eta d\theta \\
&= U(p,s)V(p,s).
\end{aligned}$$

□

Lemma 2.1. [20] *The conformable double Laplace transform of β-th and α-th order fractional partial derivatives are given respectively as follows.*

$$\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[{}_xD_{\beta}u(x,t)] = pU(p,s) - U(0,s), \tag{7}$$

$$\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[{}_tD_{\alpha}u(x,t)] = sU(p,s) - U(p,0) \tag{8}$$

where ${}_xD_{\beta}u(x,t)$, ${}_tD_{\alpha}u(x,t)$ means β-th and α-th order conformable fractional partial derivatives respectively.
In the same manner the conformable double Laplace transform of mixed fractional partial derivatives

$$\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[{}_xD_{\beta}{}_tD_{\alpha}(u(x,t))] = psU(p,s) - pU(p,0) - sU(0,s) + U(0,0). \tag{9}$$

Proof. One can easily see the proof by using the definition of conformable fractional integral and Theorem 2.2 in Ref. [5]. □

Theorem 2.3. [20] *Let $0 < \alpha, \beta \leq 1$ and $m, n \in \mathbb{N}$ such that $u(x,t) \in C^l(\mathbb{R}^+ \times \mathbb{R}^+)$, $l =$max(m,n). Also let the conformable Laplace transforms of the functions $u(x,t)$, ${}_xD_{\beta}^{(i)}u(x,t)$ and ${}_tD_{\alpha}^{(j)}u(x,t)$ $i = 1, ..., m$, $j = 1, ..., n$ exist. Then*

$$\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[{}_xD_{\beta}^{(m)}u(x,t)] = p^mU(p,s) - p^{m-1}U(0,s) - \sum_{i=1}^{m-1} p^{m-1-i}\mathscr{L}_t^{\alpha}[{}_xD_{\beta}^{(i)}U(0,t)], \tag{10}$$

$$\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[{}_tD_{\alpha}^{(n)}u(x,t)] = s^nU(p,s) - s^{n-1}U(p,0) - \sum_{j=1}^{n-1} s^{n-1-j}\mathscr{L}_x^{\beta}[{}_tD_{\alpha}^{(j)}U(x,0)]. \tag{11}$$

In the same manner conformable double Laplace transform of mixed partial derivative

$$\begin{aligned}\mathscr{L}_t^{\alpha}\mathscr{L}_x^{\beta}[{}_xD^{(m)\beta}{}_tD^{(n)\alpha}(u(x,t))] &= p^ms^n\big(U(p,s) - s^{-1}U(p,0)\\ &- p^{-1}U(0,s) - \sum_{j=1}^{n-1} s^{-j-1}\mathscr{L}_x^{\beta}[{}_tD^{(j)\alpha}U(x,0)]\\ &- \sum_{i=1}^{m-1} p^{-i-1}\mathscr{L}_t^{\alpha}[{}_xD^{(i)\beta}U(0,t)]\\ &+ \sum_{j=1}^{n-1} s^{-j-1}p^{-1}{}_tD^{(j)\alpha}U(0,0)\\ &+ \sum_{i=1}^{m-1} s^{-1}p^{-i-1}{}_xD^{(i)\beta}U(0,0)\\ &+ \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} s^{-j-1}p^{-i-1}{}_tD^{(j)\alpha}{}_xD^{(i)\beta}U(0,0)\\ &+ p^{-1}s^{-1}U(0,0)\big)\end{aligned}$$

where ${}_xD_{\beta}^{(m)}u(x,t)$, ${}_tD_{\alpha}^{(n)}u(x,t)$ denotes m, n times conformable fractional derivatives of function $u(x,t)$ with order β and α respectively.

Functions $f(x,t)$	Conformable Double Laplace Transform $f(p,s)$
ab	$\frac{ab}{ps}$
xt	$\beta^{\frac{1}{\beta}}\alpha^{\frac{1}{\alpha}}\frac{\Gamma(1+\frac{1}{\beta})\Gamma(1+\frac{1}{\alpha})}{p^{\frac{\beta+1}{\beta}}s^{\frac{\alpha+1}{\alpha}}}$
$\frac{x^\beta}{\beta}\frac{t^\alpha}{\alpha}$	$\frac{1}{p^2s^2}$
$\frac{x^{m\beta}}{\beta}\frac{t^{n\alpha}}{\alpha}$, *m,n are natural numbers*	$\frac{m!n!}{p^{m+1}s^{n+1}}$
$e^{\frac{x^\beta}{\beta}+\frac{t^\alpha}{\alpha}}$	$\frac{1}{(s-1)(p-1)}$
$e^{\frac{x^\beta}{\beta}+\frac{t^\alpha}{\alpha}}\frac{x^{m\beta}}{\beta}\frac{t^{n\alpha}}{\alpha}$, *m,n are natural numbers*	$\frac{m!n!}{(p-1)^{m+1}(s-1)^{n+1}}$
$\cos\left(\omega\frac{x^\beta}{\beta}\right)\cos\left(\omega\frac{t^\alpha}{\alpha}\right)$	$\frac{ps}{(w^2+s^2)(w^2+p^2)}$
$\sin\left(\omega\frac{x^\beta}{\beta}\right)\sin\left(\omega\frac{t^\alpha}{\alpha}\right)$	$\frac{1}{(w^2+s^2)(w^2+p^2)}$
$\sqrt{\frac{x^\beta}{\beta}\frac{t^\alpha}{\alpha}}$	$\frac{\pi}{4(ps)^{3/2}}$
$e^{\frac{x^\beta}{\beta}+\frac{t^\alpha}{\alpha}}\sinh\left(\frac{x^\beta}{\beta}\right)\sinh\left(\frac{t^\alpha}{\alpha}\right)$	$\frac{1}{(p-2)p(s-2)s}$
$e^{\frac{x^\alpha}{\alpha}+\frac{t^\beta}{\beta}}\cosh\left(\frac{x^\alpha}{\alpha}\right)\cosh\left(\frac{t^\beta}{\beta}\right)$	$\frac{(p-1)(s-1)}{(p-2)p(s-2)s}$
$I_\alpha^a I_\beta^a(f(x,t))$	$\frac{f(p,s)}{ps}$

Table 1. Conformable double Laplace transform of some basic functions.

Proof. The proof follows from Lemma 2.1. □

3. Illustrative examples

Example 3.1. *Consider the following conformable fractional partial integro-differential equation*

$$D_t^{2\alpha}u(x,t)-D_x^{2\beta}u(x,t)+u(x,t)+\int_0^x\int_0^t e^{\frac{(x-\nu)^\beta}{\beta}+\frac{(t-\mu)^\alpha}{\alpha}}u(\nu,\mu)d_\beta\nu d_\alpha\mu = e^{\frac{x^\beta}{\beta}+\frac{t^\alpha}{\alpha}}\left(1+\frac{x^\beta}{\beta}\frac{t^\alpha}{\alpha}\right) \tag{12}$$

with the conditions

$$\begin{aligned} u(0,t) &= e^{\frac{t^\alpha}{\alpha}}, \\ u(x,0) &= e^{\frac{x^\beta}{\beta}}, \\ D_x^\beta u(0,t) &= e^{\frac{t^\alpha}{\alpha}} \\ D_t^\alpha u(x,0) &= e^{\frac{x^\beta}{\beta}} \end{aligned} \tag{13}$$

where $0<\beta\le 1$, $0<\alpha\le 1$, $x>0$, $t>0$, $D_t^{2\alpha}$, $D_x^{2\beta}$ *denotes the two times* α*-th and* β*-th order conformable fractional derivative of function* $u(x,t)$ *and the integrals are in conformable sense. First of all applying the double conformable Laplace transform to Eq. (12) and using the convolution theorem led to*

$$\begin{aligned} &s^2U(p,s)-sU(p,0)-D_t^\alpha U(p,0)-p^2U(p,s)+pU(0,s)+D_x^\beta U(0,s)+U(p,s) \\ &+\frac{U(p,s)}{(s-1)(p-1)}=\frac{1}{(s-1)(p-1)}+\frac{1}{(s-1)^2(p-1)^2} \end{aligned} \tag{14}$$

where $U(p,s)$ is the double conformable Laplace transformed version of the function $u(x,t)$. Then applying the conformable Laplace transform to the conditions (13) yields

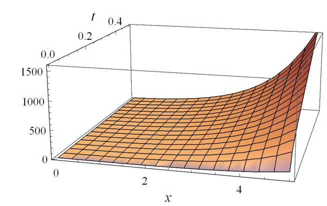
$$\begin{aligned}
\mathscr{L}_t^{\alpha}[u(0,t)] &= \mathscr{L}_t^{\alpha}\left[e^{\frac{t^{\alpha}}{\alpha}}\right] = \frac{1}{s-1},\\
\mathscr{L}_x^{\beta}[u(x,0)] &= \mathscr{L}_x^{\beta}\left[e^{\frac{x^{\beta}}{\beta}}\right] = \frac{1}{p-1},\\
\mathscr{L}_t^{\alpha}\left[D_x^{\beta}u(0,t)\right] &= \mathscr{L}_t^{\alpha}\left[e^{\frac{t^{\alpha}}{\alpha}}\right] = \frac{1}{s-1}\\
\mathscr{L}_x^{\beta}\left[D_t^{\alpha}u(x,0)\right] &= \mathscr{L}_x^{\beta}\left[e^{\frac{x^{\beta}}{\beta}}\right] = \frac{1}{p-1}.
\end{aligned} \tag{15}$$

By using Eqns.(15) and making some algebraic operations to Eqn. (14) we get

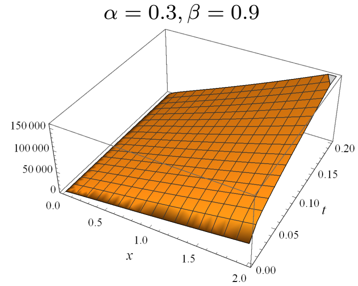
$$U(p,s) = \frac{1}{(p-1)(s-1)}.$$

So $u(x,t)$ can be obtained as

$$u(x,t) = e^{\frac{t^{\alpha}}{\alpha}+\frac{x^{\beta}}{\beta}}.$$



(A) $u(x,t)$ for $\alpha = 0.3, \beta = 0.9$



(B) $u(x,t)$ for $\alpha = 0.4, \beta = 0.1$

FIGURE 1. Behavior of $u(x,t)$ for different values of α and β

Example 3.2. *Consider the following conformable fractional partial integro-differential equation*

$$D_t^{\alpha}u + D_t^{3\alpha}u - \int_0^t \sinh\left(\frac{(t-y)^{\alpha}}{\alpha}\right) D_x^{3\beta}u(x,y)d_{\alpha}y = 0 \tag{16}$$

via the conditions

$$u(0,t) = 0, D_x^\beta u(0,t) = \sin\frac{t^\alpha}{\alpha}, D_x^{2\beta} u(0,t) = 0,$$
$$u(x,0) = 0, D_t^\alpha u(x,0) = \frac{x^\beta}{\beta}, D_t^{2\alpha} u(x,0) = 0 \tag{17}$$

with $0 < \beta \leq 1$, $0 < \alpha \leq 1$, $x > 0$, $t > 0$, $D_t^{3\alpha}$, $D_x^{3\beta}$ *denotes the three times* α *and* β *order conformable fractional derivative of function* $u(x,t)$ *and the integrals are by means of conformable sense. Employing the conformable double Laplace transform for Eq. (16) produces*

$$sU(p,s) - U(p,0) + s^3 U(p,s) - s^2 U(p,0) - sD_t^\alpha U(p,0) - D_t^{2\alpha} U(p,0) - \frac{1}{s^2+1}$$
$$\left(p^3 U(p,s) - p^2 U(0,s) - pD_x^\beta U(0,s) - D_x^{2\beta} U(0,s)\right) = 0. \tag{18}$$

Afterwards implementing conformable Laplace transform to the conditions (17)

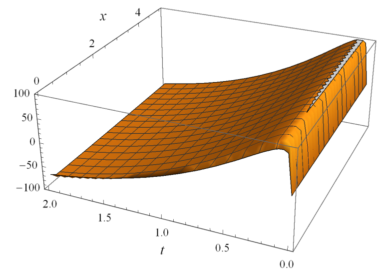
$$\mathscr{L}_t^\alpha[u(0,t)] = 0, \mathscr{L}_t^\alpha[D_x^\beta u(0,t)] = \frac{1}{s^2+1}, \mathscr{L}_t^\alpha[D_x^{2\beta} u(0,t)] = 0,$$
$$\mathscr{L}_x^\beta[u(x,0)] = 0, \mathscr{L}_x^\beta[D_t^\alpha u(x,0)] = \frac{1}{p^2}, \mathscr{L}_x^\beta[D_t^{2\alpha} u(x,0)] = 0 \tag{19}$$

Gathering all the acquired results (18),(19) and making some algebraic arrangements led to

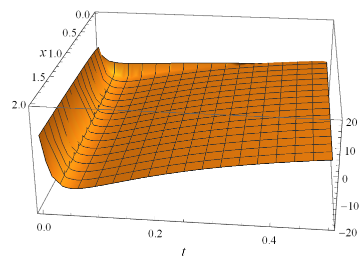
$$U(p,s) = \frac{1}{p^2(s^2+1)}.$$

Thus we can get the function $u(x,t)$ as

$$u(x,t) = \frac{x^\beta}{\beta}\sin\left(\frac{t^\alpha}{\alpha}\right).$$



(A) $u(x,t)$ for $\alpha = 0.3, \beta = 0.01$



(B) $u(x,t)$ for $\alpha = 0.05, \beta = 0.1$

FIGURE 2. Effect of the different values for α and β to the analytical solution $u(x,t)$

4. Conclusions

In this article we introduce the conformable fractional double Laplace transform, give some basic properties and convolution theorem for this new transform. We give the table of conformable double Laplace transform for some basic functions. Then the solutions partial integro-differential equations are obtained with the aid of conformable double Laplace transform. Thereby conformable double Laplace transform is newly defined, there are many open problems on this subject. This transform can be an efficient technique for finding the solutions of conformable fractional partial integro-differential equations which can correspond to physical end engineering models.

References

[1] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, (John Wiley &Sons, New York, 1993).

[2] A.Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, (Elsevier, San Diego, 2006).

[3] I. Podlubny, Fractional Differential Equations, (Academic Press,San Diego, 1999).

[4] K. Oldham, J. Spanier, The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier, 1974.

[5] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264, 65 (2014).

[6] D. Zhao, M. Luo, General conformable fractional derivative and its physical interpretation, Calcolo, 1-15 (2017).

[7] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math. 279, 57 (2015).

[8] M. Eslami, H. Rezazadeh, M. Rezazadeh, S.S. Mosavi, Exact solutions to the space-time fractional Schrdinger-Hirota equation and the spacetime modified KDV-Zakharov-Kuznetsov equation, Opt Quant Electron., 49, 279, (2017).

[9] A. Korkmaz, Explicit exact solutions to some one-dimensional conformable time fractional equations, Waves in Random and Complex Media, 1-14, (2017).

[10] M.B. Hossen, H. Roshid, M.Z. Ali, Modified Double Sub-equation Method for Finding Complexiton Solutions to the $(1+1)$ Dimensional Nonlinear Evolution Equations, International Journal of Applied and Computational Mathematics, 3(1), 679, (2017).

[11] M. Al-Refai, T. Abdeljawad, Fundamental Results of Conformable Sturm-Liouville Eigenvalue Problems, Complexity, vol. 2017, Article ID 3720471, 7 pages, (2017).

[12] A. Korkmaz, K. Hosseini, Exact solutions of a nonlinear conformable time-fractional parabolic equation with exponential nonlinearity using reliable methods, Optical and Quantum Electronics, 49(8), 278, (2017).

[13] K. Hosseini, P. Mayeli, R. Ansari, Bright and singular soliton solutions of the conformable time-fractional KleinGordon equations with different nonlinearities, Waves in Random and Complex Media, 1-9, (2017).

[14] F.S. Khodadad, F. Nazari, M. Eslami, H. Rezazadeh, Soliton solutions of the conformable fractional Zakharov-Kuznetsov equation with dual-power law nonlinearity, Opt. Quant. Electron., 49, 384, (2017).

[15] W. Cao, Q. Xu, Z. Zheng, Solution of two-dimensional time-fractional Burgers equation with high and low Reynolds numbers, Adv. Differ. Equ., 2017, 338, (2017).

[16] M. Yavuz, Novel solution methods for initial boundary value problems of fractional order with conformable differentiation. An International Journal of Optimization and Control: Theories and Applications (IJOCTA), 8(1), 1-7, (2017).

[17] M. Kaplan, Applications of two reliable methods for solving a nonlinear conformable time-fractional equation, Opt. Quant. Electron., 49, 312, (2017).

[18] A. Souahi, O. Naifar, A.B .Makhlouf, M. A. Hammami, Discussion on Barbalat Lemma extensions for conformable fractional integrals. International Journal of Control.,90, 1-8, (2017).

[19] D. Kumar, K. Hosseini, F.Samadani, The sine-Gordon expansion method to look for the traveling wave solutions of the Tzitzica type equations in nonlinear optics, Optik-International Journal for Light and Electron Optics, 149, 439, (2017).

[20] O. Özkan, A. Kurt, On conformable double Laplace transform. Optical and Quantum Electronics, 50(2), 103, (2018).
[21] E. Love, Journal of the Australian Mathematical Society, 11(4), 421-432. doi:10.1017/S1446788700007904.

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